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Theory of functions / *Théorie des fonctions*

Polynomials with real zeros via special polynomials

Polynômes à racines réelles via des polynômes spéciaux

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Abstract. In this paper, we use particular polynomials to establish some results on the real rootedness of polynomials. The considered polynomials are Bell polynomials and Hermite polynomials. *To cite this article: M. Mihoubi and S. Taharbouchet, C. R. Acad. Sci. Paris, Ser. I 340 (2021).*

Résumé. Dans ce papier, nous utilisons des polynômes particuliers pour établir quelques résultats sur les polynômes à racines réelles. Les polynômes considérés sont des polynômes de Bell et des polynômes de Hermite. *Pour citer cet article : M. Mihoubi and S. Taharbouchet, C. R. Acad. Sci. Paris, Ser. I 340 (2021).*

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1. Introduction

The real rootedness of polynomials has attracted researchers great interest. One of the reasons is that any polynomial of real zeros implies the log-concavity and the unimodality of its coefficients, which appear in various fields of mathematics, see [4, 13].

Let $\mathbb{RZ}$ be the set of all polynomials having only real zeros and $\mathbb{NRZ}$ be the set of all polynomials having only non-positive real zeros. In a previous paper, we showed that for any polynomial f, if $f(\mathbf{B_x}) \in \mathbb{RZ}$, then $(\mathbf{B_x})_r f(\mathbf{B_x}) \in \mathbb{RZ}$, where $\mathbf{B_x}$ is the generalized Bell umbra [2]. Motivated by this work, we use in this paper the umbrae of special polynomials to give new results on the real rootedness of polynomials. The tools used here are classical umbral calculus, theorem 1 of Wang and Yeh [14] and Rolle's theorem.

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To begin, let $(\mathscr{P}_n(x))$ be a sequence of polynomials with $\deg\mathscr{P}_n = n$ and $\mathscr{P}_0(x) = 1$ and let $\mathbf{P}_\mathbf{x}^n$ be the umbra defined by $\mathbf{P}_\mathbf{x}^n := \mathscr{P}_n(x)$. The conditions $\deg\mathscr{P}_n = n$ and $\mathscr{P}_0(x) = 1$ show that the system of polynomials

$$\{\mathscr{P}_0(x), \mathscr{P}_1(x), \mathscr{P}_2(x), \ldots\}$$

is a free system and ensure that any polynomial with degree n can be expressed in terms of $1, \mathbf{P}_\mathbf{x}, \ldots, \mathbf{P}_\mathbf{x}^n$. We consider here some sequences of polynomials $(\mathscr{P}_n(x))$ satisfying $\mathscr{P}_0(x) = 1$, $\deg\mathscr{P}_n = n$ and a relation of the form

$$\mathscr{P}_{n+1}(x) = (u_1 x + u_2)\mathscr{P}_n(x) + (v_1 x + v_2)\frac{d}{dx}\mathscr{P}_n(x), \tag{1}$$

for some known real numbers u_1, u_2, v_1 and v_2.

For more information about classical umbral calculus, see [2, 7, 11, 12].

We use below the notations

$$(x)_0 := 1,\ (x)_n := x(x-1)\cdots(x-n+1),\ n \geq 1,$$
$$\langle x\rangle_0 := 1,\ \langle x\rangle_n := x(x+1)\cdots(x+n-1),\ n \geq 1.$$

The paper is organized as follows: in the following Section 2 we give the main results and in the last Section 3 we present some of their applications.

2. Real-rooted polynomials via special polynomials

Let $\mathscr{B}_n(x)$ and $\mathscr{H}_n(x)$ be, respectively, the n^{th} Bell and Hermite polynomials defined by

$$\sum_{n\geq 0}\mathscr{B}_n(x)\frac{t^n}{n!} = \exp\big(x(\exp(t)-1)\big),$$
$$\sum_{n\geq 0}\mathscr{H}_n(x)\frac{t^n}{n!} = \exp\big(2xt - t^2\big),$$

and let $\mathbf{B}_\mathbf{x}$ and $\mathbf{H}_\mathbf{x}$ be the umbrae defined respectively by

$$\mathbf{B}_\mathbf{x}^n = \mathscr{B}_n(x) \quad \text{and} \quad \mathbf{H}_\mathbf{x}^n = \mathscr{H}_n(x).$$

These polynomials satisfy the following differential recurrence relations

$$\mathscr{B}_{n+1}(x) - x\mathscr{B}_n(x) = x\frac{d}{dx}\mathscr{B}_n(x),$$
$$\mathscr{H}_{n+1}(x) - 2x\mathscr{H}_n(x) = -\frac{d}{dx}\mathscr{H}_n(x),$$

which can be written as

$$\mathbf{B}_\mathbf{x}^{n+1} - x\mathbf{B}_\mathbf{x}^n = x\frac{d}{dx}\mathbf{B}_\mathbf{x}^n, \quad \mathbf{H}_\mathbf{x}^{n+1} - 2x\mathbf{H}_\mathbf{x}^n = -\frac{d}{dx}\mathbf{H}_\mathbf{x}^n.$$

Hence, for any polynomial f, these relations are equivalent to

$$\mathbf{B}_\mathbf{x} f(\mathbf{B}_\mathbf{x}) - x f(\mathbf{B}_\mathbf{x}) = x\frac{d}{dx} f(\mathbf{B}_\mathbf{x}), \tag{2}$$
$$\mathbf{H}_\mathbf{x} f(\mathbf{H}_\mathbf{x}) - 2x f(\mathbf{H}_\mathbf{x}) = -\frac{d}{dx} f(\mathbf{H}_\mathbf{x}). \tag{3}$$

2.1. *Real-rooted polynomials via Wang–Yeh's theorem*

The results of this subsection are based on [14, Theorem 1] of Wang and Yeh and the principal result is given by the following theorem.

Theorem 1. *Let f be a polynomial and m be a positive integer. Then, if $f(\mathbf{B_x}) \in \mathbb{RZ}$, there holds*

$$f(\mathbf{B_x}) \prod_{k=1}^{m} (\mathbf{B_x} + a_k x + b_k) \in \mathbb{RZ}, \quad a_k \in \mathbb{R},\ b_k \in [0,\infty[, \quad k = 1,\dots,m,$$

and if $f(\mathbf{H_x}) \in \mathbb{RZ}$, there holds

$$f(\mathbf{H_x}) \prod_{k=1}^{m} (\mathbf{H_x} + a_k x + b_k) \in \mathbb{RZ}, \quad a_k \in [-2,\infty[,\ b_k \in \mathbb{R}, \quad k = 1,\dots,m.$$

Proof. Let g be a polynomial. Relation (2) implies

$$\big(\mathbf{B_x} + g(x) - x\big) f(\mathbf{B_x}) = g(x) f(\mathbf{B_x}) + x \frac{d}{dx} f(\mathbf{B_x}). \tag{4}$$

The choice $g(x) = (a_1+1)x + b_1$ in the last relation gives

$$f_1(\mathbf{B_x}) = (\mathbf{B_x} + a_1 x + b_1) f(\mathbf{B_x}) = ((a_1+1)x + b_1) f(\mathbf{B_x}) + x \frac{d}{dx} f(\mathbf{B_x}).$$

For $(a_1, b_1) \in \mathbb{R} \times [0,\infty[$ apply [14, Theorem 1] of Wang and Yeh to deduce that $f_1(\mathbf{B_x}) \in \mathbb{RZ}$. So, by repeating the same process, we conclude that

$$\begin{aligned} f_2(\mathbf{B_x}) &= (\mathbf{B_x} + a_2 x + b_2) f_1(\mathbf{B_x}) \\ &= (\mathbf{B_x} + a_2 x + b_2)(\mathbf{B_x} + a_1 x + b_1) f(\mathbf{B_x}) \in \mathbb{RZ}, \end{aligned}$$

and so on. The second result can be proved similarly on using the relation (3). □

Proposition 2. *Let f be a polynomial such that $f(\mathbf{B_x}) \in \mathbb{RZ}$. Then, for any polynomial $g(x) \in \mathbb{NRZ}$, there holds*

$$g(\mathbf{B_x} + ax) f(\mathbf{B_x}) \in \mathbb{RZ},\ a \in \mathbb{R},$$

and, when $g(x) \in \mathbb{RZ}$ and $f(\mathbf{H_x}) \in \mathbb{RZ}$, there holds

$$g(\mathbf{H_x} + ax) f(\mathbf{H_x}) \in \mathbb{RZ},\ a \geq -2.$$

Proof. Let $g(x) = \prod_{k=1}^{m} (x + b_k)$. Then, for the choices $a_1 = \cdots = a_m = a$ in Theorem 1, we obtain the desired results. □

Proposition 3. *Let g be a polynomial. Then, if $g(x) \in \mathbb{NRZ}$, then there holds*

$$g(\mathbf{B_x}) \in \mathbb{RZ},$$

and if $g(x) \in \mathbb{RZ}$, then there holds

$$g(\mathbf{H_x}) \in \mathbb{RZ}.$$

Proof. For $a_1 = \cdots = a_m = 0$ and $f(x) = x + b_0$, $b_0 > 0$, in Theorem 1 and by setting $g(x) = \prod_{k=0}^{m} (x + b_k)$, we obtain $g(\mathbf{B_x}) \in \mathbb{RZ}$ when $g(x) \in \mathbb{NRZ}$, and $g(\mathbf{H_x}) \in \mathbb{RZ}$ when $g(x) \in \mathbb{RZ}$. □

Remark 4. Since for any non-negative integer k we have [1]

$$(\mathbf{B_x})_k = x^k,\ \langle \mathbf{B_x} \rangle_k = \mathbb{L}_k(x) \text{ and } (\mathbf{B_x} - 1)_k = (-1)^k \mathbb{D}_k(1-x),$$

Proposition 3 shows that if one of the polynomials

$$\sum_{k=0}^{m} g_1(m,k)(x)_k, \quad \sum_{k=0}^{m} g_2(m,k)\langle x \rangle_k, \quad \sum_{k=0}^{m} g_3(m,k)(x-1)_k$$

has only non-positive real zeros, then its correspondent polynomial given between the polynomials

$$\sum_{k=0}^{m} g_1(m,k)\,x^k,\quad \sum_{k=0}^{m} g_2(m,k)\,\mathbb{L}_k(x),\quad \sum_{k=0}^{m} (-1)^k g_3(m,k)\,\mathbb{D}_k(1-x),$$

has also only real zeros, where

$$\mathbb{L}_n(x) = \sum_{k=1}^{n} \frac{n!}{k!}\binom{n-1}{k-1} x^k,\quad n \geq 1,\ \mathbb{L}_0(x) = 1,$$
$$\mathbb{D}_n(x) = \sum_{j=0}^{n} \binom{n}{j} j!\,(x-1)^{n-j}$$

are, respectively, the Lah and derangement polynomials.

Remark 5. Proposition 3 shows that if the polynomial $g(x) = \sum_{k=0}^{m} g(m,k)x^k$ has only non-positive real zeros, then the polynomial

$$g(\mathbf{B_x}) = \sum_{k=0}^{m} g(m,k)\,\mathscr{B}_k(x)$$

has only real zeros, and if the polynomial $g(x)$ has only real zeros, then the polynomial

$$g(\mathbf{H_x}) = \sum_{k=0}^{m} g(m,k)\,\mathscr{H}_k(x)$$

has only real zeros.

Remark 6. From the above remarks, it follows that the linear transformations

$$\begin{aligned} (x)_k &\to x^k, \\ \langle x\rangle_k &\to \mathbb{L}_k(x), \\ (x-1)_k &\to (-1)^k \mathbb{D}_k(1-x), \\ x^k &\to \mathscr{B}_k(x) \end{aligned}$$

map $\mathbb{NRZ}$ onto $\mathbb{RZ}$ and $x^k \to \mathscr{H}_k(x)$ maps $\mathbb{RZ}$ onto $\mathbb{RZ}$.

For the choice $g(x) = x^q (f(x))^{n-1}$ in Proposition 3 we obtain:

Corollary 7. *Let f be a polynomial. If $f(x) \in \mathbb{NRZ}$, then there hold*

$$\mathbf{B}_\mathbf{x}^q \left(f(\mathbf{B_x})\right)^n \in \mathbb{RZ},\quad n = 1,2,\ldots,\ q = 0,1,\ldots,$$

and, if $f(x) \in \mathbb{RZ}$, then there holds

$$\mathbf{H}_\mathbf{x}^q \left(f(\mathbf{H_x})\right)^n \in \mathbb{RZ},\quad n = 1,2,\ldots,\ q = 0,1,\ldots.$$

Proposition 8. *Let f be a polynomial such that $f(\mathbf{B_x}) \in \mathbb{RZ}$. Then, for any non-negative integers $r_1,\ldots,r_p$, there holds*

$$\langle \mathbf{B_x} + a_1 x\rangle_{r_1} \cdots \left\langle \mathbf{B_x} + a_p x\right\rangle_{r_p} f(\mathbf{B_x}) \in \mathbb{RZ},\ a_1,\ldots,a_p \in \mathbb{R}.$$

Furthermore, if $r_1 + \cdots + r_p \geq 1$, and if $g(x) \in \mathbb{NRZ}$, there holds

$$\langle \mathbf{B_x}\rangle_{r_1} \cdots \langle \mathbf{B_x}\rangle_{r_p} g(\mathbf{B_x} + ax) \in \mathbb{RZ},\ a \in \mathbb{R}.$$

Proof. For the first part of the proposition choice in Proposition 2 $g(x)$ to be $\langle x\rangle_{r_1}$ and replace a by a_1 to obtain

$$f_1(\mathbf{B_x}) := \langle \mathbf{B_x} + a_1 x\rangle_{r_1} f(\mathbf{B_x}) \in \mathbb{RZ},\ a_1 \in \mathbb{R}.$$

Now, choice in Proposition 2 the polynomial $g(x)$ to be $\langle x\rangle_{r_2}$ and replace a by a_2 and f by f_1 to obtain

$$f_2(\mathbf{B_x}) := \langle \mathbf{B_x} + a_2 x\rangle_{r_2} f_1(\mathbf{P_x}) \in \mathbb{RZ},\ a_2 \in \mathbb{R},$$

and so on. For the second part of the proposition, by choosing $a_1 = \cdots = a_p = 0$ and $f(x) = x + i$ for such integer $i \in \{r_1, \ldots, r_p\}$, the first part of this proposition proves that

$$\langle \mathbf{B_x}\rangle_{r_1} \cdots \langle \mathbf{B_x}\rangle_{r_p} \in \mathbb{RZ},$$

with $r_1 + \cdots + r_p \geq 1$ and $n \geq 1$. So, the choice $f(x) = \langle x\rangle_{r_1} \cdots \langle x\rangle_{r_p}$ in the first part of Proposition 2 completes the proof. □

One can prove similarly the following Proposition 9.

Proposition 9. *Let f be a polynomial such that $f(\mathbf{H_x}) \in \mathbb{RZ}$. Then, for any non-negative integers $r_1, \ldots, r_p$, there hold*

$$(\mathbf{H_x} + a_1 x)_{r_1} \cdots \left(\mathbf{H_x} + a_p x\right)_{r_p} f(\mathbf{H_x}) \in \mathbb{RZ},\ a_1 \geq -2, \ldots, a_p \geq -2,$$

$$\langle \mathbf{H_x} + a_1 x\rangle_{r_1} \cdots \left\langle \mathbf{H_x} + a_p x\right\rangle_{r_p} f(\mathbf{H_x}) \in \mathbb{RZ},\ \ a_1 \geq -2, \ldots, a_p \geq -2.$$

Furthermore, if $r_1 + \cdots + r_p \geq 1$, $n \geq 1$, and $g(x) \in \mathbb{RZ}$, there hold

$$(\mathbf{H_x})_{r_1} \cdots (\mathbf{H_x})_{r_p}\, g(\mathbf{H_x} + ax) \in \mathbb{RZ},\ a \geq -2,$$

$$\langle \mathbf{H_x}\rangle_{r_1} \cdots \langle \mathbf{H_x}\rangle_{r_p}\, g(\mathbf{H_x} + ax) \in \mathbb{RZ},\ a \geq -2.$$

2.2. *Real-rooted polynomials via Rolle's theorem*

The results of this subsection are based on Rolle's theorem and the principal result is given by the following Theorem 10.

Theorem 10. *Let $a_1, \ldots, a_m, b_1, \ldots, b_m$ be real numbers and let f be a polynomial. Then, if $f(\mathbf{B_x}) \in \mathbb{RZ}$, there hold*

$$f(\mathbf{B_x}) \prod_{k=1}^{m} \left(\mathbf{B_x} + a_k x^2 + b_k x\right) \in \mathbb{RZ}, \quad a_1 \leq 0, \ldots, a_m \leq 0.$$

Proof. Let n be the degree of the polynomial f.

The choice $g(x) = (a_1 x + b_1 + 1)x$ in (4) gives

$$\left(\mathbf{B_x} + a_1 x^2 + b_1 x\right) f(\mathbf{B_x}) = x \exp\left(-a_1 \frac{x^2}{2} - (b_1 + 1)x\right) \frac{d}{dx}\left(\exp\left(a_1 \frac{x^2}{2} + (b_1 + 1)x\right) f(\mathbf{B_x})\right).$$

Case $a_1 < 0$. The application of Rolle's theorem on the function

$$T(x) = \exp\left(a_1 \frac{x^2}{2} + (b_1 + 1)x\right) f(\mathbf{B_x})$$

shows that the polynomial $f_1(\mathbf{B_x}) = (\mathbf{B_x} + a_1 x^2 + b_1 x) f(\mathbf{B_x})$, which is of degree $n+2$, vanishes $n-1$ times between the zeros of $f(\mathbf{B_x})$. But since $\lim_{x\to\pm\infty} T(x) = 0$ it results that $f_1(\mathbf{B_x})$ vanishes also on $]-\infty, x_1[$ and on $]x_n, \infty[$, where x_1 and x_n are, respectively, the minimum and the maximum zeros of $f(\mathbf{B_x})$. Hence, the polynomial $f_1(\mathbf{B_x})$ vanishes $n+1$ times, the missing zero must be necessarily real.

Case $a_1 = 0$: When $b_1 + 1 = 0$ the Theorem 10 is evident, otherwise, the application of Rolle's theorem on the function

$$T(x) = \exp((b_1 + 1)x) f(\mathbf{B_x})$$

shows that the polynomial $f_1(\mathbf{B_x}) = (\mathbf{B_x} + b_1 x) f(\mathbf{B_x})$ vanishes $n-1$ times between the zeros of $f(\mathbf{B_x})$. But since $\lim_{x\to+\infty} T(x) = 0$ or $\lim_{x\to-\infty} T(x) = 0$ it results that $f_1(\mathbf{B_x})$ vanishes also on $]-\infty, x_1[$ or $]x_n, \infty[$. Hence, the polynomial $f_1(\mathbf{B_x})$ which is of degree $n+1$ vanishes n times, the missing zero must be necessarily real.

To obtain the first part of the Theorem 10, one can replace $f(\mathbf{B_x})$ by $f_1(\mathbf{B_x})$ and proceed as above, and so on. □

Proposition 11. *Let a be a real number and let f, g be polynomials with* $\deg g = m$. *Then, if $f(\mathbf{B_x}) \in \mathbb{RZ}$ and $g(x) \in \mathbb{RZ}$, we get*

$$x^m g\left(\frac{\mathbf{B_x}}{x} + ax\right) f(\mathbf{B_x}) \in \mathbb{RZ}, \quad a \leq 0.$$

Proof. The proposition 11 follows form Theorem 10 by taking $a_1 = \cdots = a_m = a \leq 0$ and by setting $g(x) = \prod_{k=1}^{m} (x + b_k)$. □

Remark 12. Let $q, r_1 \leq \cdots \leq r_p$ be non-negative integers not all null.

For $f(x) = (x)_{r_1} \cdots (x)_{r_p} x^q$ we get

$$f(\mathbf{B_x}) = (\mathbf{B_x})_{r_p} \cdots (\mathbf{B_x})_{r_p} \mathbf{B_x}^q = x^{r_p} \mathscr{B}_q(x; r_1, \cdots, r_p) \in \mathbb{RZ}$$

and for $g(x) = \sum_{k=0}^{m} g(m,k) x^k \in \mathbb{RZ}$ and $a = 0$ in Proposition 11 we get

$$x^{m-r_p} g\left(\frac{\mathbf{B_x}}{x}\right) f(\mathbf{B_x}) = \sum_{k=0}^{m} g(m,k)\, x^{m-k} \mathscr{B}_{k+q}(x; r_1, \cdots, r_p) \in \mathbb{RZ},$$

where $\mathscr{B}_n(x; r_1, \cdots, r_p)$ is the $n^{\text{th}}(r_1, \cdots, r_p)$ – Bell polynomial introduced by Mihoubi and Maamra [8, 10], see also [1].
In other words, for $0 \leq k \leq m$, the linear transformation

$$x^k \to x^{m-k} \mathscr{B}_{k+q}(x; r_1, \ldots, r_p), \quad q + r_1 + \cdots + r_p \geq 1,$$

map the set $\mathbb{RZ}$ on to $\mathbb{RZ}$.

3. Some applications

In this section, we present some applications of the above results.

Corollary 13. *Let G be a simple graph with n vertices and let*

$$P(G,x) = \sum_{k=\chi(G)}^{n} \alpha_k(G)(x)_k$$

be its chromatic polynomial such that $P(G,x) \in \mathbb{RZ}$. Then, the polynomial

$$P(G, -\mathbf{B_x}) = \sum_{k=\chi(G)}^{n} (-1)^k \alpha_k(G) \mathbb{L}_k(x)$$

has only real zeros, where $\chi(G)$ is the chromatic number of G.

Proof. The chromatic polynomial $P(G,x)$ can also be written as

$$P(G,x) = x \sum_{k=1}^{n-1} (-1)^k t_k(G)(x-1)^{k-1},$$

where $t_1(G), \ldots, t_{n-1}(G)$ are non-negative integers, see [6]. This shows that the zeros of the polynomial $P(G,x)$ are not in $]-\infty, 1[-\{0\}$. Then, the zeros of the polynomial $P(G,-x)$ are non-positive real numbers. Since [1]

$$(\mathbf{B_x})_k f(\mathbf{B_x}) = x^k f(\mathbf{B_x} + k) \quad \text{and} \quad \langle \mathbf{B_x} \rangle_k = \mathbb{L}_k(x),\ k = 0, 1, 2, \ldots,$$

then, by Proposition 3, it follows that $P(G, -\mathbf{B_x}) \in \mathbb{RZ}$, with

$$P(G,-\mathbf{B_x}) = \sum_{k=\chi(G)}^{n} (-1)^k \alpha_k(G) \langle \mathbf{B_x} \rangle_k = \sum_{k=\chi(G)}^{n} (-1)^k \alpha_k(G) \mathbb{L}_k(x).$$

□

Example 14. For $q=0$, $f(x)=x$ in Corollary 7, we get

$$\left(f(\mathbf{B_x})\right)^n = \mathscr{B}_n(x) \in \mathbb{RZ}, \quad \left(f(\mathbf{H_x})\right)^n = \mathscr{H}_n(x) \in \mathbb{RZ}, \quad n=1,2,\ldots.$$

Also, since $\mathscr{B}_n(x)$ has non-negative coefficients, it follows that its zeros are non-positive and by Proposition 3, we get

$$\mathscr{B}_n(\mathbf{P_x}) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} \mathscr{P}_k(x) \in \mathbb{RZ},$$
$$\mathscr{H}_n(\mathbf{P_x}) = \sum_{k=0}^{[n/2]} (-1)^k \frac{2^{n-2k} n!}{k!(n-2k)!} \mathscr{P}_{n-2k}(x) \in \mathbb{RZ},$$

where $[x]$ is the largest integer $\leq x$, $\left\{ {n \atop k} \right\}$ is the $(n,k)^{\text{th}}$ Stirling number of the second kind, and, $\mathbf{P}_\mathbf{x}^n = \mathscr{P}_n(x)$ is any one of the polynomials $\mathscr{B}_n(x)$ and $\mathscr{H}_n(x)$.

Example 15. For $f(x)=x+y$ we get $f(\mathbf{B_x}) = f(\mathbf{H_x}) = x+y \in \mathbb{RZ}$ and via Corollary 7, we obtain

$$\mathbf{B}_\mathbf{x}^q \left(f(\mathbf{B_x})\right)^n = \sum_{k=0}^{n} \binom{n}{k} y^{n-k} \mathscr{B}_{k+q}(x) \in \mathbb{RZ},\ y \geq 0,$$
$$\mathbf{H}_\mathbf{x}^q \left(f(\mathbf{H_x})\right)^n = \sum_{k=0}^{n} \binom{n}{k} y^{n-k} \mathscr{H}_{k+q}(x) \in \mathbb{RZ},\ y \in \mathbb{R}.$$

Example 16. From Proposition 8 it follows, for any real number a and any polynomial f such that $f(\mathbf{B_x}) \in \mathbb{RZ}$, that $(\mathbf{B_x}+ax)^{n-2} f(\mathbf{B_x}) \in \mathbb{RZ}$, $n \geq 2$. Then, for $a=-1$, $f(\mathbf{B_x}) = (\mathbf{B_x}-x)^2 = x$ we get the known result [3]

$$\mathscr{V}_n(x) := (\mathbf{B_x}-x)^n = \sum_{k=0}^{n} \binom{n}{k} (-x)^{n-k} \mathscr{B}_k(x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}^{2\uparrow} x^k \in \mathbb{RZ},$$

where $\left\{ {n \atop k} \right\}^{2\uparrow}$ is the number of ways to partition the set $\{1,\ldots,n\}$ into k blocs without singletons.

More generally, for $f(\mathbf{B_x}) = (\mathbf{B_x}+ax)^2 = x((a+1)^2x+1)$ we get

$$\mathscr{V}_{n,a}(x) := (\mathbf{B_x}+ax)^n = \sum_{k=0}^{n} B_{n,k}(a+1,1,1,\ldots) x^k \in \mathbb{RZ}.$$

We note here that $\sum_{n\geq 0} \mathscr{V}_{n,a}(x) \frac{t^n}{n!} = \exp(x(e^t-1+at))$, where $B_{n,k}(a_1,a_2,\ldots)$ are the partial Bell polynomials, see [5,9].

Example 17. For $r_p = \max\{r_1,\ldots,r_p\}$, $a_1 = \cdots = a_p = 0$ and $f(x) = x^n$ in the first identity of Proposition 8, it follows that the polynomial

$$\mathbb{L}_{n;r_1,\ldots,r_p}(x) := \sum_{k=0}^{n+r_1+\cdots+r_{p-1}} (-1)^k \left\{ {n+r_1+\cdots+r_p \atop k+r_p} \right\}_{r_1,\ldots,r_p} \mathbb{L}_k(x)$$

has only real roots. Indeed, from [1] we can write

$$
\begin{aligned}
&\langle \mathbf{B_x}\rangle_{r_1}\cdots\langle \mathbf{B_x}\rangle_{r_p}\mathbf{B}_{\mathbf{x}}^n\\
&\qquad = (-1)^{n+r_1+\cdots+r_p}(-\mathbf{B_x})_{r_1}\cdots(-\mathbf{B_x})_{r_p}(-\mathbf{B_x})^n\\
&\qquad = (-1)^{n+r_1+\cdots+r_p}\sum_{k=0}^{n+r_1+\cdots+r_{p-1}}\left\{ {n+r_1+\cdots+r_p \atop k+r_p} \right\}_{r_1,\ldots,r_p}(-\mathbf{B_x})_k\\
&\qquad = (-1)^{n+r_1+\cdots+r_p}\sum_{k=0}^{n+r_1+\cdots+r_{p-1}}(-1)^k\left\{ {n+r_1+\cdots+r_p \atop k+r_p} \right\}_{r_1,\ldots,r_p}\langle \mathbf{B_x}\rangle_k\\
&\qquad = (-1)^{n+r_1+\cdots+r_p}\sum_{k=0}^{n+r_1+\cdots+r_{p-1}}(-1)^k\left\{ {n+r_1+\cdots+r_p \atop k+r_p} \right\}_{r_1,\ldots,r_p}\mathbb{L}_k(x)\\
&\qquad = (-1)^{n+r_1+\cdots+r_p}\mathbb{L}_{n;r_1,\ldots,r_p}(x),
\end{aligned}
$$

where $\left\{ {n \atop k} \right\}_{r_1,\ldots,r_p}$ is the $(r_1,\ldots,r_p)$-Stirling numbers of the second kind introduced by Mihoubi and Maamra [8, 10].

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