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Quasihyperbolic mappings in length metric spaces

Qingshan Zhou [a], Yaxiang Li [b] and Yuehui He [*, c]

[a] School of Mathematics and Big Data, Foshan university, Foshan, Guangdong 528000, People's Republic of China

[b] Department of Mathematics, Hunan First Normal University, Changsha, Hunan 410205, People's Republic of China

[c] Department of Mathematics, Shantou University, Shantou, Guangdong 515063, People's Republic of China

E-mails: q476308142@qq.com (Q. Zhou), yaxiangli@163.com (Y. Li), 18yhhe1@stu.edu.cn (Y. He)

Abstract. In this paper, we discuss the local properties of quasihyperbolic mappings in metric spaces, which are related to an open problem raised by Huang et al in 2016. Our result is a partial solution to this problem, which is also a generalization of the corresponding result obtained by Huang et al in 2016.

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1. Introduction and main result

Roughly speaking, (locally) M-quasihyperbolic (briefly, M-QH) mappings are homeomorphisms between two domains in metric spaces, which are (locally) M-bilipschitz in the quasihyperbolic metric. Their precise definitions will be given in Section 2. The quasihyperbolicity of mappings implies their quasiconformality (cf. [10]). Also, it is known that this class of mappings is useful for the study of the theory of quasiconformal (briefly, QC) mappings in $\mathbb{R}^n$. For example, Beurling and Ahlfors constructed a QC extension of a quasisymmetric mapping on the real axis to the upper half plane, which is actually QH [1]. Tukia and Väisälä proved that two domains of $\mathbb{R}^n$ with $n \neq 4$ are quasihyperbolically equivalent if and only if they are quasiconformally equivalent [8].

[*] Corresponding author.

In his theory of (dimension) free quasiconformal (briefly, FQC) mappings in real Banach spaces, Väisälä also investigated QH mappings, and proved that every M-QH mapping is fully $4M^2$-QH [9, Theorem 4.7]. In [10], Väisälä asked that whether the converse of [9, Theorem 4.7] is true or not (see [10, 13.2]). Recently, Huang et al considered this problem in the setting of metric spaces, and obtained the following.

Theorem A ([4, Theorem 1.10]). *Let X be a c_1-quasi-convex and dense metric space and let Y be a c_2-quasi-convex, dense and proper metric space. Let $G \subsetneq X$ and $G' \subsetneq Y$ be two domains. Suppose that a homeomorphism $f : G \to G'$ is both semi-locally M-QH and semi-locally η-QS, where $M > 1$ is a constant and $\eta : [0,\infty) \to [0,\infty)$, $\eta(0) = 0$, is a homeomorphism. Then f is an M_1-QH mapping on G with $M_1 = M_1(M,\eta,c_1,c_2)$.*

In [4], the authors also asked the following Open Problem 1.

Open Problem 1 ([4, 1.12]). *Can one strengthen the above Theorem A? That is, can one deduce that a homeomorphism $f : G \to G'$ which is locally M-QH is a global M_1-QH mapping in some suitable metric spaces?*

The purpose of this paper is to discuss Väisälä's open problem and Open problem 1 further. The following is our main result.

Theorem 2. *Suppose that (X_1,d_1) is a length metric space and (X_2,d_2) is a c-quasi-convex and complete metric space, and that $G \subsetneq X_1$ and $G' \subsetneq X_2$ are two domains. Suppose that a homeomorphism $f : G \to G'$ is both semi-locally M-QH and semi-locally relatively η-QS, where $M > 1$ is a constant and $\eta : [0,\infty) \to [0,\infty)$, $\eta(0) = 0$, is a homeomorphism. Then f is an M_1-QH mapping on G with $M_1 = M_1(M,\eta,c)$.*

Remark 3. We have the following remarks on Theorem 2.

(1) The assumption "the source space being quasiconvex and dense" in Theorem A is replaced by the one "the source space being length" in Theorem 2. In Corollary 11 below, we prove a relation between dense spaces and length spaces. That is, for a complete space, it is dense if and only if it is length.
(2) In Theorem A, the target space is assumed to be quasiconvex, dense and proper. But in Theorem 2, the target space is assumed only to be quasiconvex and complete.
(3) Since every Banach space of infinite dimension is not proper (cf. [5, Theorem 1.22]), we see that, in the setting of Banach spaces, the assumptions in Theorem A force the spaces in Theorem A to be of finite dimension. But in Theorem 2, this weak point is overcome.
(4) By [7, Theorem 2.25], we see that under the condition of Theorem 2, a quasisymmetric map on A has a homeomorphism extension to its closure $\overline{A}$ which is also quasisymmetric. Thus the semi-local quasisymmetry implies semi-locally relative quasisymmetry. Therefore, the requirements in Theorem 2 on the mappings are weaker than that of Theorem A.

The rest of this paper is organized as follows. In Section 2, we introduce the necessary terminology and prove several auxiliary results, and in Section 3, the proof of Theorem 2 is presented. Section 4 is an Appendix A that is devoted to show an auxiliary result in [2], i.e., Lemma 12.

2. Preliminaries and auxiliary results

In the following, we always use (X,d) or (X_i,d_i) ($i \in \{1,2\}$) to denote a connected metric space. The open (resp. closed) metric ball with center $x \in X$ and radius $r > 0$ is denoted by $\mathbb{B}(x,r) = \{z \in$

$X: d(z,x) < r\}$ (resp. $\overline{\mathbb{B}}(x,r) = \{z \in X: d(z,x) \le r\}$), and the metric sphere by $\mathbb{S}(x,r) = \{z \in X: d(z,x) = r\}$. For a curve $\gamma \subset X$, we denote its length by $\ell(\gamma)$.

The space (X,d) is called *c-quasi-convex* if there is a constant $c \ge 1$ such that any pair of points $x, y \in X$ can be joined by a curve γ such that $\ell(\gamma) \le cd(x,y)$. The curve is also called *c-quasi-convex*.

Suppose that (X,d) is non-complete and that $0 < \lambda \le \frac{1}{2}$ and $c \ge 1$ are constants. (X,d) is said to be *locally (λ,c)-quasi-convex* if for any $x \in X$ and all points $y,z \in B(x,\lambda d(x))$, there is a c-quasi-convex curve in X joining y and z. Here and hereafter, $d(x) = d(x,\partial X)$ denotes the distance from x to the boundary ∂X of X.

A domain $G \subset X$ is an open and connected nonempty set. We use $\overline{G}$ to denote the metric closure of G, and let ∂G stand for the boundary of G. If $\partial G \neq \emptyset$, then $d_G(x)$ denotes the distance from x to ∂G.

The *quasihyperbolic length* of a rectifiable curve γ in G is the number:

$$\ell_{k_G}(\gamma) = \int_\gamma \frac{|dz|}{d_G(z)}.$$

For any x, y in G, the *quasihyperbolic distance* $k_G(x,y)$ between x and y is defined by $k_G(x,y) = \inf\{\ell_{k_G}(\gamma)\}$, where the infimum is taken over all rectifiable curves γ joining x to y in G.

Definition 4. *Suppose that $G \subsetneq X_1$ and $G' \subsetneq X_2$ are domains and $M \ge 1$. We say that a homeomorphism $f: G \to G'$ is*

(1) M-QH *if for all $x, y \in G$,*

$$\frac{1}{M} k_G(x,y) \le k_{G'}\big(f(x), f(y)\big) \le M k_G(x,y),$$

i.e., f is M-bilipschitz in the quasihyperbolic metric.

(2) semi-locally M-QH *if the restriction $f|_{B_x}$ is M-QH for each $x \in G$. Here and hereafter, $B_x = \mathbb{B}(x, d_G(x))$.*

(3) η-QS *if there exists a self-homeomorphism η of $[0,\infty)$ such that*

$$d_1(x,a) \le t d_1(x,b) \quad \textit{implies} \quad d_2\big(f(x), f(a)\big) \le \eta(t) d_2\big(f(x), f(b)\big)$$

for each $t > 0$ and for each triplet $\{x, a, b\}$ of points in G.

(4) semi-locally η-QS *if for each $x \in G$, the restriction $f|_{B_x}$ is η-QS.*

(5) relatively η-QS *if f has a continuous extension to the boundary ∂G, where the extended mapping on $\overline{G}$ is still denoted by f, and there is a self-homeomorphism η of $[0,\infty)$ such that*

$$d_1(x,a) \le t d_1(x,b) \quad \textit{implies} \quad d_2\big(f(x), f(a)\big) \le \eta(t) d_2\big(f(x), f(b)\big)$$

for each $t > 0$ and for each triplet $\{x, a, b\}$ of points in $\overline{G}$ with $x \in \partial G$ or $a, b \in \partial G$.

(6) semi-locally relatively η-QS *if for each $x \in G$, the restriction $f|_{B_x}$ is relatively η-QS.*

For x, y in X, the *length metric* $\mathfrak{d}$ is defined by

$$\mathfrak{d}(x,y) = \inf\big\{\ell(\alpha): \alpha \subset X \text{ is a rectifiable curve joining } x \text{ and } y\big\}.$$

If $d = \mathfrak{d}$, then (X,d) is called *a length space.*

Obviously, for any domain $G \subsetneq X$, the inequality

$$\operatorname{dist}(x, X \setminus G) \le d_G(x) \tag{1}$$

holds true for any $x \in G$. It is not difficult to find that, in general, the strict case $\operatorname{dist}(x, X \setminus G) < d_G(x)$ in (1) may occur. But this does not happen in length spaces as indicated in the following Lemma 5.

Lemma 5. *Suppose that (X,d) is a length space and that $G \subsetneq X$ is a domain. Then for all $x \in G$ and $0 < r \le d_G(x)$, the following statements hold:*

(1) $\mathrm{dist}(x, X \setminus G) = d_G(x)$.
(2) $\mathbb{B}(x,r) \subset G$.
(3) $\mathbb{B}(x,r)$ *is open and rectifiably connected.*

Proof. To prove the statement (1), it follows from inequality (1) that we only need to show that

$$\mathrm{dist}\,(x, X \setminus G) \ge d_G(x).$$

To this end, let $a \in X \setminus G$ and $\varepsilon > 0$. Since (X,d) is a length space, there must be a curve γ in X connecting a and x such that $\ell(\gamma) \le (1+\varepsilon)d(a,x)$, and thus, $d_G(x) \le (1+\varepsilon)d(a,x)$. By letting $\varepsilon \to 0$, we get $d_G(x) \le d(a,x)$, and so, the arbitrariness of a ensures that $\mathrm{dist}(x, X \setminus G) \ge d_G(x)$.

Next, we verify (2). Suppose on the contrary that there is a point $y \in B_x$ such that $y \in X \setminus G$. Then by the statement (1) of the Lemma 5, one gets $d_G(x) \le d(x,y) < d_G(x)$, which is the needed contradiction.

The openness of $\mathbb{B}(x,r)$ follows from the statement (2). To prove the rectifiable connectedness of $\mathbb{B}(x,r)$, let $z \in \mathbb{B}(x,r)$. Then there is a curve $\alpha \subset X$ connecting z and x such that $\ell(\alpha) \le d(x,z) + (r - d(x,z))/2 < r$. By the arbitrariness of z, we see that $\mathbb{B}(x,r)$ is rectifiably connected. This completes the proof of Lemma 5. □

Remark 6. Lemma 5 ensures that $\mathbb{B}(x, d_G(x))$ is a domain contained in G when X is a length space. Thus the local conditions for the mappings both in Theorem A and Theorem 2 are well-defined. Note that $\mathbb{B}(x, d_G(x))$ may not lie in G even if X is quasi-convex, for more discussions see [3,4].

Definition 7. *In [4], (X,d) is said to be* dense *if for any two points $x,y \in X$ and two positive constants r_1, r_2 with $d(x,y) < r_1 + r_2$, we have $\mathbb{B}(x,r_1) \cap \mathbb{B}(y,r_2) \neq \emptyset$.*

Lemma 8. *Every length space (X,d) is dense.*

Proof. Suppose on the contrary that there are $x,y \in X$ and r_1, $r_2 > 0$ such that $d(x,y) < r_1 + r_2$, but $\mathbb{B}(x,r_1) \cap \mathbb{B}(y,r_2) = \emptyset$. Let $\varepsilon = (r_1+r_2)/d(x,y) - 1$. Then $\varepsilon > 0$ and there is a curve $\gamma \subset X$ joining x and y such that $\ell(\gamma) \le (1+\varepsilon/2)d(x,y)$. Let $\gamma_1 = \gamma \cap \mathbb{B}(x,r_1)$ and $\gamma_2 = \gamma \cap \mathbb{B}(y,r_2)$. Then $\gamma_1 \cup \gamma_2 \subsetneq \gamma$. Assume $w \in \gamma \setminus (\gamma_1 \cup \gamma_2)$. Then we get

$$(1+\varepsilon/2)d(x,y) \ge \ell(\gamma) \ge d(x,w) + d(y,w) \ge r_1 + r_2 = (1+\varepsilon)d(x,y),$$

which is the desired contradiction. □

Definition 9. *Suppose $\varepsilon \ge 0$ is a constant. We say that (X,d) has the ε-*midpoint property *if for any two points $x,y \in X$, there is a point $z \in X$ such that* $\max\{d(x,z), d(z,y)\} \le d(x,y)/2 + \varepsilon$. *Such a point z is called an ε-*midpoint *of x and y. (cf. [6])*

Lemma 10. *(X,d) is dense if and only if it has the ε-midpoint property for any $\varepsilon > 0$.*

Proof. The necessity part of the Lemma 10 is obvious. To prove this lemma, we only need to show the sufficiency part. Now, assume that (X,d) has the ε-midpoint property for any $\varepsilon > 0$. Let x and y be two points in X, and let r_1, r_2 be two positive numbers with $r_1 + r_2 > d(x,y)$. Without loss of generality, we may assume that

$$d(x,y) = 1 \quad \text{and} \quad 0 < r_1, r_2 < 1.$$

Assume further that $m > 0$ is an integer such that $r_1 + r_2 \ge 1 + 2^{-m}$. Let $n = m+1$. Then there is an integer $p \in (0, 2^n)$ such that

$$\frac{p}{2^n} < r_1 \le \frac{p+1}{2^n},$$

and so, we have $r_2 \geq 1+2^{-m}-r_1 \geq 1-p\cdot 2^{-n}+2^{-n}$.

Let $\tau=\min\{r_1/p-2^{-n}, \frac{r_2}{2^n-p}-2^{-n}\}$. Then $\tau>0$. By taking $\varepsilon=\tau/2$, we see that there is a point $z\in X$ such that $\max\{d(x,z), d(z,y)\} \leq 1/2+\tau/2$.

Let $x_{1,1}=x$, $x_{1,2}=z$ and $x_{1,3}=y$. By taking $\varepsilon=\tau/2^2$, we know that there are two points z_1 and $z_2\in X$ such that for $i\in\{1,2\}$, $\max\{d(x_{1,i},z_i), d(z_i,x_{1,i+1})\}\leq 1/2^2+\tau/2$.

Let $x_{2,1}=x_{1,1}=x$, $x_{2,2}=z_1$, $x_{2,3}=x_{1,2}$, $x_{2,4}=z_2$ and $x_{2,5}=x_{1,3}=y$. By taking $\varepsilon=\tau/2^3$, we know that there are four points $w_i\in X (i\in\{1,\dots,4\})$ such that $\max\{d(x_{2,i},w_i), d(w_i,x_{2,i+1})\}\leq 1/2^3+\tau/2$.

After repeating this procedure n times, we shall get 2^n+1 points $x_{n,i}$ $(i\in\{1,\dots,2^n+1\})$ such that $\max\{d(x_{n,i},w_i), d(w_i,x_{n,i+1})\}\leq 1/2^n+\tau/2$, where $x_{n,1}=x$ and $x_{n,2^n+1}=y$.

Since $d(x,x_{n,p+1})\leq p(1/2^n+\tau/2)<r_1$ and $d(x_{n,p+1},y)\leq(2^n-p)(1/2^n+\tau/2)<r_2$, we know that $x_{n,p+1}\in\mathbb{B}(x,r_1)\cap\mathbb{B}(y,r_2)$. This completes the proof of Lemma 10. ☐

Corollary 11. *Suppose that (X,d) is complete. Then X is length $\Leftrightarrow$ X is dense $\Leftrightarrow$ X has the ε-midpoint property for any $\varepsilon>0$.*

Proof. This follows from Lemmas 8, 10 and [6, Lemma 2.1]. ☐

We remark that the assumption "(X,d) being completeness" in Corollary 11 cannot be removed. For example, let $\mathbb{Q}\subset\mathbb{R}$ denote the set of all rational numbers endowed with Euclidean metric. Then the metric space $(\mathbb{Q},|\cdot|)$ is dense and has the ε-midpoint property for any $\varepsilon>0$, but it is noncomplete and not length either.

3. Local characterization of quasihyperbolic mappings

The aim of this section is to prove Theorem 2. Before proceeding, we need two auxiliary results.

Lemma 12. *Let (X_1,d_1) and (X_2,d_2) be c-quasi-convex metric spaces. Suppose that $G\subsetneq X_1$ and $G'\subsetneq X_2$ are domains, and that $f:G\to G'$ is a homeomorphism. Then f is L-QH if and only if there is a constant $\lambda_0\in(0,\frac{1}{6cL}]$ such that for all x, $y\in G$ with $d_1(x,y)\leq\lambda_0 d_G(x)$,*

$$\frac{1}{C}\frac{d_1(x,y)}{d_G(x)}\leq\frac{d_2(x',y')}{d_{G'}(x')}\leq C\frac{d_1(x,y)}{d_G(x)}. \tag{2}$$

The constants L and C depend on each other and c. Here and hereafter, $x'=f(x)$ for all $x\in G$.

Proof. Since (X_i,d_i) are c-quasi-convex metric spaces for $i=1,2$, G and G' are clearly locally (λ,c)-quasi-convex with $\lambda=1/(2c)$, rectifiably connected and non-complete as metric spaces. Thus Lemma 12 follows from [2, Lemma 1.1]. (Since the manuscript [2] is not available for the reader yet, we provide the proof of [2, Lemma 1.1] (see Lemma 19 below) in the Appendix A.) ☐

Lemma 13. *Let (X_1,d_1) be a length space, and let (X_2,d_2) be a c-quasi-convex metric space. Suppose that $G\subsetneq X_1$ and $G'\subsetneq X_2$ are domains, and that $f:G\to G'$ is a homeomorphism. Then f is L-QH if and only if there are constants $L_1\geq 1$ and $L_2\geq 1$ such that for all $x\in G$, the restrictions $f|_{B_x}$ is L_1-QH and $d_{G'}(x')\leq L_2 d_{f(B_x)}(x')$, where the constants L, L_1 and L_2 depend on each other, and c.*

Proof. Since (X_1,d_1) is a length space, by Lemma 5, we see that for any $x\in G$,

$$B_x\subset G \quad\text{and}\quad d_{B_x}(x)=d_G(x). \tag{3}$$

For the proof of the necessity part, we assume that f is L-QH. Since the assumption that (X_1,d_1) is length implies that (X_1,d_1) is 2-quasi-convex, it follows from [4, Theorem 3.7] that for each $x\in G$, $f|_{B_x}$ is L_1-QH with $L_1=L_1(c,L)$.

For $x \in G$, let $y \in \overline{\mathbb{B}}(x, \lambda_0 d_G(x))$, where $\lambda_0 \in (0, \frac{1}{6cL}]$ is the constant from Lemma 12. Since f is L-QH, by Lemma 12, it follows that

$$d_{G'}(x') \le C \frac{d_G(x)}{d_1(x,y)} d_2(x', y'), \tag{4}$$

where $C = C(c, L) \ge 1$.

Since $f|_{B_x}$ is L_1-QH, again by Lemma 12, together with (3), we have

$$d_2(x', y') \le C_1 \frac{d_1(x,y)}{d_G(x)} d_{f(B_x)}(x'),$$

where $C_1 = C_1(c, L_1) \ge 1$. By taking $L_2 = CC_1$, we get

$$d_{G'}(x') \le L_2 d_{f(B_x)}(x'),$$

and thus, the necessity part is proved.

Next, we prove the sufficiency part. Since for any $x \in G$, $f|_{B_x}$ is L_1-QH, once more, Lemma 12 along with (3) guarantees that there exist constants $L_3 = L_3(c, L_1) \ge 1$ and $\lambda_1 \in (0, \frac{1}{6cL}]$ such that for any $y \in \overline{\mathbb{B}}(x, \lambda_1 d_G(x))$,

$$\frac{1}{L_3} \frac{d_1(x,y)}{d_G(x)} \le \frac{d_2(x', y')}{d_{f(B_x)}(x')} \le L_3 \frac{d_1(x,y)}{d_G(x)}. \tag{5}$$

Moreover, it follows from [3, Lemma 3.3(1)] that

$$d_{f(B_x)}(x') \le c d_{G'}(x'),$$

because (X_2, d_2) is c-quasi-convex. Then by the assumption $d_{G'}(x') \le L_2 d_{f(B_x)}(x')$, we get from (5) that for all $y \in \overline{\mathbb{B}}(x, \lambda_1 d_G(x))$,

$$\frac{1}{L_2 L_3} \frac{d_1(x,y)}{d_G(x)} \le \frac{d_2(x', y')}{d_{G'}(x')} \le c L_3 \frac{d_1(x,y)}{d_G(x)}.$$

Thus Lemma 12 ensures that f is L-QH with $L = L(c, L_1, L_2)$. This ends the proof of Lemma 13. □

3.1. *Proof of Theorem 2*

Proof. By Lemma 13, we see that, to prove this theorem, it is sufficient to show that for any $x_0 \in G$,

$$d_{G'}\left(x_0'\right) \le 2\eta(1) d_{f(B_{x_0})}\left(x_0'\right). \tag{6}$$

Fix $x_0 \in G$, and let $\varepsilon \in (0, 1/100)$ be small enough such that

$$\big(1 + \eta(1)\big)\eta(3\varepsilon/2) \le \frac{1}{2}, \tag{7}$$

and for $i \in \{0, 1, 2, \ldots\}$, let $\varepsilon_i = \varepsilon/2^i$. Based on the sequence $\{\varepsilon_i\}_{i=0}^{\infty}$, we determine a (finite or infinite) sequence of points in $\overline{G}$ as stated in the following claim.

Claim 14. *There is a sequence of points $\{x_i\}_{i=0} \subset \overline{G}$, which satisfies the following.*

(1) *For $i \ge 0$, if $x_{i+1} \ne x_i$, then $x_{i+1} \in \mathbb{S}(x_i, d_G(x_i))$ and*

$$d_G(x_{i+1}) \le 3\varepsilon_{i+1} d_G(x_i) \le \frac{1}{2} d_G(x_i). \tag{8}$$

(2) *There is a point $a \in \partial G$ such that*
   (a) *if the sequence is infinite, i.e.,*

$$\{x_i\}_{i=0} = \{x_i\}_{i=0}^{\infty}, \quad \text{then} \quad \lim_{n\to\infty} x_n = a.$$

   (b) *if the sequence is finite, i.e., there is an integer $k_0 \ge 1$ such that $\{x_i\}_{i=0} = \{x_i\}_{i=0}^{k_0}$, then $x_{k_0} = a$. Further, for all $0 \le i < k_0$, $x_i \ne a$ and $x_i \ne x_{i+1}$.*

We determine the needed sequence $\{x_i\}_{i=0}$ of points as follows:

If there exists some point $x_1 \in \partial G$ such that $d_1(x_0, x_1) = d_G(x_0)$, let the sequence be $\{x_0, x_1\}$, i.e., $k_0 = 1$. Otherwise, we choose a point $y_1 \in \partial G$ such that $d_1(x_0, y_1) \le (1+\varepsilon_1)d_G(x_0)$. Since (X_1, d_1) is a length space, there is a curve α_0 connecting x_0 and y_1 with

$$\ell(\alpha_0) \le (1+\varepsilon_1)d_1(x_0, y_1) \le (1+\varepsilon_1)^2 d_G(x_0).$$

Let $x_1 \in \alpha_0 \cap \mathbb{S}(x_0, d_G(x_0))$ with $\alpha_0[x_0, x_1) \subset \mathbb{B}(x_0, d_G(x_0))$, where $\alpha_0[x_0, x_1)$ denotes the subcurve of α_0 between x_0 and x_1 with the point x_1 deleted. Then $x_1 \in G$, and it follows from the fact

$$d_G(x_1) \le d_1(x_1, y_1) \le \ell(\alpha_0) - d_1(x_0, x_1)$$

that

$$d_G(x_1) \le 3\varepsilon_1 d_G(x_0) \le d_G(x_0)/2.$$

By repeating this procedure, we get a sequence $\{x_i\}_{i=0}$ of points in $\overline{G}$ such that if $x_{i+1} \ne x_i$, then $d_G(x_{i+1}) \le 3\varepsilon_{i+1} d_G(x_i) \le d_G(x_i)/2$, and so,

$$d_1(x_{i+1}, x_i) = d_G(x_i) \le 2^{-i} d_G(x_0).$$

We divide the discussions into two cases. For the case when the sequence $\{x_i\}_{i=0}$ is infinite, we know that $\{x_i\}_{i=0} \subset G$ is a Cauchy sequence. Thus there is a point $a \in \partial G$ such that $\lim_{n\to\infty} x_n = a$.

For the remaining case, that is, the sequence $\{x_i\}_{i=0}$ is finite, i.e., there is an integer $k_0 \ge 1$ such that $\{x_i\}_{i=0} = \{x_i\}_{i=0}^{k_0}$, by the construction, we see that $x_{k_0} \in \partial G$. Further, for all $0 \le i < k_0$, $x_i \ne x_{k_0}$ and $x_i \ne x_{i+1}$.

By letting $a = x_{k_0}$, we see that Claim 14 is proved.

Assume that the sequence $\{x_i\}_{i=0}$ of points in $\overline{G}$ and the sequence $\{\alpha_i\}_{i=0}$ of curves in $\overline{G}$ are constructed in the proof of Claim 14. Then we have the following claim.

Claim 15. *Suppose there is an integer $r \ge 0$ such that $x_{r+2} \ne x_{r+1}$. Then*

$$\alpha_r\,[x_r, x_{r+1}] \cap \mathbb{S}\,(x_{r+1}, d_G\,(x_{r+1})) \ne \emptyset, \tag{1}$$

and

$$d_2\left(x'_{r+1}, x'_{r+2}\right) \le \frac{1}{2} d_2\left(x'_r, x'_{r+1}\right). \tag{2}$$

The first assertion in the claim directly follows from (8). In the following, we prove the second assertion. It follows from the first assertion of the claim that there is a point $z_{r+1} \in G$ such that $z_{r+1} \in \alpha_r[x_r, x_{r+1}] \cap \mathbb{S}(x_{r+1}, d_G(x_{r+1}))$, which implies that $z_{r+1} \in B_{x_r}$. Since f is relatively η-quasi-symmetric on B_{x_r}, we get

$$\frac{d_2\left(x'_{r+1}, z'_{r+1}\right)}{d_2\left(x'_{r+1}, x'_r\right)} \le \eta\left(\frac{d_1\left(x_{r+1}, z_{r+1}\right)}{d_1\left(x_{r+1}, x_r\right)}\right) \le \eta\left(3\varepsilon_{r+1}\right). \tag{10}$$

If $z_{r+1} = x_{r+2}$, then the second assertion in the claim follows from (7) and (10).

For the case when $z_{r+1} \ne x_{r+2}$, since f is relatively η-quasi-symmetric on $B_{x_{r+1}}$, we have

$$\frac{d_2\left(x'_{r+2}, x'_{r+1}\right)}{d_2\left(z'_{r+1}, x'_{r+1}\right)} \le \eta\left(\frac{d_1\left(x_{r+2}, x_{r+1}\right)}{d_1\left(z_{r+1}, x_{r+1}\right)}\right) = \eta(1). \tag{11}$$

It follows from (10) that

$$d_2\left(x'_{r+1}, x'_{r+2}\right) \le \eta(1) d_2\left(x'_{r+1}, z'_{r+1}\right) \le \eta(1)\eta(3\varepsilon_{r+1}) d_2\left(x'_r, x'_{r+1}\right),$$

and so, (7) leads to

$$d_2\left(x'_{r+1}, x'_{r+2}\right) \le \frac{1}{2} d_2\left(x'_r, x'_{r+1}\right).$$

This proves the assertion (2) in the claim, and so, the proof of the Claim 15 is complete.

Claim 16. *$d_2(x_0', a') \le 2d_2(x_0', x_1')$, where the point a is from Claim 14.*

We divide the proof into two cases. The first case is that the sequence $\{x_i\}_{i=0}$ is finite. Then it follows from Claim 14 (2) that there is an integer $k_0 \ge 1$ such that the sequence is $\{x_i\}_{i=0}^{k_0}$ and $x_{k_0} = a \in \partial G$.

If $k_0 = 1$, i.e., $x_1 = a$, then the inequality in the claim is obvious. If $k_0 > 1$, then Claim 15 (2) guarantees that

$$d_2\left(x_0', a'\right) \le \sum_{i=0}^{k_0-1} d_2\left(x_i', x_{i+1}'\right) < 2d_2\left(x_0', x_1'\right).$$

For the remaining case, that is, the sequence $\{x_i\}_{i=0}$ is infinite, let $n \ge 2$ be an integer. Again, Claim 15 (2) gives that

$$d_2\left(x_0', x_{n+1}'\right) \le \sum_{i=0}^{n} d_2\left(x_i', x_{i+1}'\right) < 2d_2\left(x_0', x_1'\right),$$

which implies that

$$d_2\left(x_0', a'\right) \le 2d_2\left(x_0', x_1'\right),$$

since, by Claim 14 (2), $\lim_{n\to\infty} x_n = a$. Hence the claim is true.

We are ready to finish the proof of the Theorem 2. It follows from Claim 16 that

$$d_{G'}\left(x_0'\right) \le 2d_2\left(x_0', x_1'\right), \tag{12}$$

since the fact $a' \in \partial G'$ implies that $d_{G'}(x_0') \le d_2(x_0', a')$.

To continue the discussions, we consider two cases. For the case when $G = B_{x_0}$ and $\partial G = \{x_1\}$, (6) follows from (12). For the remaining case, that is, $B_{x_0} \subsetneq G$ or $\partial G \setminus \{x_1\} \ne \emptyset$, the assumption that f is relatively η-quasi-symmetric on B_{x_0} implies that for any $w \in \mathbb{S}(x_0, d_G(x_0))$,

$$\frac{d_2\left(x_0', x_1'\right)}{d_2\left(w', x_0'\right)} \le \eta\left(\frac{d_1\left(x_0, x_1\right)}{d_1\left(w, x_0\right)}\right) = \eta(1),$$

which gives

$$d_2\left(x_0', x_1'\right) \le \eta(1) d_{f(B_{x_0})}\left(x_0'\right).$$

Hence (12) shows that

$$d_{G'}\left(x_0'\right) \le 2\eta(1) d_{f(B_{x_0})}\left(x_0'\right).$$

Therefore, this proves (6), and thus, the proof of Theorem 2 is complete. □

Appendix A. Metric characterization of quasihyperbolic maps

In this appendix, we establish a metric characterization of quasihyperbolic maps. Our goal is to show Lemma 19 which is needed in the proof of Lemma 12.

First, we introduce a basic property for quasihyperbolic metric.

Lemma 17 ([2, Lemma 2.2]). *Suppose that (X, d) is locally (λ, c)-quasi-convex, rectifiably connected and non-complete. Let $x, y \in X$. For any $\tau \in (0, \lambda]$, if either $d(x, y) \le \frac{\tau}{3c} d(x)$ or $k_X(x, y) \le \tau$, then*

$$\frac{1}{1+\tau}\frac{d(x,y)}{d(x)} \le k_X(x, y) \le 3c\frac{d(x,y)}{d(x)}. \tag{13}$$

Proof. We first consider the case when $d(x, y) \le \frac{\tau}{3c} d(x)$. Since X is locally (τ, c)-quasi-convex, there is a curve α in X joining x to y such that

$$\ell(\alpha) \le c d(x, y) \le \frac{\tau}{3} d(x).$$

For any $z \in \alpha$, we have

$$d(z) \ge d(x) - d(x, z) \ge \frac{3-\tau}{3} d(x),$$

which implies that

$$k_X(x,y) \le \int_\alpha \frac{|dz|}{d(z)} \le \frac{3}{3-\tau}\frac{\ell(\alpha)}{d(x)} \le \frac{3c}{3-\tau}\frac{d(x,y)}{d(x)}.$$

To prove the left hand side of (13), we only need to show that for any rectifiable curve γ in X with endpoints x and y,

$$\ell_{k_X}(\gamma) = \int_\gamma \frac{|dz|}{d(z)} \ge \frac{1}{1+\tau}\frac{d(x,y)}{d(x)}.$$

Indeed, if $\gamma \subset \overline{\mathbb{B}}(x,\tau d(x))$, then for any $z \in \gamma$, we obtain

$$d(z) \le d(x) + d(x,z) \le (1+\tau)d(x),$$

and thus,

$$\ell_{k_X}(\gamma) \ge \frac{1}{1+\tau}\frac{\ell(\gamma)}{d(x)} \ge \frac{1}{1+\tau}\frac{d(x,y)}{d(x)}. \tag{14}$$

If $\gamma \not\subset \overline{\mathbb{B}}(x,\tau d(x))$, then there must exist a subcurve γ_1 of γ such that

$$\gamma_1 \subset \overline{\mathbb{B}}(x,\tau d(x)) \quad \text{and} \quad \gamma_1 \cap \mathbb{S}(x,\tau d(x)) \ne \emptyset.$$

Thus we deduce from (14) that

$$\ell_{k_X}(\gamma) \ge \ell_{k_X}(\gamma_1) \ge \frac{1}{1+\tau}\frac{\ell(\gamma_1)}{d(x)} \ge \frac{3c}{1+\tau}\frac{d(x,y)}{d(x)},$$

since $\ell(\gamma_1) \ge \tau d(x) \ge 3c d(x,y)$.

To prove the Lemma 17, it remains to consider the case when $k_X(x,y) \le \tau$ and $d(x,y) \ge \frac{\tau}{3c} d(x)$. Under these assumptions, the right hand side of (13) is obvious. By [4, Theorem 2.5(1)], we have

$$d(x,y) \le \left(e^{k_X(x,y)} - 1\right) d(x).$$

Then the left hand side of (13) follows from the fact

$$e^t \le 1 + (1+\tau)t$$

for $t \in (0,\tau)$. □

Let $f : (X_1, d_1) \to (X_2, d_2)$ be a homeomorphism, and let x be a non-isolated point of X_1. We write the maximal stretching and the minimal stretching of f at x as follows:

$$L_{d_1}(x,f) = \limsup_{y\to x} \frac{d_2(x',y')}{d_1(x,y)} \quad \text{and} \quad l_{d_1}(x,f) = \liminf_{y\to x} \frac{d_2(x',y')}{d_1(x,y)}.$$

The following Lemma 18 concerns the stretching.

Lemma 18. *Let (X_i, d_i) be locally (λ, c)-quasi-convex, rectifiably connected and non-complete metric spaces ($i \in \{1,2\}$), and let $f : (X_1, d_1) \to (X_2, d_2)$ be a homeomorphism. Then*

(1) *for all $x \in X_1$,*

$$\frac{1}{6c}\frac{d_1(x)}{d_2(x')} L_{d_1}(x,f) \le L_{k_{X_1}}(x,f) \le 6c\frac{d_1(x)}{d_2(x')} L_{d_1}(x,f)$$

and

$$\frac{1}{6c}\frac{d_1(x)}{d_2(x')} l_{d_1}(x,f) \le l_{k_{X_1}}(x,f) \le 6c\frac{d_1(x)}{d_2(x')} l_{d_1}(x,f).$$

(2) *$f : (X_1, k_{X_1}) \to (X_2, k_{X_2})$ is L-quasi-isometric if and only if there is a constant $M \ge 1$ such that for all $x \in X_1$,*

$$\frac{1}{M} \le l_{k_{X_1}}(x,f) \le L_{k_{X_1}}(x,f) \le M,$$

where the constants L and M depends on each other, together with the given constants λ and c.

Proof. By Lemma 17, we have

$$\begin{aligned} L_{k_{X_1}}(x,f) &= \limsup_{y\to x} \frac{k_{X_2}(x',y')}{k_{X_1}(x,y)} \\ &= \limsup_{y\to x}\left(\frac{k_{X_2}(x',y')}{d_2(x',y')}\frac{d_2(x',y')}{d_1(x,y)}\frac{d_1(x,y)}{k_{X_1}(x,y)}\right) \\ &\le 6c\frac{d_1(x)}{d_2(x')}L_{d_1}(x,f) \end{aligned}$$

and

$$\begin{aligned} L_{d_1}(x,f) &= \limsup_{y\to x} \frac{d_2(x',y')}{d_1(x,y)} \\ &= \limsup_{y\to x}\left(\frac{d_2(x',y')}{k_{X_2}(x',y')}\frac{k_{X_2}(x',y')}{k_{X_1}(x,y)}\frac{k_{X_1}(x,y)}{d_1(x,y)}\right) \\ &\le 6c\frac{d_2(x')}{d_1(x)}L_{k_{X_1}}(x,f). \end{aligned}$$

These show that the first chain of inequalities in the assertion (1) is true.

Similarly, we can show that the second chain of inequalities in the assertion (1) is also true.

The assertion (2) in the Lemma 18 follows from [10, Lemma 5.5] and the obvious fact $l_{k_{X_1}}(x,f)L_{k_{X_2}}(x',f^{-1}) = 1$ for all $x \in X_1$ since for each $i \in \{1,2\}$, the space (X_i, k_{X_i}) is μ-quasi-convex for any $\mu > 1$. □

Lemma 19 ([2, Lemma 1.1]). *Suppose that for each $i \in \{1, 2\}$, (X_i, d_i) is a locally (λ, c)-quasi-convex, rectifiably connected and non-complete space, and $f : (X_1, d_1) \to (X_2, d_2)$ is a homeomorphism. Then $f : (X_1, k_{X_1}) \to (X_2, k_{X_2})$ is L-quasi-isometric if and only if there is a constant $\lambda_0 \in (0, \frac{\lambda}{3cL}]$ such that for all $x, y \in X_1$ with $d_1(x,y) \le \lambda_0 d_1(x)$,*

$$\frac{1}{C}\frac{d_1(x,y)}{d_1(x)} \le \frac{d_2(x',y')}{d_2(x')} \le C\frac{d_1(x,y)}{d_1(x)}. \tag{15}$$

The constants L and C depend on each other, and the parameters λ and c. Here and hereafter, $d_i(z) = d_i(z,\partial X_i)$ denotes the distance from z to the boundary ∂X_i of X_i, $i \in \{1,2\}$.

Proof. *Necessity:* Assume that $f : (X_1, k_{X_1}) \to (X_2, k_{X_2})$ is L-quasi-isometric, and let $x, y \in X$ be such that

$$d_1(x,y) \le \lambda_0 d_1(x),$$

where $\lambda_0 \in (0, \frac{\lambda}{3cL}]$. By Lemma 17, we have

$$k_{X_1}(x,y) \le 3c\frac{d_1(x,y)}{d_1(x)} \le \frac{\lambda}{L},$$

and thus, by [4, Theorem 2.5(1)], we obtain

$$\begin{aligned} \log\left(1+\frac{d_2(x',y')}{d_2(x')}\right) &\le k_{X_2}(x',y') \\ &\le L k_{X_1}(x,y) \\ &\le \min\left\{3cL\frac{d_1(x,y)}{d_1(x)}, \lambda\right\}. \end{aligned} \tag{16}$$

This leads to

$$d_2(x',y') \le (e^{\lambda}-1)d_2(x'),$$

which implies

$$\frac{d_2(x',y')}{d_2(x')} \le e^{\lambda}\log\left(1+\frac{d_2(x',y')}{d_2(x')}\right). \tag{17}$$

Then (16) gives

$$\frac{d_2(x',y')}{d_2(x')} \le 3cLe^{\lambda} \frac{d_1(x,y)}{d_1(x)}. \tag{18}$$

On the other hand, we deduce from (16) that $k_{X_2}(x',y') \le \lambda$. Then it follows from [4, Theorem 2.5(1)] and Lemma 17 that

$$\begin{aligned}\log\left(1+\frac{d_1(x,y)}{d_1(x)}\right) &\le k_{X_1}(x,y)\\ &\le Lk_{X_2}(x',y')\\ &\le \min\left\{3cL\frac{d_2(x',y')}{d_2(x')}, L\lambda\right\},\end{aligned}$$

which implies

$$d_1(x,y) \le \left(e^{L\lambda}-1\right) d_1(x),$$

and so, we have

$$\frac{d_1(x,y)}{d_1(x)} \le e^{L\lambda}\log\left(1+\frac{d_1(x,y)}{d_1(x)}\right) \le 3cLe^{L\lambda}\frac{d_2(x',y')}{d_2(x')}.$$

This, together with (17), yields (2) with $C = 3cLe^{L\lambda}$.

Sufficiency. *Assume that there is a constant $\lambda_0 \in (0, \frac{\lambda}{3cL}]$ such that for all x, $y \in X_1$ with $d_1(x,y) \le \lambda_0 d_1(x)$,*

$$\frac{1}{C}\frac{d_1(x,y)}{d_1(x)} \le \frac{d_2(x',y')}{d_2(x')} \le C\frac{d_1(x,y)}{d_1(x)}.$$

This implies that

$$\frac{1}{C}\frac{d_2(x')}{d_1(x)} \le L_{d_1}(x,f) \le C\frac{d_2(x')}{d_1(x)} \quad \text{and} \quad \frac{1}{C}\frac{d_2(x')}{d_1(x)} \le l_{d_1}(x,f) \le C\frac{d_2(x')}{d_1(x)},$$

which, together with Lemma 18 (1), gives

$$\frac{1}{6cC} \le l_{k_{X_1}}(x,f) \le L_{k_{X_1}}(x,f) \le 6cC.$$

By taking $M = 6cC$, we see from Lemma 18 (2) that the sufficiency is true. □

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