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Harmonic Analysis / Analyse harmonique

Fourier Quasicrystals with Unit Masses

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Abstract. The sum of $\delta$-measures sitting at the points of a discrete set $\Lambda \subset \mathbb{R}$ forms a Fourier quasicrystal if and only if $\Lambda$ is the zero set of an exponential polynomial with imaginary frequencies.

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1. Introduction

By a Fourier quasicrystal one usually means a complex measure with discrete support and spectrum. This concept goes back to works of Yves Meyer in the 1970-ies and it reappeared later in connection with an unexpected phenomenon in crystallography discovered by Dan Shechtman in the 1980-ies, see [5].

More precisely, following [6] we call a measure $\mu$ on $\mathbb{R}$ a crystalline measure, if it is an atomic measure which is a tempered distribution, its distributional Fourier transform $\hat{\mu}$ is an atomic measure and both the support $\Lambda$ and the spectrum $S$ of $\mu$ are locally finite sets. If in addition the measures $|\mu|$ and $|\hat{\mu}|$ are also tempered, then $\mu$ is called a Fourier quasicrystal (FQ).

The classical example of an FQ is the Dirac comb (the crystal)

$$\mu = \sum_{k \in \mathbb{Z}} \delta_k,$$

where $\delta_x$ is the unit mass at point $x$. Then the Poisson summation formula reads $\hat{\mu} = \mu$.

Examples of aperiodic quasicrystals were presented in [3] and then in [1, 6, 7]. Recently a new progress was achieved by P. Kurasov and P. Sarnak [2] who discovered examples of FQs with unit masses

$$\mu = \sum_{\lambda \in \Lambda} \delta_\lambda,$$

(1)

where $\Lambda \subset \mathbb{R}$ is a uniformly discrete aperiodic set. An alternative construction of such measures was suggested by Y. Meyer [8].

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Below we present one more construction and prove that it characterizes all FQs of form (1). A preliminary publication of our results was given in arXiv [9, 10].

The Theorem 1 below reveals a fundamental connection between FQs with unit masses and the zero sets \( Z(p) := \{ z \in \mathbb{C} : p(z) = 0 \} \) of exponential polynomials \( p \) with imaginary frequencies.

**Theorem 1.**

(i) Let \( p \) be an exponential polynomial

\[
p(t) = \sum_{1 \leq j \leq N} c_j e^{2\pi i \gamma_j t}, \quad N \in \mathbb{N}, c_j \in \mathbb{C}, \gamma_j \in \mathbb{R},
\]

which has only simple real zeros. Then the measure \( \mu \) defined in (1) with \( \Lambda = Z(p) \) is an FQ.

(ii) Conversely, let \( \mu \) be an FQ of form (1). Then there is an exponential polynomial \( p \) of form (2) with real simple zeros such that \( \Lambda = Z(p) \).

We will sketch the proof of part (iii), see [10] for the proof of part (i). Using Theorem 1 (i) one may construct simple examples of aperiodic FQs.

**Lemma 2.** Fix a real number \( \epsilon \) satisfying \( 0 < |\epsilon| \leq 1/2 \) and set

\[
p_\epsilon(t) := \sin(\pi t) + \epsilon \sin t.
\]

Then \( p_\epsilon \) has only simple real zeros and

\[
Z(p_\epsilon) = \{ k + \epsilon_k : k \in \mathbb{Z} \}, \quad \epsilon_k \in [-1/6, 1/6].
\]

For a proof see [10].

Theorem 1 and Lemma 2 show that the sum of \( \delta \)-measures sitting at the points of \( Z(p_\epsilon) \) is an FQ.

Let \( p_\epsilon \) be given in (3). One may check that the numbers \( \epsilon_k \) in Lemma 2 satisfy \( \max_k |\epsilon_k| \to 0 \) as \( \epsilon \to 0 \). Therefore, the set \( Z(p_\epsilon) \) “approaches” the set of integers \( \mathbb{Z} \):

**Corollary 3.** For every \( \epsilon > 0 \) there is an aperiodic set

\[
\Lambda = \{ k + \epsilon_k : k \in \mathbb{Z} \}, \quad 0 < |\epsilon_k| < \epsilon, k \in \mathbb{Z},
\]

such that the corresponding measure in (1) is an FQ.

**2. Proof of Part (ii) of Theorem 1**

In what follows we consider the standard form of the Fourier transform

\[
\hat{h}(u) := \int_\mathbb{R} e^{-2\pi i u t} h(t) \, dt, \quad h \in L^1(\mathbb{R}).
\]

Let us start with a result which may have intrinsic interest:

**Proposition 4.** Let \( \mu \) be a positive measure which is a tempered distribution, such that its distributional Fourier transform \( \hat{\mu} \) is a measure satisfying

\[
|\hat{\mu}|(-R, R) = O(R^m), \quad R \to \infty, \quad \text{for some } m > 0,
\]

which means that \( |\hat{\mu}| \) is a tempered distribution. Then there exists \( C \) such that

\[
\mu(a, b) \leq C(1 + b - a), \quad -\infty < a < b < \infty.
\]
Proof. It suffices to prove (5) for every interval \((a, b)\) satisfying \(b - a \geq 2\).

Fix any non-negative Schwartz function \(g(x)\) supported by \([-1/2, 1/2]\) and such that
\[
\int_{\mathbb{R}} g(x) \, dx = 1.
\]
Set
\[
f(x) := (g * 1_{(a-1/2, b+1/2)})(x) \in S(\mathbb{R}).
\]
Clearly,
\[
|\hat{f}(t)| = |\hat{g}(t)\hat{1}_{(a-1/2, b+1/2)}(t)| \leq (1 + b - a) |\hat{g}(t)|.
\]
Using this inequality and (4), we get
\[
\int_{\mathbb{R}} f(x) \mu(dx) = \int_{\mathbb{R}} \hat{f}(t) \hat{\mu}(dt) \leq (1 + b - a) \int_{\mathbb{R}} |\hat{g}(t)| |\hat{\mu}|(dt) = C(1 + b - a).
\]
On the other hand, clearly,
\[
f(x) = g(x) * 1_{(a-1/2, b+1/2)}(x) = 1, \quad x \in (a, b).
\]
Hence,
\[
\int_{\mathbb{R}} f(t) \mu(dt) \geq \mu(a, b),
\]
which proves the Proposition 4.

Recall that a set \(\Lambda \subset \mathbb{R}\) is called uniformly discrete, if
\[
\inf_{\lambda', \lambda \in \Lambda, \lambda \neq \lambda'} |\lambda - \lambda'| > 0.
\]
A set \(\Lambda\) is called relatively uniformly discrete if it is a union of finite number of uniformly discrete sets.

Proposition 4 implies

**Corollary 5.** Let \(\mu\) be a measure of form (1) whose distributional Fourier transform is a measure satisfying (4). Then its support \(\Lambda\) is a relatively uniformly discrete set.

Assume \(\mu\) is an FQ of form (1). This means that a Poisson-type formula
\[
\sum_{\lambda \in \Lambda} f(\lambda) = \sum_{s \in S} a_s \hat{f}(s), \quad f \in S(\mathbb{R}),
\]
is true where \(S(\mathbb{R})\) denotes the Schwartz space, \(S\) is locally finite set and the coefficients \(a_s\) satisfy
\[
\sum_{s \in S, |s| < R} |a_s| \leq CR^m, \quad R > 1, \quad \text{for some } C, m > 0.
\]
To prove part (ii) of Theorem 1 we have to show that \(\Lambda = Z(p)\) for some exponential polynomial \(p\) of form (2). We will prove this under the additional restrictions that \(\Lambda\) is a symmetric set, \(-\Lambda = \Lambda\) and \(0 \notin \Lambda\). For the general case see [9].

Set
\[
\psi(z) := \prod_{\lambda \in \Lambda} \left(1 - \frac{z}{\lambda}\right) = \prod_{\lambda \in \Lambda, \lambda > 0} \left(1 - \frac{z^2}{\lambda^2}\right), \quad z \in \mathbb{C}.
\]
The product converges (uniformly on compacts) due to Corollary 5.

**Lemma 6.** \(\psi\) is an entire function of order one and finite type, i.e. there exist \(C, \sigma > 0\) such that
\[
|\psi(z)| \leq Ce^{\sigma|z|}, \quad z \in \mathbb{C}.
\]
This lemma follows from Corollary 5 and the symmetry of \(\Lambda\) by standard estimates.

**Lemma 7.** The following representation is true:
\[
\frac{\psi'(z)}{\psi(z)} = -2\pi i \left( a_0/2 + \sum_{s \in S \cap (-\infty, 0)} a_s e^{-2\pi i s z} \right), \quad \text{Im } z > 0,
\]
where \(a_s\) are the coefficients in (6).
By (7), the series in (9) converges absolutely for every \( z, \text{Im} \, z > 0 \).
Let us sketch a proof of Lemma 7. It follows from (8) that
\[
\frac{\psi'(z)}{\psi(z)} = \sum_{\lambda \in \Lambda} \frac{1}{z - \lambda}, \quad z \in \mathbb{C}.
\] (10)

The next step is to check that
\[
\sum_{\lambda \in \Lambda} \frac{1}{z - \lambda} = -2\pi i \left( a_0/2 + \sum_{s \in S \cap (-\infty, 0)} a_s e^{-2\pi i sz} \right), \quad \text{Im} \, z > 0.
\] (11)

This can be done as follows: For every fixed \( z, \text{Im} \, z > 0 \), set
\[
e_z(u) = \begin{cases} 2\pi e^{-2\pi i zu} & u < 0 \\ 0 & u \geq 0 \end{cases}
\]

Then the inverse Fourier transform of \( e_z \) is the function \( i/(z - t) \). Fix any function \( h \in S(\mathbb{R}) \) such that \( h(0) = 1 \) and the Fourier transform \( \hat{H} := \hat{h} \) is even, non-negative and vanishes outside \((-1, 1)\).
Then use (6) with \( f(t) = h(e_t)/(z - t) \):
\[
\sum_{\lambda \in \Lambda} \frac{h(e_{\lambda})}{z - \lambda} = -i \sum_{s \in S} a_s \left( e_z(u) * \frac{1}{e} H(u/e) \right)(s).
\]
Finally, to prove Lemma 7 one lets \( \epsilon \to 0 \) and checks that the right and left hand-sides above converge to the corresponding sides of (11).

Now, it follows from (9) that there exists \( K \in \mathbb{C} \) such that
\[
\psi(z) = K \exp \left( -\pi i a_0 z + \sum_{s \in S \cap (-\infty, 0)} (a_s/s) e^{-2\pi i sz} \right), \quad \text{Im} \, z > 0.
\]

Set
\[
p(z) := e^{\pi i a_0 z} \psi(z)/K = \exp \left( \sum_{s \in S \cap (-\infty, 0)} (a_s/s) e^{-2\pi i sz} \right), \quad \text{Im} \, z > 0.
\] (12)

Recall that \( S \) is a locally finite set. Therefore, by (7) the series above converges absolutely for every \( z, \text{Im} \, z > 0 \).
Denote by \( S_k \) the sets
\[
S_1 := S \cap (-\infty, 0), \quad S_2 := S_1 + S_1, \quad S_3 := S_1 + S_1 + S_1, \ldots
\]
Denote by \( a_{s,k} \) the coefficients of the series
\[
\frac{1}{k!} \left( \sum_{s \in S \cap (-\infty, 0)} (a_s/s) e^{-2\pi i sz} \right)^k = \sum_{s \in S_k} a_{s,k} e^{-2\pi i sz}, \quad k \in \mathbb{N}, \text{Im} \, z > 0.
\]
Then by (12) we get a representation
\[
p(z) = 1 + \sum_{k=1}^{\infty} \sum_{s \in S_k} a_{s,k} e^{-2\pi i sz},
\]
where the double series converges absolutely for every \( z, \text{Im} \, z > 0 \). Set
\[
U := [0] \cup \bigcup_{j=1}^{\infty} S_j \subset (-\infty, 0].
\]
One may check that \( U \) is a locally finite set and that \( p \) admits a representation
\[
p(z) = \sum_{u \in U} d_u e^{-2\pi i uz}, \quad \text{Im} \, z > 0,
\] (13)
where the series converges absolutely.
To prove part (ii) of Theorem 1 it remains to check that the series in the right hand-side of (13) contains only a finite number of terms. This can be done as follows: Since \( \psi \) is an entire
function of order one and finite type, the same is true for \( p \). By (13), \( p \) is bounded on every line \( \text{Im} \, z = \text{const} > 0 \). It follows that (see [4, Lecture 6, Theorem 2]) \( p \) is an entire function of exponential type, i.e. it satisfies
\[
|p(x + iy)| \leq C e^{\sigma |y|}, \quad x, y \in \mathbb{R},
\]
with some \( C, \sigma > 0 \). Now, to check that in (13) we have \( d_u = 0 \) for every \( u \in U, |u| > \sigma \), one simply integrates both sides against \( e^{2\pi i uz} (\sin \epsilon z / \epsilon z)^2 \), where \( \epsilon > 0 \) is so small that \( |u| - \epsilon > \sigma \) and \( U \cap (u - \epsilon, u + \epsilon) = \{u\} \).

We note that one can extend Theorem 1 to measures with integer masses,
\[
\mu = \sum_{\lambda \in \Lambda} c_\lambda \delta_\lambda, \quad c_\lambda \in \mathbb{N}, \lambda \in \Lambda.
\]  

(14)

**Theorem 8.**

(i) **If a measure \( \mu \) of form (14) is an FQ, then there is an exponential polynomial \( p \) of form (2) with real zeros such that \( \Lambda = Z(p) \) and \( c(\lambda) \) is the multiplicity of zero \( \lambda \).**

(ii) **Conversely, let \( p \) be an exponential polynomial of form (2) with real zeros and let \( c(\lambda) \) be the multiplicity of zero \( \lambda \). Then the measure \( \mu \) of form (14) where \( \Lambda = Z(p) \) is an FQ.**

**References**