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Complex analysis and geometry / *Analyse et géométrie complexes*

On the Erdős–Lax Inequality

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Abstract. The Erdős–Lax Theorem states that if $P(z)=\sum_{\nu=1}^{n} a_\nu z^\nu$ is a polynomial of degree n having no zeros in $|z|<1$, then

$$\max_{|z|=1}|P'(z)| \le \frac{n}{2}\max_{|z|=1}|P(z)|. \tag{1}$$

In this paper, we prove a sharpening of the above inequality (1). In order to prove our result we prove a sharpened form of the well-known Theorem of Laguerre on polynomials, which itself could be of independent interest.

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1. Introduction

If $P(z)$ is a polynomial of degree n, then the well-known Bernstein's inequalities [2] on polynomials are given by

$$\max_{|z|=1}|P'(z)| \le n\max_{|z|=1}|P(z)| \tag{2}$$

and

$$\max_{|z|=R}|P(z)| \le R^n\max_{|z|=1}|P(z)|, \tag{3}$$

whenever $R \ge 1$.

The inequality (2) is a direct consequence of Bernstein's Theorem on the derivative of a trigonometric polynomial [9] and the inequality (3) follows from the maximum modulus theorem (see [8, Problem 269]). For the class of polynomials having no zeros inside the unit circle, it was Erdős [3] who conjectured, and later proved by Lax [6] that, if $P(z)$ is a polynomial of degree n having no zeros in $|z|<1$, then

$$\max_{|z|=1}|P'(z)| \le \frac{n}{2}\max_{|z|=1}|P(z)|. \tag{4}$$

Equality holds in (4) if all zeros of $P(z)$ lie on the circle $|z|=1$.

Although, the above inequality (4) is best possible with equality holding for polynomials $P(z)=\sum_{\nu=1}^{n} a_\nu z^\nu$ having no zeros in $|z|<1$, satisfying $|a_0|=|a_n|$, in particular for polynomials having all its zeros on $|z|=1$, it should be possible to improve upon the bound for polynomials $P(z)=\sum_{\nu=1}^{n} a_\nu z^\nu$ having no zeros in $|z|<1$, satisfying $|a_0|\ne|a_n|$. Here we consider this problem and in this regard prove the following.

Theorem 1. *If $P(z)=\sum_{\nu=1}^{n} a_\nu z^\nu$ is a polynomial of degree n, having no zeros in $|z|<1$, then*

$$\max_{|z|=1}|P'(z)| \le \frac{n}{2}\left[1-\frac{|a_0|-|a_n|}{n(|a_0|+|a_n|)}\right]\max_{|z|=1}|P(z)| \tag{5}$$

whenever $\max_{|z|=1}|P(z)|\ge 2|a_0|$, *and* (4) *otherwise.*

Since $P(z)$ has all its zeros in $|z|\ge 1$, hence for any polynomial $P(z)=\sum_{\nu=1}^{n} a_\nu z^\nu$, we have $\frac{|a_0|-|a_n|}{n(|a_0|+|a_n|)}\ge 0$. Therefore, for all polynomials satisfying the hypotheses of Theorem 1 excepting those satisfying $|a_0|=|a_n|$, and so in particular for polynomials satisfying $P(z)\equiv z^n\overline{P(1/\overline{z})}$ which includes polynomials having all their zeros on $|z|=1$, our above inequality (5) sharpens Erdős–Lax Inequality (4).

Ankeny and Rivlin [1] improved the inequality (3) for the class of polynomials having no zeros in the unit disk by proving that, if $P(z)$ is a polynomial of degree n having no zeros in $|z|<1$, then

$$\max_{|z|=R}|P(z)| \le \frac{1+R^n}{2}\max_{|z|=1}|P(z)|, \tag{6}$$

for any $R\ge 1$. Ankeny and Rivlin used Erdős–Lax Inequality (4) in order to prove the above Inequality (6), and so if instead of (4) we use its sharpened form (5) we easily get the following which is a refinement of (6).

Corollary 2. *If $P(z)=\sum_{\nu=1}^{n} a_\nu z^\nu$ is polynomial of degree n having no zeros in $|z|<1$ then*

$$\max_{|z|=R}|P(z)| \le \frac{(1+R^n)-\lambda(R^n-1)}{2}\max_{|z|=1}|P(z)|, \tag{7}$$

for any $R\ge 1$, and $\lambda=\frac{|a_0|-|a_n|}{n(|a_0|+|a_n|)}$ whenever $\max_{|z|=1}|P(z)|\ge 2|a_0|$, *and* (6) *otherwise.*

2. Lemmas

The first result in this section is a simple exercise, and can be easily verified by principle of induction on n and hence the proof is omitted.

Lemma 3. *Let $x_i\ge 1, 1\le i\le n$. Then*

$$\sum_{i=1}^{n}\frac{1}{1+x_i}\le\frac{n-1}{2}+\frac{1}{1+x_1x_2\dots x_n}.$$

The following few results seek attention to some well-known facts associated with the polynomial $P(z)$ having no zeros in the unit disc. In that aspect, the next lemma describes one of the fundamental properties of the class of polynomials having no zeros in the unit disc.

Lemma 4. *If $P(z)=\sum_{\nu=1}^{n} a_\nu z^\nu$ is a polynomial of degree $n\ge 1$ having no zeros in $|z|<1$, then for all z on $|z|=1$ for which $P(z)\ne 0$*

$$\mathrm{Re}\left(\frac{zP'(z)}{P(z)}\right)\le\frac{n}{2}-\frac{|a_0|-|a_n|}{2(|a_0|+|a_n|)}. \tag{8}$$

Proof. Suppose $P(z)=a_n\prod_{k=1}^{n}(z-z_k)$. By hypothesis each z_k satisfies $|z_k|\ge 1, 1\le k\le n$. Then for all z on $|z|=1$ for which $P(z)\ne 0$, we will have

$$\mathrm{Re}\left(\frac{zP'(z)}{P(z)}\right)=\sum_{k=1}^{n}\mathrm{Re}\frac{z}{z-z_k}\le\sum_{k=1}^{n}\frac{1}{1+|z_k|}.$$

Now it follows from the Lemma 3 that

$$\mathrm{Re}\left(\frac{zP'(z)}{P(z)}\right)\le\frac{n-1}{2}+\frac{1}{1+\frac{|a_0|}{|a_n|}} \tag{9}$$

for all z on $|z|=1$ for which $P(z)\ne 0$. Note that (9) and (8) are equivalent and hence the proof is complete. □

The well-known Theorem of Laguerre [5, 10] states that, if $P(z)$ is a polynomial of degree n having no zeros in the disc $|z| < 1$, then the polynomial $nP(z) + (\alpha - z)P'(z)$ has no zeros in $|z| < 1$ for every complex number α with $|\alpha| < 1$. What happens if we replace n by $n - \delta$ where δ is a non-negative real number less than n? This question is very intriguing and is answered in the following result for some specific values of δ that depends on the coefficients of the underlying polynomial.

Lemma 5. *If $P(z) = \sum_{\nu=1}^{n} a_\nu z^\nu$ is a polynomial of degree n having no zeros in the disc $|z| < 1$, then the polynomial*

$$n\left(1 - \frac{|a_0| - |a_n|}{n(|a_0| + |a_n|)}\right) P(z) + (\alpha - z)P'(z)$$

has no zeros in $|z| < 1$ for every α with $|\alpha| < 1$.

Proof. From Lemma 4, it follows that, for all z on $|z| = 1$, for which $P(z) \neq 0$, we have

$$\operatorname{Re}\left(\frac{zP'(z)}{n\left(1 - \frac{|a_0|-|a_n|}{n(|a_0|+|a_n|)}\right)P(z)}\right) \le \frac{1}{2},$$

and hence

$$\left|1 - \left(\frac{zP'(z)}{n\left(1 - \frac{|a_0|-|a_n|}{n(|a_0|+|a_n|)}\right)P(z)}\right)\right| \ge \left|\left(\frac{zP'(z)}{n\left(1 - \frac{|a_0|-|a_n|}{n(|a_0|+|a_n|)}\right)P(z)}\right)\right|$$

for all z on $|z| = 1$, for which $P(z) \neq 0$. Therefore

$$\left|n\left(1 - \frac{|a_0| - |a_n|}{n(|a_0| + |a_n|)}\right)P(z) - zP'(z)\right| \ge |P'(z)|$$

for all z on $|z| = 1$. But then for any α with $|\alpha| < 1$ and $|z| < 1$, we have

$$\left|n\left(1 - \frac{|a_0| - |a_n|}{n(|a_0| + |a_n|)}\right)P(z) - zP'(z)\right| > |\alpha P'(z)|,$$

or in other words, for any α with $|\alpha| < 1$ and $|z| < 1$, we have

$$n\left(1 - \frac{|a_0| - |a_n|}{n(|a_0| + |a_n|)}\right)P(z) - zP'(z) + \alpha P'(z) \neq 0. \qquad \square$$

Lemma 5 is crucial in proving the next result as well as the main theorem in this paper. Note that Lemma 5 holds also true if we replace $|a_0|$ by any positive real number x such that $|a_0| \ge x$.

Lemma 6. *If D is the open unit disc and S is an arbitrary point set of complex numbers, and $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ is a polynomial of degree n satisfying $\max_{|z|=1} |P(z)| \ge |s|, \forall s \in S$ then we have for any $z \in D$ and $\alpha \in D$ that*

$$\frac{(\alpha - z)P'(z)}{n\left(1 - \frac{|a_0|-|a_n|}{n(|a_0|+|a_n|)}\right)} + P(z) \in S,$$

whenever $\max_{|z|=1} |P(z)| \ge 2|a_0|$.

Proof. Suppose δ is outside S. Then $P(z) \neq \delta$ for any $z \in D$. Now applying Lemma 5 to the polynomial $P(z) - \delta$, with the fact that $|\delta| > \max_{|z|=1} |P(z)| \ge 2|a_0|$, it follows that

$$(\alpha - z)P'(z) + n\left(1 - \frac{|a_0| - |a_n|}{n(|a_0| + |a_n|)}\right)P(z) \neq n\left(1 - \frac{|a_0| - |a_n|}{n(|a_0| + |a_n|)}\right)\delta,$$

which is equivalent to

$$\frac{(\alpha - z)P'(z)}{n\left(1 - \frac{|a_0|-|a_n|}{n(|a_0|+|a_n|)}\right)} + P(z) \neq \delta$$

for all $z \in D$, $\alpha \in D$ and any $\delta \notin S$. This completes the proof. $\square$

3. Proof of Theorem 1

Since $P(z) \neq 0$ in $|z| < 1$, we have from Lemma 5

$$\zeta P'(z) \neq zP'(z) - n\left(1 - \frac{|a_0| - |a_n|}{n(|a_0| + |a_n|)}\right) P(z) \tag{10}$$

for $|\zeta| < 1$, and $|z| < 1$. With an appropriate choice of the argument of ζ in (10) we can get

$$|\zeta||P'(z)| \neq |zP'(z) - n\left(1 - \frac{|a_0| - |a_n|}{n(|a_0| + |a_n|)}\right) P(z)|. \tag{11}$$

Thus we have

$$|\zeta||P'(z)| < |zP'(z) - n\left(1 - \frac{|a_0| - |a_n|}{n(|a_0| + |a_n|)}\right) P(z)| \tag{12}$$

for $|\zeta| < 1$, and $|z| < 1$, because otherwise inequality is violated for sufficiently small values of $|\zeta|$. Taking $|\zeta| \to 1$, in (12) we get for $|z| \leq 1$

$$|P'(z)| \leq |zP'(z) - n\left(1 - \frac{|a_0| - |a_n|}{n(|a_0| + |a_n|)}\right) P(z)|. \tag{13}$$

On the other hand, from Lemma 6, for $|\alpha| < 1$ and $|z| < 1$,

$$\frac{(\alpha - z)P'(z)}{n\left(1 - \frac{|a_0| - |a_n|}{n(|a_0| + |a_n|)}\right)} + P(z) \in S \tag{14}$$

where S is as defined in Lemma 6. Hence for $|z| = 1$ and taking $|\alpha| \to 1$ with appropriate choice of argument of α we obtain

$$|zP'(z)| + |zP'(z) - n\left(1 - \frac{|a_0| - |a_n|}{n(|a_0| + |a_n|)}\right) P(z)| \leq n\left(1 - \frac{|a_0| - |a_n|}{n(|a_0| + |a_n|)}\right) \max_{|z|=1} |P(z)|. \tag{15}$$

From (13) and (15), the result follows.

4. Polynomials having no zeros in $|z| < K,\ K \geq 1$

It is quite natural to seek the extension of Theorem 1 to the class of polynomials having no zeros in any open disc of radius $K \geq 1$. Let $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial of degree n having no zeros in $|z| < K,\ K \geq 1$. Then Malik [7] (see also [4]) proved that

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+K} \max_{|z|=1} |P(z)|. \tag{16}$$

Let us sharpen (16) as well. Observe that $P(Kz)$ is a polynomial of degree n having no zeros in $|z| < 1$. Now using $P(Kz)$ in place of $P(z)$ in the proof of Theorem 1 and proceeding similarly by noting that the inequality (13) becoming for $|z| \leq 1$,

$$K|P'(z)| \leq |zP'(z) - n\left(1 - \frac{|a_0| - |a_n|K^n}{n(|a_0| + |a_n K^n|)}\right) P(z)|,$$

through some simple steps, we will obtain the following generalization of Theorem 1.

Theorem 7. *If $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ is a polynomial of degree n, having no zeros in $|z| < K, K \geq 1$, then*

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+K}\left[1 - \frac{|a_0| - |a_n|K^n}{n(|a_0| + |a_n|K^n)}\right] \max_{|z|=1} |P(z)| \tag{17}$$

whenever $\max_{|z|=K} |P(z)| \geq 2|a_0|$, *and* (16) *otherwise.*

Theorem 7 is the sharpened form of result due to Malik [7] (see also [4]) on the generalization of Erdős–Lax inequality (4) to the class of polynomials having no zeros in $|z| < K,\ K \geq 1$.

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