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Académie des sciences

# *Comptes Rendus*

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## *Mathématique*

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Volume 358, issue 11-12 (2020), p. 1179-1185

Published online: 25 January 2021

<https://doi.org/10.5802/crmath.137>



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*Les Comptes Rendus. Mathématique* sont membres du  
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e-ISSN : 1778-3569



Number Theory / *Théorie des nombres*

# Hyperharmonic integers exist

## *Des entiers hyperharmoniques existent*

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*Dedicated to the memories of John H. Conway and Richard K. Guy*

**Abstract.** We show that there exist infinitely many hyperharmonic integers, and this refutes a conjecture of Mező. In particular, for  $r = 64 \cdot (2^\alpha - 1) + 32$ , the hyperharmonic number  $h_{33}^{(r)}$  is integer for 153 different values of  $\alpha \pmod{748440}$ , where the smallest  $r$  is equal to  $64 \cdot (2^{2659} - 1) + 32$ .

**2020 Mathematics Subject Classification.** 11B83, 05A10, 11B75.

*Manuscript received 12th February 2020, revised 20th July 2020 and 22nd October 2020, accepted 23rd October 2020.*

### Version française abrégée

Dans [4], Conway et Guy ont introduit des nombres hyperharmoniques qui sont une généralisation des nombres harmoniques ordinaires. Mező [8] a d'abord conjecturé que les nombres hyperharmoniques n'étaient pas des entiers. Plusieurs articles [1–3, 5] dans la littérature soutiennent cette conjecture; cependant, aucun d'entre eux ne la prouve. Dans cette note, nous prouvons qu'il existe une infinité d'entiers hyperharmoniques, et cela réfute la conjecture de Mező. En particulier, nous montrons que pour  $r = 64 \cdot (2^\alpha - 1) + 32$ , le nombre hyperharmonique  $h_{33}^{(r)}$  est un entier pour 153 valeurs différentes de  $\alpha \pmod{748440}$ , où le plus petit  $r$  est  $64 \cdot (2^{2659} - 1) + 32$ .

### 1. Introduction

Any partial sum of the harmonic series is called a harmonic number. More precisely, the  $n^{\text{th}}$  harmonic number is the sum of the reciprocals of the first  $n$  positive integers, that is to say

$$h_n := \sum_{k=1}^n \frac{1}{k}.$$

In 1915, Theisinger [11] proved that  $h_n$  is never an integer, when  $n > 1$ . Moreover, Kürschák [7] showed that for any different positive integers  $m, n \geq 1$ , the corresponding difference of harmonic numbers  $h_m - h_n$  is also a non-integer rational number.

A generalization of harmonic numbers was introduced by Conway and Guy in [4]. They defined the  $n^{\text{th}}$  hyperharmonic number of order  $r$  as

$$h_n^{(r)} := \sum_{k=1}^n h_k^{(r-1)},$$

for given natural numbers  $n, r \geq 1$ , where the initial case being  $h_n^{(1)} = h_n$ . In the same book, they also showed that hyperharmonic numbers satisfy the following equality:

$$h_n^{(r)} = \binom{n+r-1}{r-1} (h_{n+r-1} - h_{r-1}). \tag{1}$$

In [8], Mezö conjectured that hyperharmonic numbers are not integers except 1 and he showed that  $h_n^{(r)}$  is not an integer for  $n > 1$  and  $r \in \{2, 3\}$ . The latter result was extended in [1, 2, 5] and it was first proved by Göral and the author that almost all hyperharmonic numbers are not integers [5]. Namely, if  $S(x)$  denotes the number of  $(n, r)$  tuples where  $h_n^{(r)}$  is not an integer for  $1 \leq n, r \leq x$ , then  $S(x) \sim x^2$ . Recently, this type of density result was improved in [3] and the current best known estimate is

$$S(x) = x^2 + \mathcal{O}_A \left( \frac{x^{\frac{80}{59}}}{(\log x)^A} \right),$$

for any  $A > 0$ . Here the implied  $\mathcal{O}$ -constant in the error term depends only on  $A$ . There are also explicit numerical values of  $n, r > 1$  for which  $h_n^{(r)} \notin \mathbb{Z}$ . For instance, it was proved in [5, Theorem 4] that  $h_n^{(r)}$  is not an integer for any  $n \in \{2, \dots, 32\}$ . By the same theorem, it was also known that  $h_{33}^{(r)} \notin \mathbb{Z}$  for any  $r \leq 20001$ . Later, an improvement on this result was given in [3, Corollary 2] which says that the lower density of the set of  $r$  values which satisfy the property  $h_{33}^{(r)} \in \mathbb{Z}$  is greater than 99%.

According to the previously mentioned results, it seems unlikely to have any hyperharmonic number which is also an integer. However in this note, it is proved that there are infinitely many of them. In particular, we proved the following Theorem 1:

**Theorem 1 (Main Theorem).** *There are infinitely many values of  $r \in \mathbb{Z}_{>0}$  such that  $h_{33}^{(r)}$  is an integer. More precisely,  $h_{33}^{(r)}$  is an integer for*

$$r = 64 \cdot \left( 2^{\alpha+k \cdot 748440} - 1 \right) + 32,$$

where  $k \geq 0$  is an integer,  $\alpha$  takes 153 different values in  $\mathbb{Z} \cap [0, 748439]$  and the minimum of these  $r$  values is equal to  $64 \cdot (2^{2659} - 1) + 32$ .

To obtain integer hyperharmonic numbers, we first observe that it is enough to get a non-negative  $p$ -adic valuation of  $h_{33}^{(r)}$ , for every prime  $p \in [2, 31]$ . After that, we give a general form of  $r$  which leads to a non-negative 2-adic valuation for  $h_{33}^{(r)}$ . Using a remark in [5], we consider each prime  $p \in [7, 31]$  except 11 and give sufficient conditions on  $r \pmod{p}$  to get  $v_p(h_{33}^{(r)}) \geq 0$ . Combining the given general form of  $r$  with each of these conditions that come from different primes  $p \in \{7, 13, 17, 19, 23, 29, 31\}$ , we obtain several common solutions for  $r$  which satisfy  $v_p(h_{33}^{(r)}) \geq 0$ , for all primes  $p \leq 33$  except  $p = 3, 5, 11$ . Analyzing each of the remaining prime cases together with the previous ones, we deduce that there are 153 different  $\alpha$  values modulo 748440 given in Appendix A, where the corresponding  $r = 64 \cdot (2^\alpha - 1) + 32$  gives us an integer  $h_{33}^{(r)}$ .

**Notation 2.** In this paper,  $p$  always denotes a prime number and  $v_p(q)$  denotes the  $p$ -adic valuation of a given rational number  $q$ ; that is for any  $q \in \mathbb{Z}$ ,

$$v_p(q) := \begin{cases} m, & \text{if } p^m \parallel q \\ \infty, & \text{if } q = 0, \end{cases}$$

and  $v_p(q) = v_p(a) - v_p(b)$ , where  $q = \frac{a}{b}$  and  $b \neq 0$ . We use  $I(n, r)$  to represent the set of integers  $\{r, \dots, n+r-1\}$ , for  $n, r \in \mathbb{Z}_{>0}$ . Also let  $\text{ord}_m(2)$  be the order of 2 in the group  $(\mathbb{Z}/m\mathbb{Z})^\times$ , for a

given odd integer  $m \geq 3$ . Denote  $\text{dlog}_2(t, n)$  as the solution  $\beta(\text{mod}(\text{ord}_n(2)))$  of the equation  $2^\beta \equiv t \pmod n$ , where  $t \in \mathbb{Z}$  and  $n \geq 3$  is odd, if such a solution  $\beta$  exists. For any prime  $p$  and a finite non-empty set  $S \subseteq \mathbb{Z}_{\geq 1}$ , define

$$\mu_p(S) := \max \{v_p(a) : a \in S\} \quad \text{and} \quad \mathcal{M}_p(S) := |S \cap p^{\mu_p(S)}\mathbb{Z}|, \tag{2}$$

as given in [5, Sections 1 and 3].

## 2. Main Result

We first restrict our set of primes  $p$  which will be used to check the  $p$ -adic valuation of  $h_{33}^{(r)}$ .

**Lemma 3.** *For any  $n, r \geq 1$ ,  $h_n^{(r)} \in \mathbb{Z}$  if and only if  $v_p(h_n^{(r)}) \geq 0$  for all primes  $p \leq n$ .*

**Proof.** The definition of the  $p$ -adic valuation leads to the following fact:  $h_n^{(r)}$  is an integer if and only if  $v_p(h_n^{(r)}) \geq 0$ , for all prime numbers  $p$ . By equation (1), note that

$$\begin{aligned} h_n^{(r)} &= \frac{r(r+1) \cdots (n+r-1)}{n!} \cdot \left( \frac{1}{r} + \frac{1}{r+1} + \cdots + \frac{1}{n+r-1} \right) \\ &= \frac{P}{n!} \cdot \left( \frac{\sum_{i=r}^{n+r-1} \frac{P}{i}}{P} \right) = \frac{P_r + P_{r+1} + \cdots + P_{n+r-1}}{n!}, \end{aligned}$$

where  $P = r(r+1) \cdots (n+r-1)$  and  $P_i = \frac{P}{i}$ , for any  $i \in \{r, r+1, \dots, n+r-1\}$ . This yields that  $v_p(h_n^{(r)}) \geq 0$  for all primes  $p > n$ . Hence, we conclude the Lemma 3.  $\square$

As a consequence of Lemma 3, it will be enough to consider the  $p$ -adic valuation of  $h_{33}^{(r)}$  for all primes  $p \leq 33$ . Before beginning with  $p = 2$ , we emphasize that

$$v_p \left( \binom{n+r-1}{r-1} \right) \leq \mu_p(I(n, r)),$$

for any prime  $p$ , as it is given in [5, Proposition 17]. Moreover it can be seen that if we obtain carries in all possible digits after adding  $n$  and  $r-1$  in their  $p$ -ary representations, then

$$v_p \left( \binom{n+r-1}{r-1} \right) = \mu_p(I(n, r)). \tag{3}$$

This fact was also mentioned in the last part of the proof of [5, Proposition 17].

**Proposition 4.** *For any  $\alpha \geq 0$ , we have  $v_2(h_{33}^{(r)}) = 0$ , if  $r$  is of the form  $64(2^\alpha - 1) + 32$ .*

**Proof.** By equation (1), we have

$$v_2(h_n^{(r)}) = v_2 \left( \binom{n+r-1}{r-1} \right) + v_2(h_{n+r-1} - h_{r-1}).$$

So the equality

$$v_2 \left( \binom{r+32}{r-1} \right) = -v_2(h_{r+32} - h_{r-1})$$

implies that  $v_2(h_{33}^{(r)}) = 0$ . Let  $\theta = \mu_2(I(33, r))$  where  $\mu_2(\cdot)$  is defined in (2). Then there exists an odd integer  $c$  such that  $c \cdot 2^\theta \in I(33, r)$ , otherwise it contradicts the fact that  $\theta = \mu_2(I(33, r))$ . Observe that neither  $(c-1)2^\theta$  nor  $(c+1)2^\theta$  lies in  $I(33, r)$ , as  $(c-1)$  and  $(c+1)$  are both even. Hence, we deduce that  $\mathcal{M}_2(I(33, r)) = 1$ , by the definition of  $\mathcal{M}_2(\cdot)$  which is given in (2). This leads to the fact that  $h_{r+32} - h_{r-1} = \frac{1}{c \cdot 2^\theta} + Q_2$ , for some  $Q_2 \in \mathbb{Q}$  where  $v_2(Q_2) \geq -(\theta - 1)$ . By the non-Archimedean property of the 2-adic valuation, we conclude that  $v_2(h_{r+32} - h_{r-1}) = -\theta$ . So in order to get  $v_2(h_{33}^{(r)}) = 0$ , we need to have  $v_2 \left( \binom{r+32}{r-1} \right) = \theta = \mu_2(I(33, r))$ . As we mentioned earlier, equation (3) holds if we write  $r-1$  and 33 in binary representations and obtain a carry in all possible places

after the addition of them. Observe that the condition on  $r$  is equivalent to  $r - 1 = 2^6(2^\alpha - 1) + 31$ , where  $31 = (1, 1, 1, 1)_2$ . Therefore we have

$$r - 1 = \overbrace{(1, 1, \dots, 1, 0, 1, 1, 1, 1)_2}^{\alpha \text{ many}}$$

$$33 = (0, 0, \dots, 0, 1, 0, 0, 0, 0, 1)_2.$$

Notice that we obtain a carry at each step in the addition of  $r - 1$  and  $33$ . Thus we conclude that  $v_2(h_{33}^{(r)}) = 0$ , if  $r = 2^6(2^\alpha - 1) + 32$  where  $\alpha \geq 0$ . □

**Remark 5.** By [6, Corollary 3.7], we know that  $v_2(h_n^{(r)}) \leq 0$  for any  $n \geq 1$ . So the form of  $r$  that we obtained in Proposition 4 is one of the best possible one in the sense of the 2-adic valuation.

Now, we consider the  $p$ -adic valuation of  $h_{33}^{(r)}$  where the prime  $p$  satisfies the inequality  $k_p := \lfloor \frac{33}{p} \rfloor < p$ . Define  $I_p(n, r) = p\mathbb{Z} \cap I(n, r)$ . Observe that  $k_p = |I_p(33, 1)|$ . By [5, Section 5], we know that  $k_p < p$  together with  $|I_p(33, r)| = k_p + 1$  imply that  $v_p(h_{33}^{(r)}) \geq 0$ . This will be a key step towards obtaining an integer  $h_{33}^{(r)}$ .

**Proposition 6.** Let  $p \in \{7, 13, 17, 19, 23, 29, 31\}$ . Assume that  $r \equiv 1 - b \pmod{p}$  for some  $b \in \{1, \dots, n_p\}$ , where  $n_p \equiv 33 \pmod{p}$  with  $n_p \in \{1, \dots, p - 1\}$ . Then  $v_p(h_{33}^{(r)}) \geq 0$ .

**Proof.** We will show that the condition given on  $r$  implies that  $|I_p(33, r)| = k_p + 1$ , for any  $p \in \{7, 13, 17, 19, 23, 29, 31\}$ . Firstly, observe that  $33 = k_p p + n_p$ , where  $0 < n_p < p$  and  $k_p$  as defined above. Note that  $k_p \geq 1$  and  $0 \geq 1 - b \geq 1 - n_p > -(p - 1)$ . If  $r = cp + (1 - b)$ , then we have  $cp - (p - 1) < r \leq cp$ . Since  $k_p > 0$ , we see that  $r + 32 = (c + k_p)p + (n_p - b) \geq (c + k_p)p > cp \geq r$ . Hence, we have  $|I_p(33, r)| = k_p + 1$ , as  $cp, \dots, (c + k_p)p \in I_p(33, r)$ . By [5, Section 5], we conclude that  $v_p(h_{33}^{(r)}) \geq 0$ , since  $k_p < p$  for all  $p \in \{7, 13, 17, 19, 23, 29, 31\}$ . □

Finally we prove our main result. The SageMath code and its consequences that are mentioned in the following proof can be found in [9].

**Proof of the Main Theorem 1.** We combine Propositions 4 and 6 to obtain a common solution  $r$  for which  $v_p(h_{33}^{(r)}) \geq 0$ , where  $p \in \{2, 7, 13, 17, 19, 23, 29, 31\}$ . Let

$$\mathcal{P} = \{7, 13, 17, 19, 23, 29, 31\}.$$

For all prime numbers  $p \in \mathcal{P}$ , we need to find a common solution for the congruence

$$64 \cdot (2^{\alpha p} - 1) + 32 \equiv 1 - b_p \pmod{p}, \tag{4}$$

where  $b_p \in \{1, \dots, n_p\}$  and  $n_p \equiv 33 \pmod{p}$  with  $n_p \in \{1, \dots, p - 1\}$ . Note that

$$2^{\alpha p} \equiv 1 - 2^{-6}(b_p + 31) \pmod{p}. \tag{5}$$

For different values of  $b_p$ , we find all solutions of congruence (4) where

$$\alpha_p \equiv \text{dlog}_2(1 - 2^{-6}(b_p + 31), p) \pmod{(\text{ord}_p(2))}. \tag{6}$$

As we mentioned in the Notation 2,  $\text{ord}_p(2)$  denotes the order of 2 in  $(\mathbb{Z}/p\mathbb{Z})^\times$  and  $\text{dlog}_2(1 - 2^{-6}(b_p + 31), p)$  represents the solution of the congruence (5) modulo  $\text{ord}_p(2)$ , if it exists. After obtaining all different solutions  $\alpha_p \pmod{(\text{ord}_p(2))}$  that come from different values of  $b_p$ , we use Chinese Remainder Theorem for non-coprime moduli  $L = \text{lcm}\{\text{ord}_p(2) : p \in \mathcal{P}\} = 27720$ . In this way, we find all common solutions  $\alpha_0 \pmod{27720}$  where each of  $\alpha_0$  is congruent to some  $\alpha_p \pmod{(\text{ord}_p(2))}$ , for any  $p \in \mathcal{P}$ . Thanks to SageMath [10], we see that the set

$$\mathcal{A}_0 := \{0 \leq \alpha_0 < 27720 : \forall p \in \mathcal{P} \alpha_0 \text{ satisfies congruence (6) for some } b_p \in \{1, \dots, n_p\}\}$$

contains 196 elements. Thus for any integer  $k \geq 0$ , any prime  $p \in \{2, 7, 13, 17, 19, 23, 29, 31\}$  and  $\alpha_0 \in \mathcal{A}_0$ , we have  $v_p(h_{33}^{(r)}) \geq 0$ , where  $r = 64 \cdot (2^{\alpha_0 + k \cdot 27720} - 1) + 32$ .

Now, we consider these solutions for the prime  $p = 11$ . As  $p \mid 33$ , we know by [5, Lemma 12] that  $|I_{11}(33, r)| = 3$ . Let  $c_{11} = \lceil \frac{r}{11} \rceil$ . Then by equation (1), we have

$$h_{33}^{(r)} = \frac{A_{11}}{B_{11}} \cdot \frac{11c_{11} \cdot 11(c_{11} + 1) \cdot 11(c_{11} + 2)}{11 \cdot 22 \cdot 33} \left( \frac{1}{11} \sum_{i=0}^2 \frac{1}{c_{11} + i} + q_{11} \right) \\ = \frac{A_{11}}{B_{11}} \cdot \frac{3c_{11}^2 + 6c_{11} + 2}{66} + \frac{A_{11}}{B_{11}} \cdot \frac{c_{11}(c_{11} + 1)(c_{11} + 2)}{6} \cdot q_{11},$$

where  $v_{11}(A_{11}) = v_{11}(B_{11}) = 0 \leq v_{11}(q_{11})$  and  $v_{11}(c_{11}(c_{11} + 1)(c_{11} + 2)) \geq 0$ . Therefore  $3c_{11}^2 + 6c_{11} + 2 \equiv 0 \pmod{11}$  implies that  $v_{11}(h_{33}^{(r)}) \geq 0$ . Note that  $x \equiv 1, 8 \pmod{11}$  are the only two roots of the polynomial  $3x^2 + 6x + 2$  modulo 11. So if

$$r \equiv 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88 \pmod{121}, \tag{7}$$

then  $c_{11} = \lceil \frac{r}{11} \rceil \equiv 1, 8 \pmod{11}$ . According to the computations performed by SageMath [10], we deduce that there are only 52 different elements in  $\mathcal{A}_0$  which satisfy congruence (7). Define the set

$$\mathcal{A}'_0 := \{ \alpha_0 \in \mathcal{A}_0 : r = 64 \cdot (2^{\alpha_0} - 1) + 32 \text{ satisfies congruence (7)} \}$$

and take any  $\alpha_0 \in \mathcal{A}'_0$ . Since the order of 2 in  $(\mathbb{Z}/121\mathbb{Z})^\times$  is  $\text{ord}_{121}(2) = 110$ , we get that

$$r = 2^6 \cdot (2^{\alpha_0 + k \cdot 27720} - 1) + 32 \equiv 64 \cdot (2^{\alpha_0} - 1) + 32 \pmod{121}, \tag{8}$$

for any integer  $k \geq 1$ . This indicates that  $v_{11}(h_{33}^{(r)}) \geq 0$  for any  $r = 2^6 \cdot (2^{\alpha_0 + k \cdot 27720} - 1) + 32$ , where  $\alpha_0 \in \mathcal{A}'_0$  and  $k \geq 0$ .

Next, we deal with the case  $p = 5$ . By SageMath [10], we obtain that  $r = 64 \cdot (2^{\alpha_0} - 1) + 32 \equiv 0, 19 \pmod{25}$ , for any  $\alpha_0 \in \mathcal{A}'_0$ . For these values of  $r$ , we also observed that

$$h_{33}^{(r)} = \binom{r+32}{r-1} (h_{r+32} - h_{r-1}) = \frac{A_5}{B_5} \cdot \frac{5c_5 \cdot 5(c_5 + 1) \cdots 5(c_5 + 6)}{5 \cdot 10 \cdots 30} \left( \frac{1}{5} \sum_{i=0}^6 \frac{1}{c_5 + i} + q_5 \right) \\ = \frac{A_5}{B_5} \cdot \frac{5c_5 \cdot 5(c_5 + 1) \cdots 5(c_5 + 6)}{5^7 \cdot 144} \\ \cdot \frac{7c_5^6 + 126c_5^5 + 875c_5^4 + 2940c_5^3 + 4872c_5^2 + 3528c_5 + 720}{5c_5(c_5 + 1) \cdots (c_5 + 6)} + Q_5 \\ = \frac{A_5}{B_5} \cdot \frac{7c_5^6 + 126c_5^5 + 875c_5^4 + 2940c_5^3 + 4872c_5^2 + 3528c_5 + 720}{5 \cdot 144} + Q_5$$

is satisfied, where  $c_5 = \lceil \frac{r}{5} \rceil$  and  $v_5(A_5) = v_5(B_5) = 0 \leq v_5(q_5)$ . Here  $Q_5$  denotes the rational number  $\frac{A_5}{B_5} \cdot \frac{c_5(c_5+1) \cdots (c_5+6)}{2^4 \cdot 3^2} \cdot q_5$ , and this yields that  $v_5(Q_5) \geq 0$ . Observe that if the polynomial

$$g_5(x) := 7x^6 + 126x^5 + 875x^4 + 2940x^3 + 4872x^2 + 3528x + 720$$

is divisible by 5 for  $x = c_5$ , then we get that  $v_5(h_{33}^{(r)}) \geq 0$ . If  $25 \mid r$ , then we see that  $g_5(c_5) \equiv 0 \pmod{5}$  as  $c_5 \equiv 0 \pmod{5}$  is a root of  $g_5(x)$  in  $\mathbb{Z}/5\mathbb{Z}$ . Similarly, for  $r \equiv 19 \pmod{25}$ , we have  $c_5 = \lceil \frac{r}{5} \rceil \equiv 4 \pmod{5}$  which is also a root of the polynomial  $g_5(x)$  modulo 5. Hence, for all previously found  $\alpha_0 \in \mathcal{A}'_0$ , we have  $v_5(h_{33}^{(r)}) \geq 0$ , where  $r = 64 \cdot (2^{\alpha_0} - 1) + 32$ . Also notice that for any integer  $k \geq 1$  and  $\alpha_0 \in \mathcal{A}'_0$ ,

$$r = 64 \cdot (2^{\alpha_0 + k \cdot 27720} - 1) + 32 \equiv 64 \cdot (2^{\alpha_0} - 1) + 32 \equiv 0, 19 \pmod{25},$$

as  $\text{ord}_{25}(2) = 20$ . Thus for any  $r = 64 \cdot (2^{\alpha_0 + k \cdot 27720} - 1) + 32$ , we conclude that  $v_5(h_{33}^{(r)}) \geq 0$ , where  $k \in \mathbb{Z}_{\geq 0}$  and  $\alpha_0 \in \mathcal{A}'_0$ .

Finally let  $p = 3$ . Define  $f_3(x) = x(x + 1) \cdots (x + 10)$ . Consider

$$\begin{aligned}
 h_{33}^{(r)} &= \frac{A_3}{B_3} \cdot \frac{3c_3 \cdot 3(c_3 + 1) \cdots 3(c_3 + 10)}{33!} \left( \frac{1}{3} \sum_{i=0}^{10} \frac{1}{c_3 + i} + q_3 \right) \\
 &= \frac{A_3}{B_3} \cdot \frac{3^{11} \cdot f_3(c_3)}{3^{15} \cdot D_3} \cdot \frac{1}{3} \cdot \frac{g_3(c_3)}{f_3(c_3)} + \frac{A_3}{B_3} \cdot \frac{3^{11} \cdot f_3(c_3)}{3^{15} \cdot D_3} \cdot q_3
 \end{aligned} \tag{9}$$

$$= \frac{A_3}{B_3} \cdot \frac{g_3(c_3)}{3^5 \cdot D_3} + \frac{A_3}{B_3} \cdot \frac{f_3(c_3)}{3^4 \cdot D_3} \cdot q_3, \tag{10}$$

where  $c_3 = \lceil \frac{r}{3} \rceil \geq 1$  and  $v_3(A_3) = v_3(B_3) = v_3(D_3) = 0 \leq v_3(q_3)$ . Here  $g_3(x)$  denotes the derivative of the polynomial  $f_3(x)$ . To see that equation (9) holds, observe that

$$\log f_3(x) = \sum_{i=0}^{10} \log(x + i), \tag{11}$$

where  $\log(\cdot)$  denotes the natural logarithm. Taking the derivative of both sides with respect to  $x$  in equation (11) yields that

$$\frac{g_3(x)}{f_3(x)} = \sum_{i=0}^{10} \frac{1}{x + i}.$$

Also note that  $v_3(f_3(c_3)) \geq 4$ , since  $c_3 \geq 1$  and

$$\binom{c_3 + 10}{11} = \frac{c_3(c_3 + 1) \cdots (c_3 + 10)}{11!} = \frac{f_3(c_3)}{2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11} \in \mathbb{Z}.$$

Therefore, we have  $v_3(\frac{A_3}{B_3} \cdot \frac{f_3(c_3)}{3^4 \cdot D_3} \cdot q_3) \geq 0$ . In order to obtain  $v_3(h_{33}^{(r)}) \geq 0$ , it is enough to have  $g_3(c_3) \equiv 0 \pmod{3^5}$ , by equation (10). Using SageMath [10], we find that there are 18 roots of the polynomial  $g_3(x)$  modulo  $3^5$ . Hence, if  $s$  is one of these roots and  $r \equiv 3s - 2, 3s - 1, 3s \pmod{3^6}$ , then we have  $(s - 1) + \ell \cdot 3^5 < \frac{r}{3} \leq s + \ell \cdot 3^5$ , for some  $\ell \in \mathbb{Z}_{\geq 0}$ . This indicates that  $c_3 = \lceil \frac{r}{3} \rceil \equiv s \pmod{243}$ . As we did in congruence (4), we will solve

$$64 \cdot (2^{\alpha_3} - 1) + 32 \equiv s_3 \pmod{729}$$

for

$$\alpha_3 \equiv \text{dlog}_2(1 + 2^{-6}(s_3 - 32), 729) \pmod{486}, \tag{12}$$

in order to obtain common solutions for primes 2 and 3, where  $s_3 \in \{3s - 2, 3s - 1, 3s\}$  and  $\text{ord}_{729}(2) = 486$ . By SageMath [10] again, we obtain 36 different solutions modulo 486 for congruence (12). Note that for any  $\alpha_3$  satisfying congruence (12), we have  $v_3(h_{33}^{(r)}) \geq 0$ , as  $r = 64 \cdot (2^{\alpha_3} - 1) + 32$ . Also recall that for any  $\alpha_0 \in \mathcal{A}'_0$ , any integer  $k \geq 0$  and any prime  $p \in [2, 31] \setminus \{3\}$ , we have  $v_p(h_{33}^{(r)}) \geq 0$ , where  $r = 64 \cdot (2^{\alpha_0 + k \cdot 27720} - 1) + 32$ . So we can see each different  $\alpha_0 \in \mathcal{A}'_0$  as an element from the corresponding equivalence class of  $\alpha_0$  in  $\mathbb{Z}/(27720)\mathbb{Z}$ . To obtain common solutions, we use generalized Chinese Remainder Theorem where we combine 52 different  $\alpha_0$  values modulo 27720 and 36 different  $\alpha_3$  values modulo 486, as  $|\mathcal{A}'_0| = 52$ . Computations give us 153 different common solutions  $\alpha$  modulo  $\text{lcm}(27720, 486) = 748440$  for which the values  $r = 64(2^\alpha - 1) + 32$  yield an integer hyperharmonic number. As given in Appendix A, the smallest of these solutions is  $\alpha = 2659$ . Thus, we conclude that the smallest  $r$  is  $64 \cdot (2^{2659} - 1) + 32$ .  $\square$

### Acknowledgements

The author would like to thank the anonymous referee and Haydar Göral who improved the quality of the paper drastically with their valuable suggestions.

## Appendix A.

As we proved above, there are 153 different  $\alpha \in \{0, \dots, 748439\}$  such that  $h_{33}^{(r)}$  is an integer for  $r = 64 \cdot (2^{\alpha+k-748440} - 1) + 32$ , where  $k \in \mathbb{Z}_{\geq 0}$ . The set of these  $\alpha$  values can be given as follows:

{2659, 23039, 28979, 30599, 30739, 36539, 36679, 38299, 44239, 64619, 70559, 72179, 72319, 78114, 78119, 78259, 79879, 85819, 106199, 112139, 113759, 113899, 119699, 119839, 121459, 127399, 147779, 153719, 155339, 155479, 161274, 161279, 161419, 195299, 196919, 197059, 202859, 202999, 204619, 210559, 230939, 236879, 238499, 238639, 163039, 168979, 189359, 244434, 244439, 244579, 246199, 252139, 272519, 278459, 280079, 280219, 286019, 286159, 287779, 293719, 314099, 320039, 321659, 321799, 327594, 327599, 327739, 329359, 335299, 355679, 361619, 363239, 363379, 369179, 369319, 370939, 376879, 397259, 403199, 404819, 404959, 410754, 410759, 410899, 412519, 418459, 438839, 444779, 446399, 446539, 452339, 452479, 454099, 460039, 480419, 486359, 487979, 488119, 493914, 493919, 494059, 495679, 501619, 521999, 527939, 529559, 529699, 535499, 535639, 537259, 543199, 563579, 569519, 571139, 571279, 577074, 577079, 577219, 578839, 584779, 605159, 611099, 612719, 612859, 618659, 618799, 620419, 626359, 646739, 652679, 654299, 654439, 660234, 660239, 660379, 661999, 667939, 688319, 694259, 695879, 696019, 701819, 701959, 703579, 709519, 729899, 735839, 737459, 737599, 743394, 743399, 743539, 745159}.

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