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Algebraic intersection for translation surfaces in the stratum $\mathscr{H}(2)$

Intersection algébrique dans la strate $\mathscr{H}(2)$ des surfaces de translation

Smail Cheboui [a, b], Arezki Kessi [a] and Daniel Massart [*, b]

[a] USTHB, Faculté de Mathématiques, Laboratoire de Systèmes Dynamiques, 16111 El-Alia BabEzzouar, Alger, Algérie

[b] IMAG, Univ Montpellier, CNRS, Montpellier, France

E-mails: scheboui@usthb.dz, akessi@usthb.dz, daniel.massart@umontpellier.fr

Abstract. We study a volume related quantity KVol on the stratum $\mathscr{H}(2)$ of translation surfaces of genus 2, with one conical point. We provide an explicit sequence $L(n,n)$ of surfaces such that $\mathrm{KVol}(L(n,n)) \to 2$ when n goes to infinity, 2 being the conjectured infimum for KVol over $\mathscr{H}(2)$.

Résumé. Nous étudions une quantité KVol liée au volume sur la strate $\mathscr{H}(2)$ des surfaces de translation de genre 2, avec une singularité conique. Nous donnons une suite explicite de surfaces $L(n,n)$ telles que $\mathrm{KVol}(L(n,n)) \to 2$ quand n tend vers l'infini, 2 étant l'infimum conjectural de KVol sur $\mathscr{H}(2)$.

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1. Introduction

Let X be a closed surface, that is, a compact, connected manifold of dimension 2, without boundary. Let us assume that X is oriented. Then the algebraic intersection of closed curves in X endows the first homology $H_1(X,\mathbb{R})$ with an antisymmetric, non degenerate, bilinear form, which we denote $\mathrm{Int}(\cdot,\cdot)$.

Now let us assume X is endowed with a Riemannian metric g. We denote $\mathrm{Vol}(X,g)$ the Riemannian volume of X with respect to the metric g, and for any piecewise smooth closed curve

[*] Corresponding author.

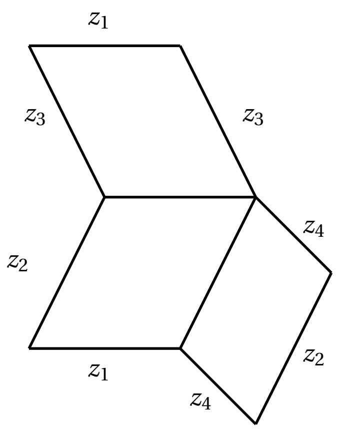

Figure 1. Unfolding an element of $\mathscr{H}(2)$

α in X, we denote $l_g(\alpha)$ the length of α with respect to g. When there is no ambiguity we omit the reference to g.

We are interested in the quantity

$$\mathrm{KVol}(X,g) = \mathrm{Vol}(X,g) \sup_{\alpha,\beta} \frac{\mathrm{Int}(\alpha,\beta)}{l_g(\alpha) l_g(\beta)} \tag{1}$$

where the supremum ranges over all piecewise smooth closed curves α and β in X. The $\mathrm{Vol}(X,g)$ factor is there to make KVol invariant to re-scaling of the metric g. See [5] as to why KVol is finite. It is easy to make KVol go to infinity, you just need to pinch a non-separating closed curve α to make its length go to zero. The interesting surfaces are those (X,g) for which KVol is small.

When X is the torus, we have $\mathrm{KVol}(X,g) \geq 1$, with equality if and only if the metric g is flat (see [5]). Furthermore, when g is flat, the supremum in (1) is not attained, but for a negligible subset of the set of all flat metrics. In [5], KVol is studied as a function of g, on the moduli space of hyperbolic (that is, the curvature of g is -1) surfaces of fixed genus. It is proved that KVol goes to infinity when g degenerates by pinching a non-separating closed curve, while KVol remains bounded when g degenerates by pinching a separating closed curve.

This leaves open the question whether KVol has a minimum over the moduli space of hyperbolic surfaces of genus n, for $n \geq 2$. It is conjectured in [5] that for almost every (X,g) in the moduli space of hyperbolic surfaces of genus n, the supremum in (1) is attained (that is, it is actually a maximum).

In this paper we consider a different class of surfaces: translation surfaces of genus 2, with one conical point. The set (or stratum) of such surfaces is denoted $\mathscr{H}(2)$ (see [3]). By [6], any surface X in the stratum $\mathscr{H}(2)$ may be unfolded as shown in Figure 1, with complex parameters z_1, z_2, z_3, z_4. The surface is obtained from the plane template by identifying parallel sides of equal length.

It is proved in [4] (see also [2]) that the systolic volume has a minimum in $\mathscr{H}(2)$, and it is achieved by a translation surface tiled by six equilateral triangles. Since the systolic volume is a close relative of KVol, it is interesting to keep the results of [4] and [2] in mind.

We have reasons to believe that KVol behaves differently in $\mathscr{H}(2)$, both from the systolic volume in $\mathscr{H}(2)$, and from KVol itself in the moduli space of hyperbolic surfaces of genus 2; that is, KVol does not have a minimum over $\mathscr{H}(2)$.

We also believe that the infimum of KVol over $\mathscr{H}(2)$ is 2. This paper is a first step towards the proof: we find an explicit sequence $L(n,n)$ of surfaces in $\mathscr{H}(2)$, whose KVol tends to 2 (see

Proposition 5). These surfaces are obtained from very thin, symmetrical, L-shaped templates (see Figure 2).

In the companion paper [1] we study KVol as a function on the Teichmüller disk (the $SL_2(\mathbb{R})$-orbit) of surfaces in $\mathscr{H}(2)$ which are tiled by three identical parallelograms (for instance $L(2,2)$), and prove that KVol does have a minimum there, but is not bounded from above. Therefore KVol is not bounded from above as a function on $\mathscr{H}(2)$. In [1] we also compute KVol for the translation surface tiled by six equilateral triangles, and find it equals 3, so it does not minimize KVol, neither in $\mathscr{H}(2)$, nor even in its own Teichmüller disk.

2. $L(n,n)$

2.1. *Preliminaries*

Following [7], for any $n \in \mathbb{N}$, $n \geq 2$, we call $L(n+1,n+1)$ the $(2n+1)$-square translation surface of genus two, with one conical point, depicted in Figure 2, where the upper and rightmost rectangles are made up with n unit squares. We call A (resp. B) the region in $L(n+1,n+1)$ obtained, after identifications, from the uppermost (resp. rightmost) rectangle, and C the region in $L(n+1,n+1)$ obtained, after identifications, from the bottom left square. Both A and B are annuli with a pair of points identified on the boundary, while C is a square with all four corners identified. We call e_1, e_2, (resp. f_1, f_2) the closed curves in $L(n+1,n+1)$ obtained by gluing the endpoints of the horizontal (resp. vertical) sides of A and B. The closed curve which sits on the opposite side of C from e_1 (resp. f_1) is called e_1' (resp. f_1'), it is homotopic to e_1 (resp. f_1) in $L(n+1,n+1)$. The closed curves in $L(n+1,n+1)$ which correspond to the diagonals of the square C are called g and h.

Figure 3 shows a local picture of $L(n+1,n+1)$ around the singular (conical) point S, with angles rescaled so the 6π fit into 2π.

Since e_1, e_2, f_1, f_2 do not meet anywhere but at S, the local picture yields the algebraic intersections between any two of e_1, e_2, f_1, f_2, summed up in the following matrix:

$$\begin{pmatrix} \text{Int} & e_2 & f_1 & e_1 & f_2 \\ e_2 & 0 & 1 & 0 & -1 \\ f_1 & -1 & 0 & 0 & 0 \\ e_1 & 0 & 0 & 0 & 1 \\ f_2 & 1 & 0 & -1 & 0 \end{pmatrix} \tag{2}$$

We call T_A (resp. T_B) the flat torus obtained by gluing the opposite sides of the rectangle made with the $n+1$ leftmost squares (resp. with the $n+1$ bottom squares), so the homology of T_A (resp. T_B) is generated by e_1 and the concatenation of f_1 and f_2 (resp. f_1 and the concatenation of e_1 and e_2).

Lemma 1. *The only closed geodesics in $L(n+1,n+1)$ which do not intersect e_1 nor f_1 are, up to homotopy, e_1, f_1, g, and h.*

Proof. Let γ be such a closed geodesic. It cannot enter, nor leave, A, B, nor C. If it is contained in A, and does not intersect e_1, then it must be homotopic to e_1, which is the soul of the annulus from which A is obtained by identifying two points on the boundary. Likewise, if it is contained in B, and does not intersect f_1, then it must be homotopic to f_1. Finally, if γ is not contained in A nor in B, it must be contained in C. The only closed geodesics contained in C are the sides and diagonals of the square from which C is obtained, which are e_1, e_1', f_1, f_1', g, and h. □

Lemma 2. *For any closed geodesic γ in $L(n+1,n+1)$, we have $l(\gamma) \geq n|\text{Int}(\gamma, e_1)|$.*

Proof. For each intersection with e_1, γ must go through A, from boundary to boundary. □

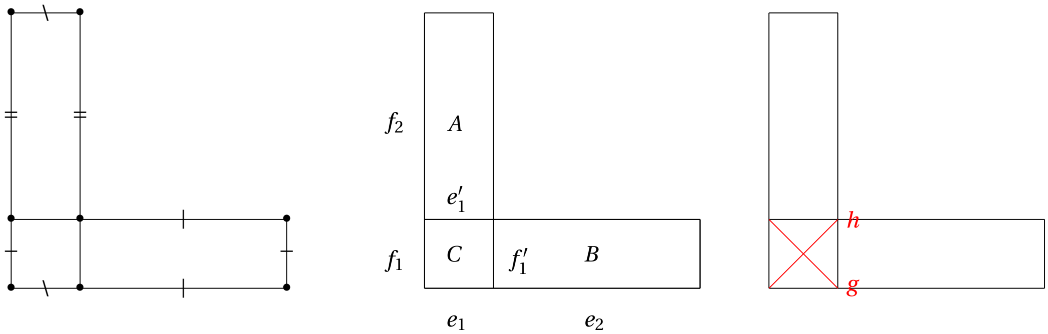

Figure 2. $L(n+1, n+1)$

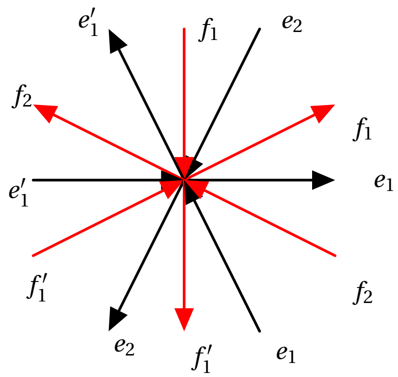

Figure 3. Local picture around the conical point

Obviously a similar lemma holds with f_1 instead of e_1. For g and h the proof is a bit different:

Lemma 3. *For any closed geodesic γ in $L(n+1, n+1)$, we have $l(\gamma) \geq n|\mathrm{Int}(\gamma, g)|$.*

Proof. First, observe that between two consecutive intersections with g, γ must go through either A or B, unless γ is g itself, or h: indeed, the only geodesic segments contained in C with endpoints on g are segments of g, or h. Obviously $\mathrm{Int}(g, g) = 0$, and from the intersection matrix (2), knowing that $[g] = [e_1] - [f_1]$, $[h] = [e_1] + [f_1]$, we see that $\mathrm{Int}(g, h) = 0$.

Thus, either $\mathrm{Int}(\gamma, g) = 0$, or each intersection must be paid for with a trek through A or B, of length at least n. □

Obviously a similar lemma holds with h instead of g. Note that Lemmata 1, 2, 3 imply that the only geodesics in $L(n+1, n+1)$ which are shorter than n are e_1, f_1, g, h, and closed geodesics homotopic to e_1 or f_1.

Lemma 4. *Let I, J be positive integers, take a_{ij}, $i = 1, \ldots, I$, $j = 1, \ldots, J$ in $\mathbb{R}_+$, and $b_1, \ldots, b_I, c_1, \ldots, c_J$ in $\mathbb{R}_+^*$. Then we have*

$$\frac{\sum_{i,j} a_{ij}}{\left(\sum_{i=1}^{I} b_i\right)\left(\sum_{j=1}^{J} c_j\right)} \leq \max_{i,j} \frac{a_{ij}}{b_i c_j}.$$

Proof. Re-ordering, if needed, the a_{ij}, b_i, c_j, we may assume

$$\frac{a_{ij}}{b_i c_j} \le \frac{a_{11}}{b_1 c_1} \quad \forall\ i = 1, \dots, I, j = 1, \dots, J.$$

Then $a_{ij} b_1 c_1 \le a_{11} b_i c_j$ $\forall\ i = 1, \dots, I, j = 1, \dots, J$, so

$$b_1 c_1 \sum_{i,j} a_{ij} \le a_{11} \sum_{i,j} b_i c_j = a_{11} \left(\sum_{i=1}^{I} b_i \right) \left(\sum_{j=1}^{J} c_j \right). \qquad \square$$

2.2. *Estimation of* KVol$(L(n,n))$

Proposition 5.

$$\lim_{n \to +\infty} \mathrm{KVol}(L(n+1, n+1)) = 2.$$

Proof. First observe that $\mathrm{Vol}(L(n+1, n+1)) = 2n+1$, $l(e_1) = 1$, $l(f_2) = n$, $\mathrm{Int}(e_1, f_2) = 1$, so

$$\mathrm{KVol}(L(n+1, n+1)) \ge 2 + \frac{1}{n}.$$

To bound KVol$(L(n+1, n+1))$ from above, we take two closed geodesics α and β; by Lemmata 2 and 3, if either α or β is homotopic to e_1, f_1, g, or h, then

$$\frac{\mathrm{Int}(\alpha, \beta)}{l(\alpha) l(\beta)} \le \frac{1}{n},$$

so from now on we assume that neither α or β is homotopic to e_1, f_1, g, h. We cut α and β into pieces using the following procedure: we consider the sequence of intersections of α with e_1, e_1', f_1, f_1', in cyclical order, and we cut α at each intersection with e_1 or e_1' which is followed by an intersection with f_1 or f_1', and at each intersection with f_1 or f_1' which is followed by an intersection with e_1 or e_1'. We proceed likewise with β. We call $\alpha_i, i = 1, \dots, I$, and $\beta_j, j = 1, \dots, J$, the pieces of α and β, respectively.

Note that

$$l(\alpha) = \sum_{i=1}^{I} l(\alpha_i),\ l(\beta) = \sum_{j=1}^{J} l(\beta_j), \quad \text{and} \quad |\mathrm{Int}(\alpha, \beta)| \le \sum_{i,j} |\mathrm{Int}(\alpha_i, \beta_j)|,$$

so Lemma 4 says that

$$\frac{|\mathrm{Int}(\alpha, \beta)|}{l(\alpha) l(\beta)} \le \max_{i,j} \frac{|\mathrm{Int}(\alpha_i, \beta_j)|}{l(\alpha_i) l(\beta_j)}.$$

We view each piece α_i (resp. β_j) as a geodesic arc in the torus T_A (resp. T_B), with endpoints on the image in T_A (or T_B) of f_1 or f_1' (resp. e_1 or e_1'), which is a geodesic arc of length 1, so we can close each α_i (resp. β_j) with a piece of f_1 or f_1' (resp. e_1 or e_1'), of length ≤ 1. We choose a closed geodesic $\widehat{\alpha}_i$ (resp. $\widehat{\beta}_j$) in T_A (resp. T_B) which is homotopic to the closed curve thus obtained. We have $l(\widehat{\alpha}_i) \le l(\alpha_i) + 1$, $l(\widehat{\beta}_j) \le l(\beta_j) + 1$, so

$$\frac{1}{l(\widehat{\alpha}_i) l(\widehat{\beta}_j)} \ge \frac{1}{(l(\alpha_i)+1)(l(\beta_j)+1)}.$$

Now recall that $l(\alpha_i), l(\beta_j) \ge n$, so $l(\alpha_i) + 1 \le (1 + \frac{1}{n}) l(\alpha_i)$, whence

$$\frac{1}{l(\widehat{\alpha}_i) l(\widehat{\beta}_j)} \ge \frac{1}{l(\alpha_i) l(\beta_j)} \left(\frac{n}{n+1} \right)^2.$$

Next, observe that $|\mathrm{Int}(\alpha_i, \beta_j)| \le |\mathrm{Int}(\widehat{\alpha_i}, \widehat{\beta_j})| + 1$, because $\widehat{\alpha}_i$ (resp. $\widehat{\beta}_j$) is homologous to a closed curve which contains α_i (resp. β_j) as a subarc, and the extra arcs cause at most one extra intersection, depending on whether or not the endpoints of α_i and β_j are intertwined. So,

$$\frac{|\mathrm{Int}(\alpha_i, \beta_j)|}{l(\alpha_i) l(\beta_j)} \le \frac{|\mathrm{Int}(\widehat{\alpha_i}, \widehat{\beta_j})| + 1}{l(\widehat{\alpha}_i) l(\widehat{\beta}_j)} \left(\frac{n+1}{n} \right)^2 \le \left(\frac{|\mathrm{Int}(\widehat{\alpha_i}, \widehat{\beta_j})|}{l(\widehat{\alpha}_i) l(\widehat{\beta}_j)} + \frac{1}{n^2} \right) \left(\frac{n+1}{n} \right)^2,$$

where the last inequality stands because $l(\widehat{\alpha}_i) \geq n$, $l(\widehat{\beta}_j) \geq n$, since $\widehat{\alpha}_i$ and $\widehat{\beta}_j$ both have to go through a cylinder A or B at least once. Finally, since $\widehat{\alpha}_i$ and $\widehat{\beta}_j$ are closed geodesics on a flat torus of volume $n+1$, we have (see [5])

$$\frac{|\mathrm{Int}(\widehat{\alpha_i}, \widehat{\beta_j})|}{l(\widehat{\alpha}_i)\, l(\widehat{\beta}_j)} \leq \frac{1}{n+1}, \text{ so}$$

$$\frac{|\mathrm{Int}(\alpha_i, \beta_j)|}{l(\alpha_i)\, l(\beta_j)} \leq \left(\frac{1}{n+1} + \frac{1}{n^2}\right)\left(\frac{n+1}{n}\right)^2 = \frac{1}{n} + o\left(\frac{1}{n}\right),$$

which yields the result, recalling that $\mathrm{Vol}(L(n+1, n+1)) = 2n+1$. □

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