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# Meromorphic solutions of generalized inviscid Burgers' equations and related PDES 

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#### Abstract

The purposes of this paper are twofold. The first one is to describe entire solutions of certain type of PDEs in $\mathbb{C}^{n}$ with the modified KdV-Burgers equation and modified Zakharov-Kuznetsov equation as the prototypes. The second one is to characterize entire and meromorphic solutions of generalized inviscid Burgers' equations in $\mathbb{C}^{2}$.


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## 1. Introduction and main results

In this paper, we firstly describe entire solutions of the following partial differential equations in $\mathbb{C}^{n}$

$$
\begin{equation*}
u_{t}+B(u) u_{x}=L\left(u_{x}\right), \tag{1}
\end{equation*}
$$

where $B(u)=\sum_{k=0}^{m} b_{k} u^{k}\left(b_{m} \not \equiv 0, m \geq 2\right)$ is a polynomial in $u$ and $L\left(u_{x}\right)$ is a partial differential polynomial in $u_{x}$ with degree at most $m-1$, and the coefficients of $B(u)$ and $L\left(u_{x}\right)$ are rational functions. Equation (1) have some prototypes, such as the modified KdV-Burgers equation and modified Zakharov-Kuznetsov ( mZK ) equation, which are stated in (2), (3) respectively.

$$
\begin{align*}
& u_{t}-\alpha u^{2} u_{x}=\beta u_{x x}-u_{x x x}  \tag{2}\\
& u_{t}+\beta u^{2} u_{x}=-\left(u_{x x x}+u_{x y y}\right), \tag{3}
\end{align*}
$$

where $\alpha$ and $\beta$ are nonzero constants. These PDEs are occurring in various areas of applied mathematics, such as fluid mechanics, nonlinear acoustics, gas dynamics, and traffic flow. There are so many approaches developed over years to analyze/solve such PDEs. For example, Hassan in [5] obtained abundant new exact solutions of modified Zakharov-Kuznetsov (mZK) equation arising in plasma and dust plasma are presented by using the extended mapping method and the availability of symbolic computation. These solutions include the Jacobi elliptic function
solutions, hyperbolic function solutions, rational solutions, and periodic wave solutions. Yuan in [26] employed complex method to obtain all traveling meromorphic exact solutions of the modified Zakharov-Kuznetsov (mZK) equation.

It is known that the study of complex entire and meromorphic solutions of partial differential equations has a long history. Given a nonlinear differential equation, it is in general difficult to find such solutions in closed form. Recently, by employing Nevanlinna theory, Li, Hu and their coworkers considered entire solutions of some certain PDEs, see e.g., [7,10, 12, 15, 17]. In this paper, inspired by their ideas, we firstly consider the solutions of (1) for complex variables. Actually, we describe entire solutions of (1) as follows.
Theorem 1. Suppose that $u$ is an entire solution of (1). Then $u$ is a polynomial. In particular, $u$ is constant if the coefficients of $B(u), L\left(u_{x}\right)$ are constant and $L\left(u_{x}\right)$ does not contain the constant term.
Remark 2. From Theorem 1, we can see that all entire solutions of (2)-(3) are constant. It is also pointed out that the conclusion of Theorem 1 is invalid for meromorphic solutions, as seen by the following Example 3, which can be found in [26, Theorem 1].
Example 3. Consider the mZK equation (3). Substituting traveling wave transformation $u(x, t, y)=w(z), z=k(x+l y)$, into (3), and integrating it yields

$$
\begin{equation*}
k^{2}\left(l^{2}+1\right) w^{\prime \prime}+\frac{\beta}{3} w^{3}=0 . \tag{4}
\end{equation*}
$$

Then, the following function is a transcendental meromorphic solution of (4).

$$
w_{d}(z)=C \frac{(-\wp+A)\left(4 \wp A^{2}+4 \wp^{2} A+2 \wp^{\prime} B-\wp g_{2}-A g_{2}\right)}{\left(\left(12 A^{2}-g_{2}\right) \wp+4 A^{3}-3 A g_{2}\right) \wp^{\prime}+4 B \wp^{3}+12 A B \wp^{2}-3 B g_{2} \wp-A B g_{2}},
$$

where

$$
C= \pm \frac{1}{2} \sqrt{-\frac{6 k^{2}\left(1+l^{2}\right)}{\beta}}
$$

$B^{2}=4 A^{3}-g_{2} A, g_{2}$ and $A$ are arbitrary constants, $\wp$ is Weierstrass elliptic function. Therefore, the function $u(x, t, y)=w(k(x+l y))$ is a meromorphic solution of (3). In fact, one can find some other transcendental meromorphic solutions to (3) in [26].
Remark 4. We emphasize that the method in Theorem 1 does not work if the degree of $L\left(u_{x}\right)$ is $m$. Unfortunately, we can not handle this case and leave it for further research. The conclusion of Theorem 1 is invalid if the degree of $L\left(u_{x}\right)$ is larger than $m$, as seen by the following Example 5 .
Example 5. The function $u(x, t)=e^{x+t}$ satisfies the equation $u_{t}+u^{2} u_{x}=\left(u_{x x}\right)^{3}+u_{x}$. But $u(x, t)$ is not a polynomial.

Next, we turn our attention to entire and meromorphic solutions of general inviscid Burgers' equation. Burgers' equation or Bateman-Burgers equation

$$
\begin{equation*}
u_{t}+u u_{x}=\beta u_{x x} \tag{5}
\end{equation*}
$$

is a fundamental partial differential equation occurring in various areas of applied mathematics. This is the simplest PDE combining both nonlinear propagation effects and diffusive effects. The Burgers' equation has been solved by many ways. In [4], Fay derived one of the most interesting solutions of Burgers' equation, in a series form. Without any auxiliary conditions, a Riccati solution for the Burgers' equation was found by Rodin in [20]. By employing a new finiteelement method, Varoglu and Finn in [24] solved the Burgers' equation which was based on the combination of the space-time elements and the characteristics. When the right term of (5) is removed, one gets the hyperbolic PDE

$$
\begin{equation*}
u_{t}+u u_{x}=0 \tag{6}
\end{equation*}
$$

which is called inviscid Burgers' equation. In [15], Li considered entire solutions of (6). More generally, Li obtained the following.

Theorem A. A function $u$ is an entire solution of the partial differential equation $u_{t}-C u^{m} u_{x}=0$ in $\mathbb{C}^{2}$, where $C \neq 0$ is a constant and $m \geq 0$ is an integer, if and only if $u$ is a constant when $m>0$; and $u=f(x+C t)$ when $m=0$, where $f$ is an entire function in the complex plane.

Naturally, one may ask what will happen if the right side 0 of the above PDE in Theorem A is replaced by a polynomial $Q$. Obviously, the equation $u_{t}-C u^{m} u_{x}=Q$ has nonconstant entire solutions when $m>0$, such as $u=x$ is a solution to $u_{t}-C u^{m} u_{x}=-C x^{m}$. Observe that $u=x$ is a polynomial. This observation leads us to ask whether all entire solutions of the above PDE are polynomials or not if $m>0$. Next, we focus on these questions and consider entire solutions of the following equation (which can be called general inviscid Burgers' equation)

$$
\begin{equation*}
u_{t}-C u^{m} u_{x}=Q, \tag{7}
\end{equation*}
$$

where $C \neq 0$ is a constant, $m \geq 0$ and $Q$ is a polynomial. Actually, combining characteristic equations for quasi-linear partial differential equations, normal family and the Nevanlinna theory, we derive that

Theorem 6. Suppose that $u$ is an entire solution to (7). Then the following assertions hold:
(1) If $m \geq 2$, then $u$ is a polynomial;
(2) If $m=1$, then either $u$ is a polynomial or

$$
u(x, t)=e^{\beta(t)}+x \alpha(t)+A,
$$

where $A$ is a constant, $\alpha(t)(\not \equiv 0), \beta(t)$ are polynomials satisfying

$$
\beta^{\prime}(t)=C \alpha(t), x\left[\alpha^{\prime}(t)-C \alpha^{2}(t)\right]-C A \alpha(t)=Q(x, t) ;
$$

(3) If $m=0$, then

$$
u(x, t)=F(t, x+C t)+f(x+C t),
$$

where $F(\theta, s)=\int_{0}^{\theta} Q(\theta,-C \theta+s) d \theta$ is a polynomial and $f$ is an entire function in the complex plane.

Remark 7. From Theorem 6 (2) one can obtain that $u_{t}-C u u_{x}=Q$ does not admit transcendental entire solutions if $\operatorname{deg}_{x} Q \neq 1$ with respective to the variable $x$. We also point out that all the cases (1)-(3) in Theorem 6 can indeed occur.

For the special case that $Q$ is constant, we can immediately derive the following Corollary 8 by Theorem 6.

Corollary 8. Suppose that $u$ is an entire solution to $u_{t}-C u^{m} u_{x}=A$, where $A$ and $C(\neq 0)$ are constant. Then the following assertions hold:
(1) If $m \geq 1$, then $u(t)=A t+B$, where $B$ is an arbitrary constant;
(2) If $m=0$, then

$$
u(x, t)=A t+f(x+C t)
$$

where $f$ is an entire function in the complex plane.
Obviously, Corollary 8 is an improvement of Theorem A. It is also pointed out that (7) admits meromorphic functions. For example, $u=\frac{1}{x+t}+x t$ is a meromorphic solution to $u_{t}-u_{x}=x-t$; $u=\frac{x}{t}+t^{2}$ is a meromorphic solution to $u_{t}+u u_{x}=3 t$. In [21], Saleeby considered meromorphic solutions of $u_{t}-C u^{m} u_{x}=0$. Below, by the characteristic equations for quasi-linear partial differential equations, we also describe meromorphic solutions to (7) as follows.

Theorem 9. Suppose that $u$ is a meromorphic solution to (7). Then the following assertions hold:
(1) If $m \geq 2$, then $u$ is a rational function;
(2) If $m=1$, then $u$ is a rational function or $u=\frac{P_{0}(t, x)+P_{1}(t, x) u}{Q_{0}(t, x)+Q_{1}(t, x) u}$, where $P_{i}(t, x), Q_{i}(t, x)$ ( $i=0,1$ ) are polynomials;
(3) If $m=0$, then

$$
u(x, t)=F(t, x+C t)+f(x+C t)
$$

where $F(\theta, s)=\int_{0}^{\theta} Q(\theta,-C \theta+s) d \theta$ is a polynomial and $f$ is a meromorphic function in $\mathbb{C}$.
Remark 10. The conclusion (2) of Theorem 9 seems unclear. Actually, the form

$$
u=\frac{P_{0}(t, x)+P_{1}(t, x) u}{Q_{0}(t, x)+Q_{1}(t, x) u}
$$

may become the trivial case $u=u$. Therefore, we can not get any useful information from the equation. Indeed, the conclusion (2) does not describe the specific forms of meromorphic solutions to (7) as in Theorem 6, and our method is invalid for this case $m=1$. Here, we leave this case for further research.

In order to prove the above results, we need some notations and results. Let $f(\xi)$ be a meromorphic function in $\mathbb{C}^{n}$. We use the following Nevanlinna characteristic function (see e.g., [13,22])

$$
T_{f}(r, s)=\int_{s}^{r} \frac{A_{f}(t)}{t} d t \text { for } r \geq s>0
$$

where

$$
A_{f}(t)=\frac{(n-1)!}{\pi^{n} t^{2 n-2}} \int_{B_{n}(t)} \frac{i}{2\left(1+|f|^{2}\right)^{2}} d f \wedge d \bar{f} \wedge \omega_{n}
$$

Here $B_{n}(t)=\left\{\xi \in \mathbb{C}^{n}:|\xi|<t\right\}$, and

$$
\omega_{n}\left(\xi_{1}, \ldots, \xi_{n}\right)=\frac{1}{(n-1)!}\left\{\sum_{j=1}^{n} \frac{i}{2} d \xi_{j} \wedge d \bar{\xi}_{j}\right\}^{n-1} .
$$

The order $\varrho(f)$ of $f$ is defined by

$$
\varrho(f)=\limsup _{r \rightarrow \infty} \frac{\log ^{+} T_{f}(r, s)}{\log r}, \text { for fixed } s>0 .
$$

Below, we also write $T_{f}(r, s)$ as $T(r, f)$. It is said that $f$ has finite (respectively infinite) order if $\varrho(f)$ is finite (respectively infinite). We remark that $f$ is rational if and only if $T(r, f)=O(\log r)$.

We are able to prove (2) of Theorem 6 by utilizing the following Proposition 11, which is itself of independent interest. And the proofs of Proposition 11 are based on normal family, which is stated in Section 3.

Proposition 11. Suppose that $u$ is an entire solution to

$$
\begin{equation*}
u_{t}-C u^{m} u_{x}=Q, \tag{8}
\end{equation*}
$$

where $C \neq 0$ is a constant, $m \geq 1$ and $Q$ is a polynomial. Then $u$ is of finite order.
Remark 12. Obviously, the conclusion is invalid if $m=0$, as seen by Theorem A and the following Example 13.
Example 13. The function $u=e^{e^{(x+C t)}}$ is an entire solution of $u_{t}-C u_{x}=0$. But $u$ is of infinite order.

Before to proceed, we assume the reader's familiarity with the basic notations of Nevanlinna theory of meromorphic functions in $\mathbb{C}^{n}$, such as the characteristic function $T(r, u)$ and the proximity function $m(r, u)$, and utilize four results of a meromorphic function $u$ in $\mathbb{C}^{n}$ (see e.g., [1,22,25]):
(a) The Nevanlinna first fundamental theorem $T\left(r, \frac{1}{u-a}\right)=T(r, u)+O(1)$ for any meromorphic function $u$ and constant $a$;
(b) The logarithmic derivative lemma

$$
\| m\left(r, \frac{u_{z_{j}}}{u}\right)=O(\log r T(r, u)),
$$

where the symbol $\|$ means that the relation holds outside a set of $r$ of finite linear measure;
(c) Suppose that $R(u)=\frac{P(u)}{Q(u)}$, where $P, Q$ are co-prime polynomials in $u$ with rational coefficients. Then

$$
\| T(r, R(u))=\max \{\operatorname{deg} P, \operatorname{deg} Q\} T(r, u)+O(\log r)
$$

(d) If $f$ is a transcendental meromorphic function in $\mathbb{C}$ and $g$ is a transcendental entire function in $\mathbb{C}^{n}$, then $\lim _{r \rightarrow \infty} \frac{T(r, f(g))}{T(r, g)}=\infty$.

## 2. Proofs of main theorems

In this section, base on the ideas in $[15,21]$, we give the proofs of Theorems 1,6 and 9.
Proof of Theorem 1. The proof of Theorem 1 is based on the following general Clunie Lemma 14 in $\mathbb{C}^{n}$, which can be seen in [11]. For the case $n=1$, see [6, Lemma 3.3]. For some special cases, see [8,9]. A general proof can be found in [14].

Lemma 14 (General Clunie Lemma.). Let f be a nonconstant meromorphic function on $\mathbb{C}^{n}$. Take a positive integer $t$ and take polynomials of $f$ and its partial derivatives:

$$
\begin{array}{ll}
P(f)=\sum_{\mathbf{p} \in I} a_{\mathbf{p}} f^{p_{0}}\left(\partial^{\mathbf{i}_{f}} f\right)^{p_{1}} \cdots\left(\partial^{i_{i}} f\right)^{p_{i}}, \quad \mathbf{p}=\left(p_{0}, \ldots, p_{l}\right) \in \mathbb{Z}_{+}^{l+1} \\
Q(f)=\sum_{\mathbf{q} \in J} c_{\mathbf{q}} f^{q_{0}}\left(\partial^{\mathbf{i}_{1}} f\right)^{q_{1}} \cdots\left(\partial^{\mathbf{i}_{s}} f\right)^{q_{s}}, & \mathbf{q}=\left(q_{0}, \ldots, q_{s}\right) \in \mathbb{Z}_{+}^{s+1}
\end{array}
$$

and

$$
B(f)=\sum_{k=0}^{t} b_{k} f^{k},
$$

where $I, J$ are finite sets of distinct elements and $a_{p}, c_{q}, b_{k}$ are rational functions on $\mathbb{C}^{n}$ with $b_{t} \not \equiv 0$. Assume that $f$ satisfies the equation

$$
B(f) Q(f)=P(f)
$$

If $\operatorname{deg}(P(f)) \leq t=\operatorname{deg}(B(f))$, then

$$
\| m(r, Q(f))=O(\log r T(r, f))
$$

Now, we give the proof of Theorem 1. Firstly, assume that $u$ is transcendental. We rewrite (1) as

$$
\begin{equation*}
B(u) u_{x}=L\left(u_{x}\right)-u_{t} \tag{9}
\end{equation*}
$$

Applying General Clunie Lemma 14 to equation (9) leads to $\| m\left(r, u_{x}\right)=O(\log r T(r, u))$. Then

$$
\| T\left(r, u_{x}\right)=m\left(r, u_{x}\right)=O(\log r T(r, u))
$$

Assume that $u_{x} \not \equiv 0$. Then the fact (a) yields that $T\left(r, \frac{1}{u_{x}}\right)=T\left(r, u_{x}\right)+O(1)=O(\log r T(r, u))$. Note that $m \geq 2$. So, the fact (c) yields that

$$
\begin{align*}
\| m T(r, u) & =T(r, B(u))+O(\log r) \leq m\left(r, B(u) u_{x}\right)+m\left(r, \frac{1}{u_{x}}\right)+O(\log r) \\
& \leq m\left(r, L\left(u_{x}\right)-u_{t}\right)+T\left(r, u_{x}\right)+O(\log r)  \tag{10}\\
& \leq(m-1) T(r, u)+O(\log r T(r, u))
\end{align*}
$$

which implies that $T(r, u)=O(\log r T(r, u))$. It forces that $u$ reduces to a polynomial, a contradiction. Thus, $u_{x} \equiv 0$. Then $u$ is a function of the complex variable $t$ only and (1) reduces to $u_{t} \equiv C(t)$, a polynomial, which implies that $u$ is also a polynomial, a contradiction. All the above discussions yields that $u$ is a polynomial. Moreover, suppose the coefficients of $B(u)$ and $L\left(u_{x}\right)$ are constant. Note that $u$ is a polynomial and $L\left(u_{x}\right)$ does not contain the constant term. By comparing the degrees of both sides of (1), one can easily deduce that $u$ is constant. Thus, the proof of Theorem 1 is finished.

Proof of Theorem 6. Due to Theorem 1, Proposition 11 and the characteristic equations for quasi-linear partial differential equations, we give the proof of Theorem 1. From Theorem 6, it is suffice to consider the cases $m=1$ and $m=0$. We consider two cases.

Case 1. [ $m=1]$
Applying General Clunie Lemma to (7) yields that $\| m\left(r, u_{x}\right)=O(\log r T(r, u))$. By Proposition 11, one gets that $f$ is of finite order. Then $T(r, u)=O\left(r^{\rho}\right)$, where $\rho$ is a non-negative finite number. Further,

$$
\| T\left(r, u_{x}\right)=m\left(r, u_{x}\right)=O(\log r T(r, u))=O(\log r)
$$

which implies that $u_{x}$ is a polynomial. Then, one can assume that

$$
\begin{equation*}
u(x, t)=P(x, t)+v(t)+A \tag{11}
\end{equation*}
$$

where $A$ is a constant, $P(x, t)$ is a polynomial and $v(t)$ is an entire function. By differentiating (11) with respect to the variables $t$ and $x$, we have

$$
u_{t}(x, t)=P_{t}(x, t)+v^{\prime}(t), u_{x}(x, t)=P_{x}(x, t)
$$

Substituting the above functions into (7) with $m=1$, one has

$$
\begin{equation*}
P_{t}(x, t)+v^{\prime}(t)-C[P(x, t)+v(t)+A] P_{x}(x, t)=Q(x, t) . \tag{12}
\end{equation*}
$$

Suppose that $v(t)$ is transcendental. Then, the equation (12) indicates that

$$
\begin{equation*}
v^{\prime}(t)=C v(t) P_{x}(x, t), P_{t}(x, t)-C P(x, t) P_{x}(x, t)-C A P_{x}(x, t)=Q(x, t) \tag{13}
\end{equation*}
$$

The former equation yields that $P_{x}(x, t)$ is a polynomial of the complex variable $t$ only. Assume that $P_{x}(x, t)=\alpha(t)$. Then, $P(x, t)=x \alpha(t)$. Furthermore, integrating the first equation of (13) leads to that

$$
v(t)=e^{C \int \alpha(t) d t}=e^{\beta(t)}
$$

where $\beta(t)$ is also a polynomial with $\beta^{\prime}(t)=C \alpha(t)$. The latter equation of (13) yields that

$$
x\left[\alpha^{\prime}(t)-C \alpha^{2}(t)\right]-C A \alpha(t)=Q(x, t)
$$

which is the desired result.
Case 2. [ $m=0$ ] Then, (7) becomes $u_{t}-C u_{x}=Q$. Here, we employ the method in [16] to deal with this case. Note that the characteristic equations for the above partial differential equation are of the forms

$$
\frac{d t}{d \theta}=1, \frac{d x}{d \theta}=-C, \frac{d u}{d \theta}=Q
$$

Using the initial condition: $t=0, x=s$, and $u=u(0, s):=f(s)$ with a parameter $s$, we obtain the following parametric representation for the solutions of the characteristic equations: $t=\theta, x=$ $-C \theta+s, u=F(\theta, s)+f(s)$, where $F(\theta, s)=\int_{0}^{\theta} Q(\theta,-C \theta+s) d \theta$ is a polynomial. Note that $\theta=t$ and $s=x+C t$. Thus, we have

$$
u(x, t)=F(t, x+C t)+f(x+C t)
$$

Hence, the proof of Theorem 6 is finished.
Proof of Theorem 9. Note that the characteristic equations for the above partial differential equation are of the forms

$$
\frac{d t}{d \theta}=1, \frac{d x}{d \theta}=-C u^{m}, \frac{d u}{d \theta}=Q
$$

As the proof in Theorem 6, using the initial condition: $t=0, x=s$, and $u=u(0, s):=f(s)$ with a parameter $s$, we obtain the following parametric representation for the solutions of the characteristic equations:

$$
t=\theta, x=-C u^{m} \theta+s, u=F\left(\theta, s, u^{m}\right)+f(s), \text { where } F\left(\theta, s, u^{m}\right)=\int_{0}^{\theta} Q\left(\theta,-C u^{m} \theta+s\right) d \theta
$$

Obviously, $F$ is a polynomial. Note that $\theta=t$ and $s=x+C u^{m} t$. Thus, we have

$$
u(x, t)=F\left(t, x+C u^{m} t, u^{m}\right)+f\left(x+C u^{m} t\right)
$$

For $m=0$, then we can rewrite $u$ as

$$
u(x, t)=F(t, x+C t)+f(x+C t),
$$

which is the conclusion of Theorem 9 (3).
Suppose that $m \geq 1$. Note that $F$ is a polynomial. By the fact (d) (which was obtained by Chang-Li-Yang in [1, Theorem 4.1]), one gets $f$ must be a rational function. Further, we can set $u$ as

$$
u=\frac{\sum_{i=0}^{\mu} P_{i}(t, x)\left(u^{m}\right)^{i}}{\sum_{j=0}^{v} Q_{j}(t, x)\left(u^{m}\right)^{j}}
$$

where $\mu, v$ are two non-negative integers, $P_{i}(t, x), Q_{j}(t, x)$ are polynomials with $P_{\mu}(t, x) \not \equiv 0$ and $Q_{v}(t, x) \not \equiv 0$. Assume that $u$ is transcendental. Then, the fact (c) yields

$$
\begin{aligned}
T(r, u) & =T\left(r, \frac{\sum_{i=0}^{\mu} P_{i}(t, x)\left(u^{m}\right)^{i}}{\sum_{j=0}^{v} Q_{j}(t, x)\left(u^{m}\right)^{j}}\right) \\
& =\max \{v, \mu\} T\left(r, u^{m}\right)+O(\log r)=m \max \{v, \mu\} T(r, u)+O(\log r)
\end{aligned}
$$

which implies that $m=1$ and $\max \{v, \mu\}=1$. Thus, we obtain that $u$ is a rational function when $m \geq 2$. This is Theorem 9 (1).

For $m=1$, one has the form $u=\frac{P_{0}(t, x)+P_{1}(t, x) u}{Q_{0}(t, x)+Q_{1}(t, x) u}$. This is Theorem 9 (2).
The proof of Theorem 9 is finished.

## 3. Proof of Proposition 11

For the proof of Proposition 11, we need the following facts of normal family, which can be found in [3].

A family $\mathscr{F}$ of holomorphic functions on a domain $\Omega \subseteq \mathbb{C}^{n}$ is normal in $\Omega$ if every sequence of functions $\left\{f_{j}\right\} \subseteq \mathscr{F}$ contains either a subsequence which converges to a limit function $f \neq \infty$ uniformly on each compact subset of $\Omega$, or a subsequence which converges uniformly to $\infty$ on each compact subset.

A family $\mathscr{F}$ is said to be normal at a point $z_{0} \in \Omega$ if it is normal in some neighbourhood of $z_{0}$. A family of holomorphic functions $\mathscr{F}$ is normal in a domain $\Omega$ if and only if $\mathscr{F}$ is normal at each point of $\Omega$.

For every function $\phi$ of class $\mathbb{C}^{2}(\Omega)$, define at each point $z \in \Omega$ a Hermitian form

$$
L_{z}(\varphi, v):=\sum_{k, l=1}^{n} \frac{\partial^{2} \varphi}{\partial z_{k} \partial \bar{z}_{l}}(z) v_{k} \bar{v}_{l}
$$

and call it the Levi form of the function $\phi$ at $z$. For a holomorphic function $f$ in $\Omega$, set

$$
\begin{equation*}
f^{\sharp}(z):=\sup _{|\nu|=1} \sqrt{L_{z}\left(\log \left(1+|f|^{2}\right), v\right)} . \tag{14}
\end{equation*}
$$

This quantity is well defined since the Levi form $L_{z}\left(\log \left(1+|f|^{2}\right), v\right)$ is non-negative for all $z \in \Omega$. In particular, for $n=1$, formula (14) takes the form

$$
f^{\sharp}(z):=\frac{\left|f^{\prime}(z)\right|}{1+|f(z)|^{2}},
$$

which is the spherical derivative of $f$ in $\mathbb{C}$.
We now recall Zalcman's Rescalling Lemma in several complex variables, which is given by Dovbush in [3].

Lemma 15. Suppose that a family $\mathscr{F}$ of functions holomorphic on $\Omega \subseteq \mathbb{C}^{n}$ is not normal at some point $z_{0} \in \Omega$. Then there exist sequences $f_{j} \in \mathscr{F}, z_{j} \rightarrow z_{0}, \rho_{j} \rightarrow 0$, such that the sequence

$$
g_{j}(z)=f_{j}\left(z_{j}+\rho_{j} z\right)
$$

converges locally uniformly in $\mathbb{C}^{n}$ to a nonconstant entire function $g$.
Remark 16. We point out that if $a_{j} \rightarrow z_{0}$ and $f_{j}^{\sharp}\left(a_{j}\right) \rightarrow \infty$, then one can choose $\rho_{j}$ in Lemma 15 satisfying

$$
\begin{equation*}
\rho_{j} \leq \frac{M}{f_{j}^{\sharp}\left(a_{j}\right)}, \tag{15}
\end{equation*}
$$

where $M$ is a fixed positive constant. For $n=1$, some similar results can be found in [19,27].
Proof of Proposition 11. Below, based on the idea in [18], we will give the proof. On the contrary, we assume that $u$ is of infinite order. Firstly, we claim that for every $N>0$, there exists a sequence $w_{n} \rightarrow \infty$ such that, if $n$ is sufficiently large

$$
\begin{equation*}
u^{\sharp}\left(w_{n}\right)>\left|w_{n}\right|^{N} . \tag{16}
\end{equation*}
$$

Assume the claim is not true. Then, there exist $N>0$ and $R_{0}>0$ such that for all $z,|z| \geq R_{0}$, one has

$$
\begin{equation*}
u^{\sharp}(z)<|z|^{N} . \tag{17}
\end{equation*}
$$

A calculation yields that

$$
\begin{equation*}
L_{z}\left(\log \left(1+|u|^{2}\right), v\right)=\frac{|d u(z) v|^{2}}{\left(1+|u(z)|^{2}\right)^{2}} \tag{18}
\end{equation*}
$$

By combining (17) and (18), one has

$$
\begin{equation*}
\frac{|d u(z) v|^{2}}{\left(1+|u(z)|^{2}\right)^{2}} \leq|z|^{2 N}, \text { for }|v|=1 \tag{19}
\end{equation*}
$$

Further, we have

$$
\begin{aligned}
A_{u}(t) & =\frac{1}{\pi} t^{-2} \int_{B_{2}(t)} \frac{i}{2\left(1+|u|^{2}\right)^{2}} d u \wedge d \bar{u} \wedge \omega_{2} \\
& \leq K^{\prime} t^{-2} \int_{B_{2}(t)} \frac{1}{\left(1+|u|^{2}\right)^{2}}\left[\left(\frac{\partial u}{\partial x}\right)^{2}+\left(\frac{\partial u}{\partial t}\right)^{2}\right] \omega_{2} \\
& \leq K^{\prime \prime} t^{2 N-2} \int_{B_{2}(t)} \omega_{2} \leq K t^{2 N+2}
\end{aligned}
$$

where $K^{\prime}, K^{\prime \prime}$ and $K$ are positive constants. Then,

$$
T_{u}(r)=\int_{s}^{r} \frac{A_{u}(t)}{t} d t \leq K \frac{r^{2 N+2}}{2 N+2}
$$

which implies that $u$ is of finite order. Thus, the claim holds.
Below, we employ the following Marty's Criterion of normal families in terms of the spherical metric on $\mathbb{C}^{n}$, which can be found in $[2,3,23]$.

Lemma 17 (Marty's Criterion). A family $\mathscr{F}$ of functions holomorphic on $\Omega$ is normal on $\Omega \subseteq \mathbb{C}^{n}$ if and only if for each compact subset $K \subseteq \Omega$ there exists a constant $M(K)$ such that at each point $z \in K$

$$
f^{\sharp}(z) \leq M(K)
$$

for all $f \in \mathscr{F}$.
For $N$ large enough, we choose a sequence $w_{n}$ satisfying (16). Define $D=\{z:|z|<1\}$ and

$$
f_{n}(z)=u\left(w_{n}+z\right)
$$

Then all $f_{n}(z)$ are holomorphic in $D$ and $f_{n}^{\sharp}(0)=w^{\sharp}\left(w_{n}\right) \rightarrow \infty$ as $n \rightarrow \infty$. It follows from Marty's criterion that $\left(f_{n}\right)_{n}$ is not normal at $z=0$. Therefore, we can apply Lemma 15 and Remark 16. Choosing an appropriate subsequence of $\left(f_{n}\right)_{n}$ if necessary, we may assume that there exist sequence $z_{n} \rightarrow 0$ and $\rho_{n} \leq \frac{M}{f_{n}^{\sharp}(0)}=\frac{M}{u^{\sharp}\left(w_{n}\right)} \rightarrow 0$ such that the sequence $g_{n}$ defined by

$$
\begin{equation*}
g_{n}(\zeta)=f_{n}\left(z_{n}+\rho_{n} \zeta\right)=u\left(w_{n}+z_{n}+\rho_{n} \zeta\right) \tag{21}
\end{equation*}
$$

locally uniformly in $\mathbb{C}^{2}$ to a nonconstant entire function $g$. Further, one has

$$
\begin{equation*}
\rho_{n} \leq \frac{M}{u^{\sharp}\left(w_{n}\right)} \leq M\left|w_{n}\right|^{-N} . \tag{22}
\end{equation*}
$$

Then,

$$
\begin{align*}
& \left(g_{n}\right)_{t}(\zeta)=\rho_{n}\left(f_{n}\right)_{t}\left(z_{n}+\rho_{n} \zeta\right)=\rho_{n} u_{t}\left(w_{n}+z_{n}+\rho_{n} \zeta\right) \rightarrow g_{t}(\zeta) \\
& \left(g_{n}\right)_{x}(\zeta)=\rho_{n}\left(f_{n}\right)_{x}\left(z_{n}+\rho_{n} \zeta\right)=\rho_{n} u_{x}\left(w_{n}+z_{n}+\rho_{n} \zeta\right) \rightarrow g_{x}(\zeta) \tag{23}
\end{align*}
$$

locally uniformly in $\mathbb{C}^{2}$. Note that the equation (8). Then,

$$
\begin{equation*}
\rho_{n} u_{t}\left(w_{n}+z_{n}+\rho_{n} \zeta\right)-C \rho_{n} u^{m}\left(w_{n}+z_{n}+\rho_{n} \zeta\right) u_{x}\left(w_{n}+z_{n}+\rho_{n} \zeta\right)=\rho_{n} Q\left(w_{n}+z_{n}+\rho_{n} \zeta\right) \tag{24}
\end{equation*}
$$

Choosing $N$ such that $N>\operatorname{deg} Q$. Let $n \rightarrow \infty$. Combining (22) and (23), then (24) become the following equation

$$
g_{t}-C g^{m} g_{x}=0
$$

Then, by Theorem A, one has $g$ is constant, a contradiction. The proof of Proposition 11 is finished.

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## References

[1] D.-C. Chang, B. Q. Li, C.-C. Yang, "On composition of meromorphic functions in several complex variables", Forum Math. 7 (1995), p. 77-94.
[2] P. V. Dovbush, "Normal functions of many complex variables", Mosc. Univ. Math. Bull. 36 (1981), no. 1, p. 44-48.
[3] -, "Zalcman's lemma in $\mathbb{C}^{n ", ~ C o m p l e x ~ V a r . ~ T h e o r y ~ A p p l . ~} 65$ (2020), no. 5, p. 796-800.
[4] R. D. Fay, "Plane sound waves of finite amplitude", J. Acoust. Soc. Am. 3 (1931), p. 222-241.
[5] M. M. M. Hassan, "New exact solutions of two nonlinear physical models", Commun. Theor. Phys. 53 (2010), no. 4, p. 596-604.
[6] W. K. Hayman, Meromorphic Functions, Oxford Mathematical Monographs, Clarendon Press, 1964.
[7] P.-C. Hu, B. Q. Li, "On meromorphic solutions of linear partial differential equations of second order", J. Math Anal Appl. 393 (2012), no. 1, p. 200-211.
[8] P.-C. Hu, C.-C. Yang, "Malmquist type theorem and factorization of meromorphic solutions of partial differential equations", Complex Var. Theory Appl. 27 (1995), no. 3, p. 269-285.
[9] , Unicity of Meromorphic Mappings, Advances in Complex Analysis and its Applications, vol. 1, Kluwer Academic Publishers, 2003.
[10] , "Global solutions of homogeneous linear partial differential equations of the second order", Mich. Math. J. 58 (2009), no. 3, p. 807-831.
[11] -, "The Tumura-Clunie theorem in several complex variables", Bull. Aust. Math. Soc. 90 (2014), no. 3, p. 444456.
[12] , "On complex solutions of certain partial differential equations", Complex Var. Elliptic Equ. 62 (2017), no. 10, p. 1492-1505.
[13] R. O. Kujala, "Functions of finite $\lambda$-type in several complex variables", Trans. Am. Math. Soc. 161 (1971), p. 327-358.
[14] B. Q. Li, "On reduction of functional-differential equations", Complex Var. Theory Appl. 31 (1996), no. 4, p. 311-324.
[15] ——, "Entire solutions of certain partial differential equations and factorization of partial derivatives", Trans. Am. Math. Soc. 357 (2005), no. 8, p. 3169-3177.
[16] —, "entire solutions of $u_{z_{1}}^{m}+u_{z_{2}}^{n}=e^{g ",}$ Nagoya Math. J. 178 (2005), p. 151-162.
[17] -, "On meromorphic solutions of $f^{2}+g^{2}=1 "$, Math. Z. 258 (2008), no. 4, p. 763-771.
[18] L. Liao, W. Su, C.-C. Yang, "A Malmquist-Yosida type of theorem for the second-order algebraic differential equations", J. Differ. Equ. 187 (2003), no. 1, p. 63-71.
[19] F. Lü, J. Xu, A. Chen, "Entire functions sharing polynomials with their first derivatives", Arch. Math. 92 (2009), no. 6, p. 593-601.
[20] E. Y. Rodin, "A Riccati solution for Burgers' equation", Q. Appl. Math. 27 (1970), p. 541-545.
[21] E. G. Saleeby, "Meromorphic solutions of generalized inviscid Burgers' equations and a family of quadratic PDEs", $J$. Math. Anal. Appl. 425 (2015), no. 1, p. 508-519.
[22] W. Stoll, Introduction to Value Distribution Theory of Meromorphic Maps, Lecture Notes in Mathematics, vol. 950, Springer, 1982.
[23] R. M. Timoney, "Bloch functions in several complex variables I", Bull London Math Soc. 12 (1980), no. 4, p. 241-267.
[24] E. Varoglu, W. D. L. Finn, "Space-time finite elements incorporating characteristics for the Burgers' equation", Int. J. Numer. Meth. Engng. 16 (1980), no. Special Issue, p. 1716-184.
[25] A. Vitter, "The lemma of the logarithmic derivative in seveal complex variables", Duke Math. J. 44 (1977), no. 1, p. 89104.
[26] W. Yuan, Y. Huang, Y. Shang, "All traveling wave exact solutions of two nonlinear physical models", Appl. Math. Comput. 219 (2013), no. 11, p. 6212-6223.
[27] L. Zalcman, "Normal families: new perspectives", Bull. Am. Math. Soc. 35 (1998), no. 3, p. 215-230.

