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Partial differential equations / *Équations aux dérivées partielles*

Improvement of conditions for boundedness in a fully parabolic chemotaxis system with nonlinear signal production

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Abstract. This paper deals with the chemotaxis system with nonlinear signal secretion

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u - S(u)\nabla v), & x\in\Omega, \quad t>0,\\ v_t = \Delta v - v + g(u), & x\in\Omega, \quad t>0,\end{cases}$$

under homogeneous Neumann boundary conditions in a bounded domain $\Omega\subset\mathbb{R}^n$ $(n\geq 2)$. The diffusion function $D(s)\in C^2([0,\infty))$ and the chemotactic sensitivity function $S(s)\in C^2([0,\infty))$ are given by $D(s)\geq C_d(1+s)^{-\alpha}$ and $0<S(s)\leq C_s s(1+s)^{\beta-1}$ for all $s\geq 0$ with $C_d, C_s>0$ and $\alpha,\beta\in\mathbb{R}$. The nonlinear signal secretion function $g(s)\in C^1([0,\infty))$ is supposed to satisfy $g(s)\leq C_g s^\gamma$ for all $s\geq 0$ with $C_g,\gamma>0$. Global boundedness of solution is established under the specific conditions:

$$0<\gamma\leq 1 \quad\text{and}\quad \alpha+\beta<\min\left\{1+\frac{1}{n}, 1+\frac{2}{n}-\gamma\right\}.$$

The purpose of this work is to remove the upper bound of the diffusion condition assumed in [9], and we also give the necessary constraint $\alpha+\beta<1+\frac{1}{n}$, which is ignored in [9, Theorem 1.1].

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1. Introduction

In the present work, we consider the following system, which describes the fully parabolic chemotaxis system with nonlinear diffusion, sensitivity and signal secretion

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u - S(u)\nabla v), & x\in\Omega, \quad t>0,\\ v_t = \Delta v - v + g(u), & x\in\Omega, \quad t>0,\\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x\in\partial\Omega, \ t>0,\\ (u,v)(x,0) = (u_0(x), v_0(x)), & x\in\Omega, \end{cases} \tag{1}$$

with homogeneous Neumann boundary conditions, where $\Omega\subset\mathbb{R}^n$ ($n\ge 2$) is a bounded domain, and $\partial/\partial\nu$ is the derivative of the normal with respect to $\partial\Omega$. In system (1), $u = u(x,t)$ and $v = v(x,t)$ represent the density of population and the concentration of chemicals, respectively. In this article, the diffusion function $D \in C^2([0,\infty))$ and the chemotactic sensitivity function $S\in C^2([0,\infty))$ with $S(0)=0$ are given by

$$D(s) \ge C_d(1+s)^{-\alpha} \quad \text{and} \quad 0\le S(s)\le C_s s(1+s)^{\beta-1} \quad \text{for all } s\ge 0 \tag{2}$$

with $C_d, C_s > 0$ and $\alpha,\beta\in\mathbb{R}$. The signal secretion function $g\in C^1([0,\infty))$ is nonnegative and satisfies

$$g(s)\le C_g s^\gamma \text{ for all } s\ge 0 \quad \text{with} \quad C_g,\gamma>0. \tag{3}$$

The well-known chemotaxis model for the chemotactic movement of one specie [4] proposed by Keller and Segel, which describes the aggregation phenomenon of the Dictyostelium discoideum, there are many results about this system [1, 3, 9, 12, 13, 15, 16]. For instance, in case $g(u) = u$, the asymptotics of $\frac{S(u)}{D(u)} \simeq u^{\frac{2}{n}}$ is critical to distinguish the blow-up and global boundedness: under the condition $\frac{S(u)}{D(u)} \le cu^{\frac{2}{n}-\epsilon}$ for all $u>1$ with $\epsilon>0$, Tao and Winkler [12] obtained the global boundedness of solution; while if $\frac{S(u)}{D(u)} \le cu^{\frac{2}{n}+\epsilon}$ for all $u>1$ [16], the solution of (1) blow-up either in infinite time or finite time. We note that in [9], global boundedness of solution is established under the conditions that $\alpha+\beta+\gamma<1+\frac{2}{n}$ and $d_0(1+u)^\alpha\le D(u)\le d_1(1+u)^{\alpha_1}$ with $d_0,d_1>0$ and $\alpha,\alpha_1\in\mathbb{R}$. The purpose of this work is to remove the upper bound of the diffusion condition and give the necessary constraint $\alpha+\beta<1+\frac{1}{n}$ that is ignored in [9, Theorem 1.1]. The main result of this article is described below.

Theorem 1. *Let $\Omega\subset R^n$ ($n\ge 2$) be a smooth bounded domain. The nonnegative initial data $(u_0,v_0)\in C^0(\overline{\Omega})\times C^1(\overline{\Omega})$. Assume that* (2) *and* (3) *hold. If $0<\gamma\le 1$ and*

$$\alpha+\beta<\min\left\{1+\frac{1}{n}, 1+\frac{2}{n}-\gamma\right\},$$

then system (1) *possesses a unique global bounded classical solution (u,v) in the sense that there exists some constant $C>0$ satisfying*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} < C \quad \textit{for all } t>0.$$

Remark 2. Compared with the previous study in [9, Theorem 1.1], we give the necessary constraint $\alpha+\beta<1+\frac{1}{n}$ that is ignored in it, and we also remove the restriction on the upper bound of the diffusion function $D(s)$.

2. Boundedness

Let us state the local existence result, which has been established in [1, 3, 8, 10, 17, 18].

Lemma 3. *Let $\Omega \subset R^n$ $(n \geq 2)$ be a smooth bounded domain. The nonnegative initial data $(u_0, v_0) \in C^0(\overline{\Omega}) \times C^1(\overline{\Omega})$. Assume that* (2) *and* (3) *hold, then there exists $t \in (0, T_{\max})$ such that system* (1) *has a unique non-negative solution and satisfies*

$$u, v \in C(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})),$$

where T_{max} denotes the maximal existence time. Moreover, if $T_{max} < \infty$, then

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} \to \infty \quad \text{as } t \nearrow T_{\max}.$$

In order to obtain the global boundedness of solution to system (1), we first establish a series of prior estimates; then we treat the dissipative terms on the right hand side of the inequality by using the Gagliardo–Nirenberg inequality; last, we get our final results by controlling the parameter range in the inequality. The ideas come from [9, 12–14].

Lemma 4. *Let $\Omega \subset R^n$ $(n \geq 2)$ be a smooth bounded domain. The nonnegative initial data $(u_0, v_0) \in C^0(\overline{\Omega}) \times C^1(\overline{\Omega})$. Assume that* (2) *and* (3) *hold, then the first term of the solution to system* (1) *satisfies*

$$\|u(\cdot, t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)} \quad \text{for all} \quad t \in (0, T_{\max}). \tag{4}$$

Furthermore, assume that $0 < \gamma \leq 1$, if $s \in \left[1, \frac{n}{(n\gamma-1)_+}\right)$, then there exits $C > 0$ such that

$$\|v(\cdot, t)\|_{W^{1,s}(\Omega)} \leq C \quad \text{for all} \quad t \in (0, T_{\max}). \tag{5}$$

Proof. Integrating the first equation of (1) over Ω, (4) can be easily obtained. From the Neumann semigroup estimates method in [5, Lemma 1], (5) can be obtained. □

Before we give the result of main part, we first select the appropriate parameters.

Lemma 5. *Let $\Omega \subset R^n$ $(n \geq 2)$ be a smooth bounded domain, the nonnegative initial data $(u_0, v_0) \in C^0(\overline{\Omega}) \times C^1(\overline{\Omega})$, assume that* (2) *and* (3) *hold. In case $0 < \gamma \leq 1$, if*

$$\alpha + \beta < \min\left\{1 + \frac{1}{n}, 1 + \frac{2}{n} - \gamma\right\},$$

then there exists $s \in \left[1, \frac{n}{(n\gamma-1)_+}\right)$ such that

$$\gamma - \frac{1}{n} < \frac{1}{s} < 1 + \frac{1}{n} - \alpha - \beta. \tag{6}$$

Moreover, let $1 < a < \min\left\{\frac{n}{n-2}, \frac{s}{(s-2)_+}\right\}$ and $b > \max\left\{\frac{n}{2}, \frac{1}{2\gamma}\right\}$, we choose some $p_\star > 1 + \frac{n\alpha}{2}$ and $q_\star > 1 + \frac{s}{2}$ such that for all $p > p_\star$ and $q > q_\star$, then we have

$$\frac{n-2}{n} \cdot \frac{p + \alpha + 2\beta - 2}{p - \alpha} < \frac{1}{a} < p + \alpha + 2\beta - 2, \tag{7}$$

$$1 - \frac{2}{s} < \frac{1}{a} < 1 - \frac{n-2}{nq}, \tag{8}$$

$$\frac{n-2}{n} \cdot \frac{2\gamma}{p - \alpha} < \frac{1}{b} < \frac{2}{n} + \frac{1}{q}\left(1 - \frac{2}{n}\right) \quad \text{and} \quad \frac{2b(q-1)}{b-1} > s. \tag{9}$$

Proof. The proof is similar to [9] (also see [12]), so we omitted it here. □

In the following lemma, we obtain the uniform boundedness of $\|u\|_{L^p(\Omega)}$ by establishing a priori estimates and taking appropriate parameters.

Lemma 6. *Let $\Omega \subset R^n$ $(n \geq 2)$ be a smooth bounded domain. The nonnegative initial data $(u_0, v_0) \in C^0(\overline{\Omega}) \times C^1(\overline{\Omega})$. Assume that* (2) - (3) *and Lemma 5 hold. If*

$$0 < \gamma \leq 1 \quad \text{and} \quad \alpha + \beta < \min\left\{1 + \frac{1}{n}, 1 + \frac{2}{n} - \gamma\right\},$$

then there exists $C > 0$ such that

$$\|u(\cdot,t)\|_{L^p(\Omega)} + \|\nabla v(\cdot,t)\|_{L^q(\Omega)} \leq C \tag{10}$$

for all $t \in (0, T_{\max})$ with all $p \in [1,\infty) > p_\star$ and $q \in \left(\frac{3}{2},\infty\right) > q_\star$.

Proof. Multiplying both sides the first equation of (1) by $p(u+1)^{p-1}$ and integrating, then using Young's inequality, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (1+u)^p &\leq -C_d p(p-1)\int_\Omega (1+u)^{p-\alpha-2}|\nabla u|^2 + C_s p(p-1)\int_\Omega (1+u)^{p+\beta-2}|\nabla u||\nabla v| \\ &\leq -\frac{C_d p(p-1)}{2}\int_\Omega (1+u)^{p-\alpha-2}|\nabla u|^2 + \frac{C_s^2 p(p-1)}{2C_d}\int_\Omega (1+u)^{p+\alpha+2\beta-2}|\nabla v|^2.\end{aligned} \tag{11}$$

The first term on the right-hand side of the inequality (11) can be expressed as

$$\frac{C_d p(p-1)}{2}\int_\Omega (1+u)^{p-\alpha-2}|\nabla u|^2 = \frac{2C_d p(p-1)}{(p-\alpha)^2}\int_\Omega \left|\nabla (1+u)^{\frac{p-\alpha}{2}}\right|^2,$$

this together with (11) which implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (1+u)^p + \frac{2C_d p(p-1)}{(p-\alpha)^2}\int_\Omega \left|\nabla (1+u)^{\frac{p-\alpha}{2}}\right|^2 \leq \frac{C_s^2 p(p-1)}{2C_d}\int_\Omega (1+u)^{p+\alpha+2\beta-2}|\nabla v|^2 \tag{12}$$

for all $t \in (0, T_{\max})$. For a prior estimate of v, one can see [9,12,13], for completeness, a brief proof is given here. Applying the second equation of (1), the point-wise identity $\Delta|\nabla v|^2 = 2\left|D^2 v\right|^2 + 2\nabla v \cdot \nabla \Delta v$ and the fact $|\Delta v|^2 \leq n\left|D^2 v\right|^2$, we derive

$$\begin{aligned}\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla v|^{2q} &+ \frac{2}{n}\int_\Omega |\nabla v|^{2(q-1)}|\Delta v|^2 + 2\int_\Omega |\nabla v|^{2q} \\ &\leq \int_\Omega |\nabla v|^{2(q-1)}\Delta|\nabla v|^2 + 2\int_\Omega |\nabla v|^{2(q-1)}\nabla v \cdot \nabla g(u) \\ &= -\left(q-1\right)\int_\Omega |\nabla v|^{2(q-2)}|\nabla|\nabla v|^2|^2 + \int_{\partial\Omega} |\nabla v|^{2(q-1)}\frac{\partial|\nabla v|^2}{\partial\nu}\mathrm{d}S \\ &\quad - 2(q-1)\int_\Omega |\nabla v|^{2(q-2)}\nabla|\nabla v|^2 \cdot \nabla v \cdot g(u) - 2\int_\Omega |\nabla v|^{2(q-1)}\Delta v \cdot g(u)\end{aligned} \tag{13}$$

for all $t \in (0, T_{\max})$. Using the property of boundary integral without the convexity of domain [6, Lemma 4.2] and the trace inequality [2, Proposition 4.22, 4.24] we have

$$\int_{\partial\Omega} |\nabla v|^{2(q-1)}\frac{\partial|\nabla v|^2}{\partial\nu}\mathrm{d}S \leq 2\kappa_\Omega\int_{\partial\Omega}|\nabla v|^{2q}dS \leq \frac{q-1}{q^2}\int_\Omega |\nabla|\nabla v|^q|^2 + C_1\int_\Omega |\nabla v|^{2q} \tag{14}$$

with some $\kappa_\Omega, C_1 > 0$. Combining (13) with (14) and using Young's inequality yield

$$\begin{aligned}\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla v|^{2q} &+ \frac{2}{n}\int_\Omega |\nabla v|^{2(q-1)}|\Delta v|^2 + 2\int_\Omega |\nabla v|^{2q} \\ &\leq -\frac{q-1}{2}\int_\Omega |\nabla v|^{2(q-2)}|\nabla|\nabla v|^2|^2 + \frac{q-1}{q^2}\int_\Omega |\nabla|\nabla v|^q|^2 + C_1\int_\Omega |\nabla v|^{2q} \\ &\quad + \frac{2}{n}\int_\Omega |\nabla v|^{2(q-1)}|\Delta v|^2 + \left(2(q-1) + \frac{n}{2}\right)\int_\Omega |\nabla v|^{2(q-1)}g^2(u) \\ &= -\frac{q-1}{q^2}\int_\Omega |\nabla|\nabla v|^q|^2 + C_1\int_\Omega |\nabla v|^{2q} + \frac{2}{n}\int_\Omega |\nabla v|^{2(q-1)}|\Delta v|^2 \\ &\quad + \left(2(q-1) + \frac{n}{2}\right)\int_\Omega |\nabla v|^{2(q-1)}g^2(u),\end{aligned}$$

thus, this together with (3) which implies

$$\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla v|^{2q} + \frac{q-1}{q^2}\int_\Omega |\nabla|\nabla v|^q|^2 \le C_g^2\left(2(q-1)+\frac{n}{2}\right)\int_\Omega u^{2\gamma}|\nabla v|^{2(q-1)} + (C_1-2)\int_\Omega |\nabla v|^{2q} \quad (15)$$

for all $t\in(0,T_{\max})$. Combining (12) and (15) we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left((1+u)^p+\frac{1}{q}|\nabla v|^{2q}\right)+\frac{q-1}{q^2}\int_\Omega|\nabla|\nabla v|^q|^2+\frac{2C_d p(p-1)}{(p-\alpha)^2}\int_\Omega|\nabla(1+u)^{\frac{p-\alpha}{2}}|^2\\ \le C_2\int_\Omega(1+u)^{p+\alpha+2\beta-2}|\nabla v|^2+C_2\int_\Omega(1+u)^{2\gamma}|\nabla v|^{2(q-1)}+C_2\int_\Omega|\nabla v|^{2q}\end{aligned} \quad (16)$$

for all $t\in(0,T_{\max})$ with $C_2:=\max\left\{\frac{C_s^2 p(p-1)}{2C_d}, C_1-2, C_g^2\left(2(q-1)+\frac{n}{2}\right)\right\}>0$. According to Lemma 5, $a,b>1$, let $a^{'}:=\frac{a}{a-1}>1$ and $b^{'}:=\frac{b}{b-1}>1$, applying Hölder's inequality to the first two terms on the right-hand side of the inequality (16), we infer

$$\int_\Omega(1+u)^{p+\alpha+2\beta-2}|\nabla v|^2\le\left(\int_\Omega(1+u)^{(p+\alpha+2\beta-2)a}\right)^{\frac{1}{a}}\left(\int_\Omega|\nabla v|^{2a^{'}}\right)^{\frac{1}{a^{'}}} \quad (17)$$

and

$$\int_\Omega(1+u)^{2\gamma}|\nabla v|^{2(q-1)}\le\left(\int_\Omega(1+u)^{2\gamma b}\right)^{\frac{1}{b}}\left(\int_\Omega|\nabla v|^{2(q-1)b^{'}}\right)^{\frac{1}{b^{'}}}. \quad (18)$$

In view of (4) and Gagliardo–Nirenberg inequality [7, 11] we have

$$\begin{aligned}\left(\int_\Omega(1+u)^{(p+\alpha+2\beta-2)a}\right)^{\frac{1}{a}}&=\left\|(1+u)^{\frac{p-\alpha}{2}}\right\|^{\frac{2(p+\alpha+2\beta-2)}{p-\alpha}}_{L^{\frac{2a(p+\alpha+2\beta-2)}{p-\alpha}}(\Omega)}\\ &\le C_3\left\|\nabla(1+u)^{\frac{p-\alpha}{2}}\right\|^{\frac{2(p+\alpha+2\beta-2)\theta}{p-\alpha}}_{L^2(\Omega)}\left\|(1+u)^{\frac{p-\alpha}{2}}\right\|^{\frac{2(p+\alpha+2\beta-2)(1-\theta)}{p-\alpha}}_{L^{\frac{2}{p-\alpha}}(\Omega)}\\ &\quad+C_3\left\|(1+u)^{\frac{p-\alpha}{2}}\right\|^{\frac{2(p+\alpha+2\beta-2)}{p-\alpha}}_{L^{\frac{2}{p-\alpha}}(\Omega)}\\ &\le C_4\left(\int_\Omega\left|\nabla(1+u)^{\frac{p-\alpha}{2}}\right|^2\right)^{\frac{p+\alpha+2\beta-2}{p-\alpha}\theta}+C_4\end{aligned} \quad (19)$$

with $C_3,C_4>0$, and $\theta=\frac{\frac{p-\alpha}{2}-\frac{p-\alpha}{2a(p+\alpha+2\beta-2)}}{\frac{1}{n}-\frac{1}{2}+\frac{p-\alpha}{2}}\in(0,1)$ is guaranteed by (7). Similarly, according to (5) and Gagliardo–Nirenberg inequality again we have

$$\begin{aligned}\left(\int_\Omega|\nabla v|^{2a^{'}}\right)^{\frac{1}{a^{'}}}=\left\||\nabla v|^q\right\|^{\frac{2}{q}}_{L^{\frac{2a^{'}}{q}}(\Omega)}&\le C_5\left\|\nabla|\nabla v|^q\right\|^{\frac{2\delta}{q}}_{L^2(\Omega)}\left\||\nabla v|^q\right\|^{\frac{2(1-\delta)}{q}}_{L^{\frac{s}{q}}(\Omega)}+C_5\left\||\nabla v|^q\right\|^{\frac{2}{q}}_{L^{\frac{s}{q}}(\Omega)}\\ &\le C_6\left(\int_\Omega|\nabla|\nabla v|^q|^2\right)^{\frac{\delta}{q}}+C_6\end{aligned} \quad (20)$$

with $C_5,C_6>0$, and $\delta=\frac{\frac{q}{s}+\frac{q}{2a}-\frac{q}{2}}{\frac{1}{n}-\frac{1}{2}+\frac{q}{s}}\in(0,1)$ is guaranteed by (8). Combining (19) and (20) with (17), there exists a positive constant $C_7>0$ such that

$$C_2\int_\Omega(1+u)^{p+\alpha+2\beta-2}|\nabla v|^2\le C_7\left(\left(\int_\Omega\left|\nabla(1+u)^{\frac{p-\alpha}{2}}\right|^2\right)^{\frac{p+\alpha+2\beta-2}{p-\alpha}\theta}+1\right)\left(\left(\int_\Omega|\nabla|\nabla v|^q|^2\right)^{\frac{\delta}{q}}+1\right). \quad (21)$$

Similarly, in view of Lemma 4, (9) and Gagliardo–Nirenberg inequality again we derive

$$\left(\int_\Omega(1+u)^{2\gamma b}\right)^{\frac{1}{b}}=\left\|(1+u)^{\frac{p-\alpha}{2}}\right\|^{\frac{4\gamma}{p-\alpha}}_{L^{\frac{4\gamma b}{p-\alpha}}(\Omega)}\le C_8\left(\int_\Omega\left|\nabla(1+u)^{\frac{p-\alpha}{2}}\right|^2\right)^{\frac{2\gamma\bar\theta}{p-\alpha}}+C_8 \quad (22)$$

and

$$\left(\int_\Omega |\nabla v|^{2(q-1)b'}\right)^{\frac{1}{b'}} = \left\| |\nabla v|^q \right\|_{L^{\frac{2(q-1)b'}{q}}(\Omega)}^{\frac{2(q-1)}{q}} \le C_9 \left(\int_\Omega |\nabla |\nabla v|^q|^2\right)^{\frac{(q-1)\bar{\delta}}{q}} + C_9 \tag{23}$$

with some $C_8, C_9 > 0$, $\bar{\theta} = \frac{\frac{p-\alpha}{2} - \frac{p-\alpha}{4\gamma b}}{\frac{1}{n} - \frac{1}{2} + \frac{p-\alpha}{2}} \in (0,1)$ and $\bar{\delta} = \frac{\frac{q}{s} + \frac{q}{2(q-1)b} - \frac{q}{2(q-1)}}{\frac{1}{n} - \frac{1}{2} + \frac{q}{s}} \in (0,1)$. Then combining (22) and (23) with (18), there exists a positive constant $C_{10} > 0$ such that

$$C_2 \int_\Omega (1+u)^{2\gamma} |\nabla v|^{2(q-1)} \le C_{10} \left(\left(\int_\Omega \left| \nabla (1+u)^{\frac{p-\alpha}{2}} \right|^2 \right)^{\frac{2\gamma\bar{\theta}}{p-\alpha}} + 1 \right) \left(\left(\int_\Omega |\nabla |\nabla v|^q|^2 \right)^{\frac{(q-1)\bar{\delta}}{q}} + 1 \right). \tag{24}$$

Therefore, using (16) in conjunction with (21) and (24), we infer

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left((1+u)^p + \frac{1}{q} |\nabla v|^{2q} \right) + \frac{2C_d p(p-1)}{(p-\alpha)^2} \int_\Omega \left| \nabla (1+u)^{\frac{p-\alpha}{2}} \right|^2 + \frac{q-1}{q^2} \int_\Omega |\nabla |\nabla v|^q|^2 \\ &\le C_{11} \left(\left(\int_\Omega \left| \nabla (1+u)^{\frac{p-\alpha}{2}} \right|^2 \right)^{\frac{p+\alpha+2\beta-2}{p-\alpha}\theta} + 1 \right) \left(\left(\int_\Omega |\nabla |\nabla v|^q|^2 \right)^{\frac{\delta}{q}} + 1 \right) \\ &\quad + C_{11} \left(\left(\int_\Omega \left| \nabla (1+u)^{\frac{p-\alpha}{2}} \right|^2 \right)^{\frac{2\gamma\bar{\theta}}{p-\alpha}} + 1 \right) \left(\left(\int_\Omega |\nabla |\nabla v|^q|^2 \right)^{\frac{(q-1)\bar{\delta}}{q}} + 1 \right) + C_2 \int_\Omega |\nabla v|^{2q} \end{aligned} \tag{25}$$

for all $t \in (0, T_{\max})$ with some $C_{11} > 0$. Thus, according to [12, Lemma 3.1] and Young's inequality, we can obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left((1+u)^p + \frac{1}{q} |\nabla v|^{2q} \right) + \frac{C_d p(p-1)}{(p-\alpha)^2} \int_\Omega \left| \nabla (1+u)^{\frac{p-\alpha}{2}} \right|^2 + \frac{q-1}{2q^2} \int_\Omega |\nabla |\nabla v|^q|^2 \\ \le C_2 \int_\Omega |\nabla v|^{2q} + C_{12} \end{aligned} \tag{26}$$

with $C_{12} > 0$ if the assumptions

$$\frac{p+\alpha+2\beta-2}{p-\alpha}\theta + \frac{\delta}{q} < 1 \quad \text{and} \quad \frac{2\gamma\bar{\theta}}{p-\alpha} + \frac{(q-1)\bar{\delta}}{q} < 1 \tag{27}$$

are satisfied. Therefore, in order for the assumptions in (27) to be satisfied, let

$$h(q) := \frac{p+\alpha+2\beta-2}{p-\alpha}\theta + \frac{\delta}{q} = \frac{\frac{p+\alpha+2\beta-2}{2} - \frac{1}{2a}}{\frac{1}{n} - \frac{1}{2} + \frac{p-\alpha}{2}} + \frac{\frac{1}{s} + \frac{1}{2a} - \frac{1}{2}}{\frac{1}{n} - \frac{1}{2} + \frac{q}{s}}$$

and

$$\bar{h}(q) := \frac{2\gamma\bar{\theta}}{p-\alpha} + \frac{(q-1)\bar{\delta}}{q} = \frac{\gamma - \frac{1}{2b}}{\frac{1}{n} - \frac{1}{2} + \frac{p-\alpha}{2}} + \frac{\frac{q-1}{s} + \frac{1}{2b} - \frac{1}{2}}{\frac{1}{n} - \frac{1}{2} + \frac{q}{s}},$$

according to the condition (6) of Lemma 5, we have

$$h(q(p)) < 1 \quad \text{and} \quad \bar{h}(q(p)) < 1$$

with $q(p) := \frac{p-\alpha}{2} s$. Since $q(p) \to +\infty$ as $p \to \infty$, for all $p \ge p_\star$, there exists $q \ge q_\star$ such that

$$h(q) < 1 \quad \text{and} \quad \bar{h}(q) < 1,$$

thus, the assumptions in (27) are satisfied. In order for the inequality (26) to satisfy the form of Gronwall's inequality, using Gagliardo–Nirenberg inequality and Lemma 4 imply

$$\int_\Omega (1+u)^p = \left\| (1+u)^{\frac{p-\alpha}{2}} \right\|_{L^{\frac{2p}{p-\alpha}}(\Omega)}^{\frac{2p}{p-\alpha}} \le C_{13} \left(\int_\Omega \left| \nabla (1+u)^{\frac{p-\alpha}{2}} \right|^2 \right)^{\frac{p\sigma}{p-\alpha}} + C_{13} \tag{28}$$

with some $C_{13} > 0$, and $\sigma = \frac{\frac{p-\alpha}{2} - \frac{p-\alpha}{2p}}{\frac{1}{n} - \frac{1}{2} + \frac{p-\alpha}{2}} \in (0,1)$ is satisfied because of the condition $p > 1 + \frac{n\alpha}{2}$ in Lemma 5. In the same way, we obtain

$$\begin{aligned}\left(\frac{1}{q} + C_2\right)\int_\Omega |\nabla v|^{2q} = \left(\frac{1}{q} + C_2\right)\left\||\nabla v|^q\right\|^2_{L^2(\Omega)} &\le C_{14}\left(\int_\Omega |\nabla|\nabla v|^q|^2\right)^{\bar{\sigma}} + C_{14} \\ &\le \frac{q-1}{2q^2}\int_\Omega |\nabla|\nabla v|^q|^2 + C_{15}\end{aligned} \tag{29}$$

with some $C_{14}, C_{15} > 0$, and $\bar{\sigma} = \frac{\frac{q}{s} - \frac{1}{2}}{\frac{1}{n} - \frac{1}{2} + \frac{q}{s}} \in (0,1)$ is satisfied because of the condition $q > 1 + \frac{s}{2}$ in Lemma 5. Therefore, combining (28) and (29) with (26), which implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left((1+u)^p + \frac{1}{q}|\nabla v|^{2q}\right) + C_{16}\left(\int_\Omega (1+u)^p\right)^{\frac{p-\alpha}{p\sigma}} + \frac{1}{q}\int_\Omega |\nabla v|^{2q} \le C_{17} \tag{30}$$

for all $t \in (0, T_{\max})$ with some $C_{16}, C_{17} > 0$, therefore, according to the ODI comparison principle with (30), which implies (10). □

Now, we can easily prove Theorem 1.

Proof of Theorem 1. In view of [12, Lemmas 3.3 and A.1], we obtain the desired results. □

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