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Differential topology / *Topologie différentielle*

An \mathbf{HP}^2 -bundle over S^4 with nontrivial \widehat{A} -genus

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Abstract. We explain the existence of a smooth \mathbf{HP}^2 -bundle over S^4 whose total space has nontrivial \widehat{A} -genus. Combined with an argument going back to Hitchin, this answers a question of Schick and implies that the space of Riemannian metrics of positive sectional curvature on a closed manifold can have nontrivial higher rational homotopy groups.

Résumé. Nous expliquons l'existence d'un fibré différentiel de base S^4 et fibre \mathbf{HP}^2 , dont l'espace total est de \widehat{A} -genre non-trivial. En combinant ce résultat avec un argument de Hitchin, ceci répond à une question de Schick et implique que l'espace de métriques riemanniennes de courbure sectionnelle positive sur une variété fermée peut avoir des groupes d'homotopie rationnelle supérieures non-triviaux.

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In view of applications to spaces of Riemannian metrics with positive curvature, there has been recent interest in constructing smooth fibre bundles over spheres whose total space has nontrivial \widehat{A} -genus. In their work on the space of positive scalar curvature metrics, Hanke–Schick–Steimle [7, Corollary 1.6] showed that such bundles exist for every dimension of the base sphere. However, as noted in [7, p. 337], their method does not yield bundles with an explicit description of the fibre, though they are able to show that the fibre may be chosen to carry a metric of positive scalar curvature using a theorem of Stolz. For applications to spaces of metrics with positive sectional or Ricci curvature it is desirable to have examples with fibre carrying such a metric. In [1, p. 3999] (see also [13, Section 9]) it is said that this “seems to be a very difficult problem”: we offer the following solution.

Theorem. *There exists a smooth oriented fibre bundle $\mathbf{HP}^2 \rightarrow E^{12} \rightarrow S^4$ with $\widehat{A}(E) \neq 0$.*

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Remark. The argument we give also shows that this fibre bundle may be assumed to have a section with trivial normal bundle (see Remark 2), and provides analogous \mathbf{HP}^n -bundles over S^4 for all even $n \geq 2$ (see Remark 3). It can certainly be extended further.

The standard Riemannian metric g_{st} on \mathbf{HP}^2 has positive sectional curvature, so pulling back g_{st} along orientation-preserving diffeomorphisms yields a map

$$(-)^* g_{\text{st}}: \text{Diff}(\mathbf{HP}^2) \longrightarrow \mathcal{R}^{\text{sec}>0}(\mathbf{HP}^2) \subset \mathcal{R}^{\text{Ric}>0}(\mathbf{HP}^2) \subset \mathcal{R}^{\text{scal}>0}(\mathbf{HP}^2)$$

from the group of diffeomorphisms of \mathbf{HP}^2 in the smooth topology to the spaces of Riemannian metrics on \mathbf{HP}^2 having positive sectional, Ricci, or scalar curvature. By an argument of Hitchin [10] (see for instance [1, p. 3999] for an explanation of this), the theorem has the following corollary, which answers a question of Schick [6, p. 30] and provides an example as asked for in [1, Remark 2.2].

Corollary. *The induced map*

$$\pi_3((-)^* g_{\text{st}}) \otimes \mathbf{Q}: \pi_3(\text{Diff}(\mathbf{HP}^2); \text{id}) \otimes \mathbf{Q} \longrightarrow \pi_3(\mathcal{R}^{\text{scal}>0}(\mathbf{HP}^2); g_{\text{st}}) \otimes \mathbf{Q}$$

is nontrivial, so in particular $\pi_3(\mathcal{R}^{\text{sec}>0}(\mathbf{HP}^2); g_{\text{st}}) \otimes \mathbf{Q} \neq 0$ and $\pi_3(\mathcal{R}^{\text{Ric}>0}(\mathbf{HP}^2); g_{\text{st}}) \otimes \mathbf{Q} \neq 0$.

Proof of the Theorem

Smooth \mathbf{HP}^2 -bundles over S^4 (together with an identification of the fibre over the base-point with \mathbf{HP}^2) are classified by $\pi_4(\text{BDiff}(\mathbf{HP}^2))$, so our task is to show that the morphism $\hat{A}: \pi_4(\text{BDiff}(\mathbf{HP}^2)) \rightarrow \mathbf{Q}$ assigning an \mathbf{HP}^2 -bundle $E \rightarrow S^4$ the \hat{A} -genus of the total space is non-trivial. This morphism admits a factorisation of the form

$$\pi_4(\text{BDiff}(\mathbf{HP}^2)) \longrightarrow \pi_4(\widehat{\text{BDiff}}(\mathbf{HP}^2)) \xrightarrow{\hat{A}} \mathbf{Q}, \tag{1}$$

where $\widehat{\text{BDiff}}(\mathbf{HP}^2)$ is the block diffeomorphism group of \mathbf{HP}^2 and the first map is induced by the canonical comparison map $\text{Diff}(\mathbf{HP}^2) \rightarrow \widehat{\text{Diff}}(\mathbf{HP}^2)$. This factorisation follows for instance from [5, Theorem 1], but there is also a more direct argument: via the canonical isomorphism $\pi_4(\text{BDiff}(\mathbf{HP}^2)) \cong \pi_3(\text{Diff}(\mathbf{HP}^2); \text{id})$, the morphism $\hat{A}: \pi_4(\text{BDiff}(\mathbf{HP}^2)) \rightarrow \mathbf{Q}$ is given by mapping a diffeomorphism $\phi: D^3 \times \mathbf{HP}^2 \rightarrow D^3 \times \mathbf{HP}^2$ that is the identity on the boundary and commutes with the projection to D^3 to the \hat{A} -genus of the glued manifold $D^4 \times \mathbf{HP}^2 \cup_{\phi \cup \text{id}} D^4 \times \mathbf{HP}^2$. This description of the morphism makes clear that it factors through the map $\pi_3(\text{Diff}(\mathbf{HP}^2); \text{id}) \rightarrow \pi_0(\text{Diff}_{\partial}(D^3 \times \mathbf{HP}^2)) \cong \pi_3(\widehat{\text{Diff}}(\mathbf{HP}^2); \text{id})$ that only remembers the underlying isotopy class of ϕ .

We thus have to show nontriviality of the composition (1). It suffices to check this after rationalisation, which makes the first map surjective:

Lemma. *The map $\pi_4(\text{BDiff}(\mathbf{HP}^2)) \otimes \mathbf{Q} \longrightarrow \pi_4(\widehat{\text{BDiff}}(\mathbf{HP}^2)) \otimes \mathbf{Q}$ is surjective.*

Proof. Choosing an embedded disc $D^8 \subset \mathbf{HP}^2$, we consider the commutative square

$$\begin{array}{ccc} \text{BDiff}_{\partial}(D^8) & \longrightarrow & \text{BDiff}(\mathbf{HP}^2) \\ \downarrow & & \downarrow \\ \widehat{\text{BDiff}}_{\partial}(D^8) & \longrightarrow & \widehat{\text{BDiff}}(\mathbf{HP}^2) \end{array}$$

whose horizontal maps are induced by extending (block) diffeomorphisms of D^8 that are the identity on the boundary to \mathbf{HP}^2 by the identity. The claim follows by showing that the third rational homotopy group of the right vertical homotopy fibre vanishes for which we note that, since \mathbf{HP}^2 is 3-connected, the square is 4-cartesian by Morlet’s lemma of disjunction [4, Corollary 3.2, p. 29], so it suffices to show that the third rational homotopy group of the homotopy

fibre of the left vertical map is trivial. Since $\pi_i(\text{BDiff}_\partial(D^{2n})) \cong \pi_0 \text{Diff}_\partial(D^{2n+i-1}) \cong \Theta_{2n+i}$ vanishes rationally as the group Θ_{2n+i} of homotopy $(2n+i)$ -spheres is finite, the claim follows from $\pi_3(\text{BDiff}_\partial(D^8)) \otimes \mathbf{Q} = 0$ which holds by [12, Theorem 4.1]. \square

Given the lemma, we are left to show that the map $\widehat{A}: \pi_4(\text{BDiff}(\mathbf{HP}^2)) \rightarrow \mathbf{Q}$ is nontrivial, which we shall do after precomposition with the map

$$\pi_4(\text{hAut}(\mathbf{HP}^2)/\widehat{\text{Diff}}(\mathbf{HP}^2); \text{id}) \longrightarrow \pi_4(\text{BDiff}(\mathbf{HP}^2))$$

induced by the inclusion of the homotopy fibre of the comparison map $\text{BDiff}(\mathbf{HP}^2) \rightarrow \text{BhAut}(\mathbf{HP}^2)$, where $\text{hAut}(\mathbf{HP}^2)$ is the topological monoid of homotopy automorphisms of \mathbf{HP}^2 . Considering this homotopy fibre is advantageous since the h -cobordism theorem provides an isomorphism

$$\pi_4(\text{hAut}(\mathbf{HP}^2)/\widehat{\text{Diff}}(\mathbf{HP}^2)) \cong \mathcal{S}_\partial(D^4 \times \mathbf{HP}^2)$$

to the *structure group* of $D^4 \times \mathbf{HP}^2$ relative to $\partial D^4 \times \mathbf{HP}^2$ in the sense of surgery theory (see [14] for background on surgery theory, especially Chapter 10), which in turn fits into the *surgery exact sequence* of abelian groups

$$0 = L_{13}(\mathbf{Z}) \xrightarrow{\partial} \mathcal{S}_\partial(D^4 \times \mathbf{HP}^2) \xrightarrow{\eta} \mathcal{N}_\partial(D^4 \times \mathbf{HP}^2) \xrightarrow{\sigma} L_{12}(\mathbf{Z}) \cong \mathbf{Z}$$

featuring the surgery obstruction map σ from the normal invariants $\mathcal{N}_\partial(D^4 \times \mathbf{HP}^2)$ to the L -group $L_{12}(\mathbf{Z}) \cong \mathbf{Z}$. The standard smooth structure on $D^4 \times \mathbf{HP}^2$ provides an isomorphism

$$\mathcal{N}_\partial(D^4 \times \mathbf{HP}^2) \cong [S^4 \wedge \mathbf{HP}_+^2, G/O],$$

where $[-, -]$ stands for based homotopy classes and G/O is the homotopy fibre of the map $\text{BO} \rightarrow \text{BG}$ classifying the underlying stable spherical fibration of a stable vector bundle.

As BG has trivial rational homotopy groups, the map

$$[S^4 \wedge \mathbf{HP}_+^2, G/O] \longrightarrow [S^4 \wedge \mathbf{HP}_+^2, \text{BO}] = \widehat{KO}^0(S^4 \wedge \mathbf{HP}_+^2)$$

is rationally an isomorphism. Furthermore the Pontrjagin character gives an isomorphism

$$\text{ph}(-) = \text{ch}(- \otimes_{\mathbf{R}} \mathbf{C}): \widehat{KO}^0(S^4 \wedge \mathbf{HP}_+^2) \otimes \mathbf{Q} \xrightarrow{\cong} \bigoplus_{i \geq 0} \widetilde{H}^{4i}(S^4 \wedge \mathbf{HP}_+^2; \mathbf{Q}) = u \cdot \mathbf{Q}[z]/(z^3),$$

where $u \in \widetilde{H}^4(S^4; \mathbf{Q})$ denotes the cohomological fundamental class, and $z \in H^4(\mathbf{HP}^2; \mathbf{Q})$ is the usual generator. Therefore for any triple $(A, B, C) \in \mathbf{Q}^3$ there exists a nonzero $\lambda \in \mathbf{Z}$ and a normal invariant $n \in \mathcal{N}_\partial(D^4 \times \mathbf{HP}^2)$ whose underlying stable vector bundle ξ has $\text{ph}(\xi) = \lambda \cdot u \cdot (A + Bz + Cz^2)$. Since $S^4 \wedge \mathbf{HP}_+^2$ has no nontrivial cup-products among elements of positive degree, we have $\text{ph}_i(\xi) = (-1)^{i+1}/(2i-1)! \cdot p_i(\xi)$ and hence

$$p_1(\xi) = \lambda A \cdot u \quad p_2(\xi) = -6\lambda B \cdot u \cdot z \quad p_3(\xi) = 120\lambda C \cdot u \cdot z^2. \tag{2}$$

To evaluate the surgery obstruction map σ , recall that a normal invariant n with underlying stable vector bundle ξ is represented by a degree 1 normal map

$$\begin{array}{ccc} v_M & \xrightarrow{\widehat{f}} & v_{D^4 \times \mathbf{HP}^2} \oplus \xi \\ \downarrow & & \downarrow \\ M^{12} & \xrightarrow{f} & D^4 \times \mathbf{HP}^2, \end{array} \tag{3}$$

where $\partial M = \partial D^4 \times \mathbf{HP}^2$ and f and \widehat{f} restrict to the identity maps on the boundary. Here $v_{(-)}$ denotes the stable normal bundle of a manifold. The surgery obstruction is unchanged by gluing into M and $D^4 \times \mathbf{HP}^2$ a copy of $D^4 \times \mathbf{HP}^2$ along the identification of their boundaries with $\partial D^4 \times \mathbf{HP}^2$, and extending f and \widehat{f} trivially, giving rise to a degree 1 normal map to $f': M' \rightarrow$

$S^4 \times \mathbf{HP}^2$. The surgery obstruction may then be expressed in terms of the signatures of these manifolds, as

$$\sigma(n) = \frac{1}{8}(\text{sign}(M') - \text{sign}(S^4 \times \mathbf{HP}^2)).$$

The signature of $S^4 \times \mathbf{HP}^2$ is trivial, and that of M' may be computed in terms of the Hirzebruch signature theorem as the evaluation $\int_{M'} L(TM')$ of the L -class. As f' has degree 1 and pulls back $\nu_{S^4 \times \mathbf{HP}^2} \oplus \xi$ to the stable inverse of TM' , we have

$$\text{sign}(M') = \int_{M'} L(TM') = \int_{S^4 \times \mathbf{HP}^2} L(TS^4) \cdot L(T\mathbf{HP}^2) \cdot L(-\xi). \tag{4}$$

The first terms of the total L -class are given as

$$L = 1 + \frac{p_1}{3} + \frac{7 \cdot p_2 - p_1^2}{45} + \frac{62 \cdot p_3 - 13 \cdot p_1 p_2 + 2 \cdot p_1^3}{945} + \dots,$$

which we combine with $p(T\mathbf{HP}^2) = 1 + 2z + 7z^2$ from [8, Satz 1] to compute

$$\begin{aligned} L(TS^4) &= 1 \\ L(T\mathbf{HP}^2) &= 1 + \frac{2}{3} \cdot z + z^2 \\ L(-\xi) &= 1 + \lambda \left(-\frac{1}{3} A \cdot u + \frac{14}{15} B \cdot (u \cdot z) - \frac{496}{63} C \cdot (u \cdot z^2) \right) \end{aligned}$$

and thus

$$8\sigma(n) = \text{sign}(M') = \lambda \left(-\frac{1}{3} A + \frac{28}{45} B - \frac{496}{63} C \right).$$

It follows that for each triple $(A, B, C) \in \mathbf{Q}^3$ satisfying $\frac{1}{3}A - \frac{28}{45}B + \frac{496}{63}C = 0$ there exists a non-zero $\lambda \in \mathbf{Z}$ and a degree 1 normal map as in (3) with f a homotopy equivalence and with ξ having Pontrjagin classes as in (2). This gives a smooth block \mathbf{HP}^2 -bundle structure on the composition

$$M' \xrightarrow{f'} S^4 \times \mathbf{HP}^2 \xrightarrow{\pi_1} S^4$$

giving rise to a class in $\pi_4(\widehat{\text{BDiff}}(\mathbf{HP}^2))$, so it remains to evaluate $\widehat{A}(M')$. As in (4), we get

$$\widehat{A}(M') = \int_{M'} \widehat{A}(TM') = \int_{S^4 \times \mathbf{HP}^2} \widehat{A}(TS^4) \cdot \widehat{A}(T\mathbf{HP}^2) \cdot \widehat{A}(-\xi),$$

which we combine with the formula for the first terms of the total \widehat{A} -class

$$\widehat{A} = 1 - \frac{p_1}{24} + \frac{-4 \cdot p_2 + 7 \cdot p_1^2}{5760} + \frac{-16 \cdot p_3 + 44 \cdot p_2 p_1 - 31 \cdot p_1^3}{967680} + \dots$$

to compute

$$\begin{aligned} \widehat{A}(TS^4) &= 1 \\ \widehat{A}(T\mathbf{HP}^2) &= 1 - \frac{1}{12} \cdot z \\ \widehat{A}(-\xi) &= 1 + \lambda \left(\frac{1}{24} A \cdot u - \frac{1}{240} B \cdot (u \cdot z) + \frac{1}{504} C \cdot (u \cdot z^2) \right) \end{aligned}$$

from which we conclude

$$\widehat{A}(M') = \lambda \left(\frac{1}{2880} B + \frac{1}{504} C \right).$$

As there are clearly triples $(A, B, C) \in \mathbf{Q}^3$ satisfying

$$-\frac{1}{3}A + \frac{28}{45}B - \frac{496}{63}C = 0 \quad \text{and} \quad \frac{1}{2880}B + \frac{1}{504}C \neq 0,$$

this finishes the argument.

Remark 1. A fibre bundle $\pi: E \rightarrow S^4$ constructed in this way is fibre homotopy equivalent to the trivial bundle $\pi_1: S^4 \times \mathbf{HP}^2 \rightarrow S^4$, and under this fibre homotopy trivialisation we have $p_1(TE) = 2 \cdot (1 \otimes z) - \lambda A \cdot (u \otimes 1)$. Thus $p_1(TE)^3 = -12\lambda A \cdot (u \otimes z^2)$, and so

$$\int_E p_1(TE)^3 = -12\lambda A \quad \text{and} \quad \int_E \widehat{A}_3(TE) = \lambda \left(\frac{1}{2880} B + \frac{1}{504} C \right).$$

This argument therefore guarantees the existence of a 2-dimensional subspace of the group $\pi_4(B\text{Diff}(\mathbf{HP}^2)) \otimes \mathbf{Q}$, detected by the characteristic numbers $\int_E p_1(TE)^3$ and $\int_E \widehat{A}_3(TE)$.

Remark 2. The Theorem can be slightly strengthened: we may in addition assume that the smooth \mathbf{HP}^2 -bundle $\pi: E^{12} \rightarrow S^4$ admits a smooth section with trivial normal bundle, which may be helpful for fibrewise surgery constructions.

To see this, note that a fibre bundle $\pi: E \rightarrow S^4$ as constructed above is fibre homotopy equivalent to the trivial bundle $\pi_1: S^4 \times \mathbf{HP}^2 \rightarrow S^4$, so it admits a smooth section $s: S^4 \rightarrow E$ corresponding to a trivial section of the trivial bundle. By the description of $p_1(TE)$ in the previous remark we have $s^* p_1(TE) = -\lambda A \cdot u$. If we choose $(A = 0, B = \frac{496}{63}, C = \frac{28}{45})$, which is a triple whose surgery obstruction vanishes, then the corresponding bundle has $s^* p_1(TE) = 0$ and $\widehat{A}(E) \neq 0$. As TS^4 is stably trivial, it follows that the normal bundle of $s(S^4) \subset E$ has trivial first Pontrjagin class. This implies that the normal bundle is trivial, since $p_1: \pi_4(\text{BSO}(8)) \rightarrow \mathbf{Z}$ is injective as $\pi_4(\text{BSO}(8)) \cong \pi_4(\text{BSO})$ is stable.

Remark 3. The argument can be generalised to prove that for any even $n \geq 2$ there is a smooth \mathbf{HP}^n -bundle $E^{4n+4} \rightarrow S^4$ with $\widehat{A}(E) \neq 0$, so the Corollary holds for \mathbf{HP}^n as well.

Indeed, the application of Morlet’s lemma only required the fibre to be at least 8-dimensional and 3-connected. Similar to the above, one argues that for any pair $(A, C) \in \mathbf{Q}^2$ there exists a nonzero $\lambda \in \mathbf{Z}$ and a normal invariant $n \in \mathcal{N}_\partial(D^4 \times \mathbf{HP}^n)$ with underlying stable vector bundle ξ such that

$$p_1(\xi) = \lambda A \cdot u \quad p_i(\xi) = 0 \text{ for } 1 < i < n, \quad p_{n+1}(\xi) = \lambda(2n+1)!(-1)^n C \cdot u \cdot z^n.$$

Using that the coefficient of z^n in $L(T\mathbf{HP}^n)$ is 1 by the Hirzebruch’s signature theorem, we see that the surgery obstruction satisfies $8\sigma(n) = \lambda(-\frac{1}{3}A + h_{n+1}(2n+1)!(-1)^{n+1}C)$ where h_n is the coefficient of p_n in the total L -class. In particular, we can always find pairs (A, C) with $C \neq 0$ and $8\sigma(n) = 0$. As the coefficient of z^n in $\widehat{A}(T\mathbf{HP}^n)$ vanishes, because \mathbf{HP}^n admits a metric of positive scalar curvature, it holds $\widehat{A}(E) = \lambda a_{n+1}(2n+1)!(-1)^{n+1}C \neq 0$ with a_n the coefficient of p_n in the total \widehat{A} -class, which is easily proved to be non-zero using [9, Chapter 1. §1 (10)].

Remark 4. If one is willing to instead consider \mathbf{HP}^n -bundles over S^{4m} for n sufficiently large compared with m , then one may replace the appeal to the Lemma by the more classical [3, Corollary D], which implies that the map

$$\pi_{4m}(\text{hAut}(\mathbf{HP}^n)/\text{Diff}(\mathbf{HP}^n); \text{id}) \otimes \mathbf{Z}[\frac{1}{2}] \longrightarrow \pi_{4m}(\text{hAut}(\mathbf{HP}^n)/\widehat{\text{Diff}}(\mathbf{HP}^n); \text{id}) \otimes \mathbf{Z}[\frac{1}{2}]$$

is (split) surjective as long as $4m - 1$ lies in the pseudoisotopy stable range for \mathbf{HP}^n (so $3m < n$ suffices, by [11]). See also [2, Theorem 1]. One must still produce an appropriate element of $\mathcal{S}_\partial(D^{4m} \times \mathbf{HP}^n)$, which may be approached as in Remark 3.

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