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Remark on the Betti numbers for Hamiltonian circle actions

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Abstract. In this paper, we establish a certain inequality in terms of Betti numbers of a closed Hamiltonian S^1-manifold with isolated fixed points.

Résumé. Dans cet article, nous établissons une certaine inégalité en termes de nombres de Betti d'une S^1-variété hamiltonienne avec des points fixes isolés.

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1. Introduction

Let (M,ω) be a $2n$-dimensional closed symplectic manifold admitting a Hamiltonian torus action with only isolated fixed points. It has been a long-standing open problem whether M admits a Kähler metric or not. Historically, Delzant [9] proved that if M admits a Hamiltonian T^n-action, where the fixed point set is automatically discrete, then M admits a T^n-invariant Kähler metric. Restricting to an S^1-action case, several results on the existence of a Kähler metric were provided in some special cases. For instance, Karshon [13] proved that every closed symplectic four manifold admitting a Hamiltonian circle action admits a Kähler metric. (In fact, the S^1-action is induced from a toric action when the fixed points are isolated.) Also if $\dim M = 6$ with $b_2(M) = 1$, then it turned out that M admits a Kähler metric, which was proved by Tolman [18] and McDuff [15]. Recently, the author has shown that any 6-dimensional monotone closed semifree Hamiltonian S^1-manifold admits a Kähler metric, see [5–7].

As a counterpart, there were "candidates" of closed Hamiltonian T-manifolds (with isolated fixed points) which possibly fail to admit Kähler metrics. Tolman [17] and Woodward [19] constructed a six-dimensional closed Hamiltonian T^2-manifold with only isolated fixed points and with no T^2-invariant Kähler metric. Surprisingly Goertsches–Kostantis–Zoller [10] have recently shown that the examples of Tolman and Woodward indeed admit Kähler metrics that are not T^2-invariant. Thus their result provides a positive evidence for the conjecture of the existence of Kähler metrics.

On the other hand, it seems reasonable to ask whether (M,ω) enjoys Kählerian properties, such as the hard Lefschetz property of the symplectic form ω or the unimodality of even Betti numbers. Recall that every closed Kähler manifold (M,ω,J) satisfies the *hard Lefschetz property*, that is,

$$[\omega]^{n-k} : H^k(M;\mathbb{R}) \to H^{2n-k}(M;\mathbb{R})$$
$$\alpha \mapsto \alpha \cup [\omega]^{n-k}$$

is an isomorphism for every $k = 0,1,\cdots,n$. This implies that

$$[\omega] : H^k(M;\mathbb{R}) \to H^{k+2}(M;\mathbb{R})$$

is injective for every k with $0 \le k < n$, and therefore the sequence of even (as well as odd) Betti numbers of M is unimodal. In other words,

$$b_k \le b_{k+2}, \quad k = 0,1,\cdots,n-1$$

where b_i denotes the i^{th} Betti number of M. In this paper we deal with the following conjecture.

Conjecture 1 ([12]). *Let (M,ω) be a $2n$-dimensional closed symplectic manifold equipped with a Hamiltonian S^1-action with only isolated fixed points. Then the sequence of even Betti numbers is* unimodal*, i.e.,*

$$b_{2i} \le b_{2i+2} \quad \text{for every} \quad 0 \le i < \left\lfloor \frac{n}{2} \right\rfloor.$$

It is worth mentioning that every odd Betti number of M vanishes by Frankel's theorem which states that a moment map is a Morse function whose critical points are of even indices. (See [2, Theorem IV.2.3].) Therefore we only need to care about even Betti numbers of M.

In [8], the author and Kim proved Conjecture 1 when $\dim M = 8$. The main goal of this article is to improve the result of [8] and prove the following inequality, which is automatically satisfied when Conjecture 1 is true.

Theorem 2. *Let (M,ω) be a closed symplectic manifold admitting a Hamiltonian circle action with only isolated fixed points where $\dim M = 8n$ or $8n+4$. Then*

$$b_2 + \cdots + b_{2+4(n-1)} \le b_4 + \cdots + b_{4+4(n-1)}.$$

In particular when $\dim M = 8$ or 12, we have

$$b_2 \le b_4.$$

2. Proof of the main Theorem 2

The main technique for proving Theorem 2 is the ABBV-localization due to Atiyah–Bott and Berline–Vergne. Recall that for an S^1-manifold M, the *equivariant cohomology* is defined by $H^*_{S^1}(M) := H^*(M \times_{S^1} ES^1)$ where ES^1 is a contractible space on which S^1 acts freely. Then $H^*_{S^1}(M)$ inherits an $H^*(BS^1)$-module structure induced from the projection

$$\pi : M \times_{S^1} ES^1 \to BS^1 := ES^1/S^1.$$

Note that $H^*(BS^1;\mathbb{R}) \cong H^*(\mathbb{C}P^\infty;\mathbb{R}) = \mathbb{R}[u]$. Moreover, for the inclusion map $i : M^{S^1} \hookrightarrow M$, we have an induced ring homomorphism

$$i^* : H^*_{S^1}(M;\mathbb{R}) \to H^*_{S^1}\left(M^{S^1};\mathbb{R}\right) \cong H^*\left(BS^1;\mathbb{R}\right) \otimes H^*\left(M^{S^1};\mathbb{R}\right).$$

When $M^{S^1} = \{p_1, \cdots, p_m\}$ is discrete, we may express as

$$H^*\left(BS^1;\mathbb{R}\right) \otimes H^*\left(M^{S^1};\mathbb{R}\right) \cong \bigoplus_{i=1}^{m} H^*\left(BS^1;\mathbb{R}\right)$$

and so

$$i^*(\alpha) = \left(f_1, \cdots, f_m\right), \quad f_i \in \mathbb{R}[u]$$

for $\alpha \in H^*_{S^1}(M;\mathbb{R})$. We denote by $\alpha|_{p_i} := f_i$ and call it the *restriction of α to p_i*. By the Kirwan's injectivity theorem [14], the map i^* is injective and hence $H^*_{S^1}(M;\mathbb{R})$ is a free $H^*(BS^1;\mathbb{R})$-module.

Theorem 3 (ABBV Localization Theorem [1,3]). *Let M be a closed S^1-manifold with only isolated fixed points and $\alpha \in H^*_{S^1}(M;\mathbb{R})$. Then we have*

$$\int_M \alpha = \sum_{p \in M^{S^1}} \frac{\alpha|_p}{\left(\prod_{i=1}^n w_i(p)\right) u^n}.$$

where $w_1(p), \cdots, w_n(p)$ denote the weights of the tangential S^1-representation at p.

To obtain Theorem 2, we will apply Theorem 3 to *canonical classes* which form a basis of $H^*_{S^1}(M;\mathbb{R})$ as an $H^*(BS^1;\mathbb{R})$-module.

Theorem 4 (16, Lemma 1.13][1]). *Let (M,ω) be a $2n$-dimensional closed Hamiltonian S^1-manifold with only isolated fixed points. For each fixed point $p \in M^{S^1}$ of index $2k$, there exists a unique class $\alpha_p \in H^{2k}_{S^1}(M;\mathbb{Z})$ such that*

- *$\alpha_p|_q = 0$ for every $q (\neq p) \in M^{S^1}$ with either $H(q) \leq H(p)$ or $\operatorname{ind}(q) \leq 2k$,*
- *$\alpha_p|_p = \prod_{i=1}^k \lambda_i u$, where $\lambda_1, \cdots, \lambda_k$ are negative weights of the S^1-action at p.*

Moreover, the set $\{\alpha_p \mid p \in M^{S^1}\}$ is a basis of $H^*_{S^1}(M;\mathbb{R})$ as an $H^*(BS^1;\mathbb{R})$-module.

Now we are ready to prove Theorem 2.

Proof of Theorem 2. We first consider the case $\dim M = 8n$. Suppose that

$$b_2 + \cdots + b_{2+4(n-1)} > b_4 + \cdots + b_{4+4(n-1)}. \tag{1}$$

Since $H^*_{S^1}(M)$ is a free module over $H^*(BS^1)$, we have

$$H^{4n-2}_{S^1}(M) \cong u^0 \otimes H^{4n-2}(M) \oplus u^1 \otimes H^{4n-4}(M) \oplus \cdots \oplus u^{(2n-1)} \otimes H^0(M)$$

which implies that

- $\dim_{\mathbb{R}} H^{4n-2}_{S^1}(M;\mathbb{R}) \cong b_0 + b_2 + \cdots + b_{4n-2}$, and
- $\{\alpha_p \cdot u^{2n-1-\frac{1}{2}\operatorname{ind}(p)} \mid p \in M^{S^1},\ \operatorname{ind}(p) \leq 4n-2\}$ is a basis of $H^{4n-2}_{S^1}(M;\mathbb{R})$ (as an $\mathbb{R}$-vector space) by Theorem 4.

Now, consider the following map

$$\begin{aligned} \Phi : H^{4n-2}_{S^1}(M;\mathbb{R}) &\to \left(\mathbb{R}^{b_0} \oplus \mathbb{R}^{b_4} \oplus \cdots \oplus \mathbb{R}^{b_{4(n-1)}}\right) \oplus \left(\mathbb{R}^{b_{4n}} \oplus \cdots \oplus \mathbb{R}^{b_{8n-4}}\right) \\ \alpha &\mapsto (\alpha_0, \cdots, \alpha_{4n-4}, \alpha_{4n}, \cdots, \alpha_{8n-4}) \end{aligned}$$

with the identification

$$\mathbb{R}^{b_{4i}} = \bigoplus_{\operatorname{ind}(p)=4i} \mathbb{R} \cdot u^{2n-1} \quad \text{and} \quad \alpha_{4i} := \left(\alpha|_p\right)_{\operatorname{ind}(p)=4i} \in \bigoplus_{\operatorname{ind}(p)=4i} \mathbb{R} \cdot u^{2n-1} \tag{2}$$

for each $i = 1, \cdots, n$. Since the dimension of the range of the map Φ satisfies

$$\dim \operatorname{Im} \Phi \leq b_0 + \cdots + b_{4n-4} + (b_{4n} + b_{4n+4} + \cdots + b_{8n-4}) < b_0 + \cdots + b_{4n-4} + (b_{4n-2} + \cdots + b_2)$$

[1] See also [18, Proposition 2.2] and [11, Lemma 2.10]

by (1) and Poincaré duality, the map Φ has a non-trivial kernel. In other words, there exists a nonzero element $\alpha \in H^{4n-2}_{S^1}(M;\mathbb{R})$ such that

$$\alpha|_p = 0$$

for every fixed point $p \in M^{S^1}$ of index $0, 4, \cdots, 8n-4$.

Now fix a moment map H for the S^1-action on (M,ω) such that H attains the maximum value 0. Denote by $p_{\max}$ the maximal fixed point and so $\mathrm{ind}(p_{\max}) = 8n$. The equivariant extension $[\omega_H] \in H^2_{S^1}(M;\mathbb{R})$ of ω with respect to the moment map H satisfies

$$[\omega_H]\,|_p = -H(p)u \in \mathbb{R}[u]$$

for every $p \in M^{S^1}$, see [4, Proposition 2.6]. Since $H(p) < 0$ for every $p \neq p_{\max}$ by the choice of H, we obtain

$$0 = \int_M \alpha^2 \cdot [\omega_H] = \sum_{p \in M^{S^1}} \frac{-\alpha^2|_p \cdot H(p)u}{\left(\prod_{i=1}^{4n} w_i(p)\right) u^{4n}} = \sum_{\mathrm{ind}(p)\equiv 2 \ (\mathrm{mod}\, 4)} \frac{-\alpha^2|_p \cdot H(p)u}{\left(\prod_{i=1}^{4n} w_i(p)\right) u^{4n}} \tag{3}$$

by Theorem 3 and the fact $[\omega_H]|_{p_{\max}} = -H(p_{\max})u = 0$. Moreover, there exists at least one fixed point $p \in M^{S^1}$ such that

$$\alpha|_p \neq 0 \quad \text{and} \quad \mathrm{ind}(p) < 8n$$

because

- $\alpha|_p \neq 0$ for some $p \in M^{S^1}$ by the Kirwan's Injectivity Theorem [14], and
- if $\alpha|_p = 0$ for every $p \in M^{S^1}$ with $p \neq p_{\max}$, then $\alpha|_{p_{\max}} \neq 0$ and it violates Theorem 3

$$0 = \int_M \alpha = \frac{\alpha|_{p_{\max}}}{\left(\prod_{i=1}^{4n} w_i(p)\right) u^{4n}} \neq 0.$$

Consequently, each summand of the rightmost equation of (3) has non-positive coefficient (of $\frac{1}{u}$) and at least one of those should be negative. Therefore it leads to a contradiction.

Now it remains to consider the case of $\dim M = 8n+4$. Under the same assumption (1), we similarly define

$$\begin{aligned}\Phi : H^{4n}_{S^1}(M;\mathbb{R}) &\to \left(\mathbb{R}^{b_0} \oplus \mathbb{R}^{b_4} \oplus \cdots \oplus \mathbb{R}^{b_{4n}}\right) \oplus \left(\mathbb{R}^{b_{4n+4}} \oplus \cdots \oplus \mathbb{R}^{b_{8n}}\right) \\ \alpha &\mapsto (\alpha_0, \cdots, \alpha_{4n}, \alpha_{4n+4}, \cdots, \alpha_{8n})\end{aligned}$$

with the same identification as in (2). Note that $\dim_{\mathbb{R}} H^{4n}_{S^1}(M;\mathbb{R}) = b_0 + b_2 + \cdots + b_{4n-2} + b_{4n}$ and

$$\begin{aligned}\dim \mathrm{Im}\,\Phi \leq b_0 + \cdots + b_{4n} + (b_{4n+4} + \cdots + b_{8n}) &= b_0 + \cdots + b_{4n} + (b_{4n} + \cdots + b_4) \\ < b_0 + \cdots + b_{4n} + (b_{4n-2} + \cdots + b_2) &= \dim_{\mathbb{R}} H^{4n}_{S^1}(M;\mathbb{R})\end{aligned}$$

by (1) and Poincaré duality. Thus Φ has a non-trivial kernel $\alpha \in H^{4n}_{S^1}(M;\mathbb{R})$ and so there exists a nonzero $\alpha \in H^{4n}_{S^1}(M;\mathbb{R})$ such that $\alpha|_p = 0$ for every fixed point p of index $0, 4, \ldots, 4n$. Therefore, we obtain

$$0 = \int_M \alpha^2 \cdot [\omega_H] = \sum_{\mathrm{ind}(p)\equiv 2 \ (\mathrm{mod}\, 4)} \frac{-\alpha^2|_p \cdot H(p)u}{\left(\prod_{i=1}^{4n+2} w_i(p)\right) u^{4n+2}} \neq 0$$

which leads to a contradiction. This completes the proof of Theorem 2. □

References

[1] M. F. Atiyah, R. Bott, "The moment map and equivariant cohomology", *Topology* **23** (1984), no. 1, p. 1-28.

[2] M. Audin, *Topology of Torus actions on symplectic manifolds Second revised edition*, Progress in Mathematics, vol. 93, Birkhäuser, 2004.

[3] N. Berline, M. Vergne, "Classes charactéristiques équivariantes. Formule de localisation en cohomologie équivariante", *C. R. Math. Acad. Sci. Paris* **295** (1982), p. 539-541.

[4] Y. Cho, "Unimodality of Betti numbers for Hamiltonian circle actions with index-increasing moment maps", *Int. J. Math.* **27** (2016), no. 5, article no. 1650043 (14 pages).

[5] ———, "Classification of six dimensional monotone symplectic manifolds admitting semifree circle actions I", *Int. J. Math.* **30** (2019), no. 6, article no. 1950032 (71 pages).

[6] ———, "Classification of six dimensional monotone symplectic manifolds admitting semifree circle actions II", https://arxiv.org/abs/1904.10962, to appear in *International Journal of Mathematics*, 2021.

[7] ———, "Classification of six dimensional monotone symplectic manifolds admitting semifree circle actions III", https://arxiv.org/abs/1905.07292, to appear in *International Journal of Mathematics*, 2021.

[8] Y. Cho, M. K. Kim, "Unimodality of the Betti numbers for Hamiltonian circle action with isolated fixed points", *Math. Res. Lett.* **21** (2014), no. 4, p. 691-696.

[9] T. Delzant, "Hamiltoniens périodiques et images convexes de l'application moment", *Bull. Soc. Math. Fr.* **116** (1988), no. 3, p. 315-339.

[10] O. Goertsches, P. Konstantis, L. Zoller, "GKM theory and Hamiltonian non-Kähler actions in dimension 6", *Adv. Math.* **368** (2020), article no. 107141.

[11] R. F. Goldin, S. Tolman, "Towards Generalizing Schubert Calculus in the Symplectic Category", *J. Symplectic Geom.* **7** (2009), no. 4, p. 449-473.

[12] L. Jeffrey, T. Holm, Y. Karshon, E. M. Lerman, E. Meinrenken, "Moment maps in various geometries", 2005, available at http://www.birs.ca/workshops/2005/05w5072/report05w5072.pdf.

[13] Y. Karshon, *Periodic Hamiltonian flows on four-dimensional manifolds*, Memoirs of the American Mathematical Society, vol. 672, American Mathematical Society, 1999.

[14] F. C. Kirwan, *Cohomology of quotients in symplectic and algebraic geometry*, Mathematical Notes, vol. 31, Princeton University Press, 1984.

[15] D. McDuff, "Some 6-dimensional Hamiltonian S^1-manifolds", *J. Topol.* **2** (2009), no. 3, p. 589-623.

[16] D. McDuff, S. Tolman, "Topological properties of Hamiltonian circle actions", *Int. Math. Res. Pap.* **2006** (2006), no. 4, article no. 72826.

[17] S. Tolman, "Examples of non-Kähler Hamiltonian torus actions", *Invent. Math.* **131** (1998), no. 2, p. 299-310.

[18] ———, "On a symplectic generalization of Petrie's conjecture", *Trans. Am. Math. Soc.* **362** (2010), no. 8, p. 3963-3996.

[19] C. Woodward, "Multiplicity-free Hamiltonian actions need not be Kähler", *Invent. Math.* **131** (1998), no. 2, p. 311-319.