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BV-operators and the secondary Hochschild complex

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\textbf{Abstract.} We introduce the notion of a BV-operator $\Delta = \{\Delta^n : V^n \to V^{n-1}\}_{n \geq 0}$ on a homotopy $G$-algebra $V^\bullet$ such that the Gerstenhaber bracket on $H(V^\bullet)$ is determined by $\Delta$ in a manner similar to the BV-formalism. As an application, we produce a BV-operator on the cochain complex defining the secondary Hochschild cohomology of a symmetric algebra $A$ over a commutative algebra $B$. In this case, we also show that the operator $\Delta^*$ corresponds to Connes’ operator.

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1. Introduction

A Gerstenhaber algebra (see [3]) consists of a graded vector space $W^\bullet = \bigoplus_{n \geq 0} W^n$ equipped with the following two structures:

(a) A dot product $x \cdot y$ of degree zero making $W^\bullet$ into an associative graded commutative algebra.

(b) A bracket $[x, y]$ of degree $-1$ making $W^\bullet$ into a graded Lie algebra satisfying the compatibility property that

$$[x, y \cdot z] = [x, y] \cdot z + (-1)^{(\deg(x)-1)\deg(y)} y \cdot [x, z].$$

Gerstenhaber algebra structures appear in a variety of situations, from Hochschild cohomology of algebras to the exterior algebra of a Lie algebra and the algebra of differential forms on a Poisson manifold.

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An operator $\partial = \{\partial^n : W^n \to W^{n-1}\}_{n \geq 0}$ on $W$ of degree $-1$ is said to generate the Gerstenhaber bracket (see Koszul [7, §2] and also [6, Definition 3.2]) if it satisfies

$$[x, y] = (-1)^{(\deg(x)-1)(\deg(y))}(\partial(x) \cdot y + (-1)^{\deg(x)} x \cdot \partial(y) - \partial(x \cdot y))$$

In particular, a Batalin–Vilkovisky algebra (or BV-algebra) consists of a Gerstenhaber algebra along with a generator $\partial$ for the bracket such that $\partial^2 = 0$.

In [4], [5], Gerstenhaber and Voronov introduced the notion of a homotopy $G$-algebra, which is a brace algebra equipped with a differential of degree 1 and a dot product of degree 0 satisfying certain conditions. In particular, the cohomology groups $H(V^\bullet)$ of a homotopy $G$-algebra $V^\bullet$ carry the structure of a Gerstenhaber algebra.

In this paper, we introduce the notion of a BV-operator $\Delta = \{\Delta^n : V^n \to V^{n-1}\}_{n \geq 0}$ on a homotopy $G$-algebra $V^\bullet$ such that the Gerstenhaber bracket on $H(V^\bullet)$ is determined by $\Delta$ in a manner similar to the BV-formalism. More explicitly, for classes $\bar{f} \in H^2(V^\bullet)$ and $\bar{g} \in H^n(V^\bullet)$, we have

$$[\bar{f}, \bar{g}] = (-1)^{(n-1)m}\{\Delta(f) \cdot g + (-1)^n f \cdot \Delta(g)- \Delta(f \cdot g)\} \in H^{m+n-1}(V^\bullet)$$

where $f \in Z^n(V^\bullet)$, $g \in Z^m(V^\bullet)$ are cocycles representing $\bar{f}$ and $\bar{g}$ respectively. We note that $\Delta$ need not be a morphism of cochain complexes and therefore may not induce any operator on $H(V^\bullet)$. As such, $\Delta$ may not descend to a generator for the Gerstenhaber bracket on $H(V^\bullet)$.

Our motivation is to introduce a BV-operator on the cochain complex defining the secondary Hochschild cohomology of a symmetric algebra $A$ over a commutative algebra $B$. For a datum $(A, B, \varepsilon)$ consisting of an algebra $A$, a commutative algebra $B$ and an extension of rings $\varepsilon : B \to A$ such that $\varepsilon(B) \subseteq Z(A)$, the secondary Hochschild cohomology $H^\bullet(A, B, \varepsilon)$ was introduced by Staic [9] in order to study deformations of algebras $A[[t]]$ having a $B$-algebra structure. More generally, Staic [9] introduced the secondary Hochschild complex $C^\bullet((A, B, \varepsilon); M)$ with coefficients in an $A$-bimodule $M$.

In [10], Staic and Stancu showed that the secondary Hochschild complex $C^\bullet(A, B, \varepsilon) := C^\bullet((A, B, \varepsilon); A)$ with coefficients in $A$ is a non-symmetric operad with multiplication, giving it the structure of a homotopy $G$-algebra. Hence, the secondary cohomology $H^\bullet(A, B, \varepsilon)$ is equipped with a graded commutative cup product and a Lie bracket which makes it a Gerstenhaber algebra. For more on the secondary cohomology, the reader may see, for instance, [1], Corrigan-Salter and Staic [2], Laubacher, Staic and Stancu [8].

Let $k$ be a field. It is well known (see Tradler [11]) that the Hochschild cohomology of a finite dimensional $k$-algebra $A$ equipped with a symmetric, non-degenerate, invariant bilinear form $\langle \cdot, \cdot \rangle : A \times A \to k$ carries the structure of a BV-algebra. For the terms $C^n((A, B, \varepsilon)) = \text{Hom}_k(A^\otimes n \otimes B^\otimes m, A)$ in the secondary Hochschild complex, we define the BV-operator $\Delta = \sum_{i=1}^{n+1} (-1)^i \Delta_i : C^{n+1}(A, B, \varepsilon) \to C^n(A, B, \varepsilon)$ by the condition (see Section 3)

$$\Delta f \otimes a_{n+1} = \left\langle f \otimes \left(\begin{array}{ccc}
      a_1 & b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\
      a_2 & b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\
      \vdots & \vdots & \vdots & \ddots & \vdots \\
      1 & 1 & 1 & \cdots & b_{n-1,n} \\
      1 & 1 & 1 & \cdots & a_n
    \end{array}\right), 1 \right\rangle$$

(1)
We then show that the Gerstenhaber bracket on the secondary Hochschild cohomology of \((A, B, ε)\) is determined by \(A\) in a manner similar to the BV-formalism.

From Tradler [11], we also know that the operator \(Δ^* : C^*(A, A) \rightarrow C^{*-1}(A, A)\) on usual Hochschild cochains inducing the BV-structure on \(H^*(A, A)\) corresponds to the operator \(N s\) on duals of Hochschild chains, where \(N\) is the “norm operator” and \(s\) is the “extra degeneracy” (see (16)). The isomorphism between the two complexes is induced by the \(k\)-module isomorphism \(A^* \cong A\) determined by the non-degenerate bilinear form \(\langle \cdot, \cdot \rangle : A × A → k\). If we pass to the cohomology and take the normalized Hochschild complex which is a quasi-isomorphic subcomplex of \(C^*(A, A)\), it follows that Tradler’s \(Δ^*\) operator corresponds to Connes’ operator on Hochschild cohomology with coefficients in \(A\).

However, in the case of secondary cohomology, we have mentioned that the operator \(Δ^*\) defined in (1) is not a morphism of complexes and we cannot pass to cohomology. Accordingly, we show that the operator \(Δ^*\) defined in (1) fits into a commutative diagram (see Theorem 10)

\[
\begin{array}{ccc}
\overline{C}^*(A, B, ε) & \xrightarrow{B} & \overline{C}^{*-1}(A, B, ε) \\
\downarrow & & \downarrow \\
C^*((A, B, ε); A) & \xrightarrow{Δ^*} & C^{*-1}((A, B, ε); A)
\end{array}
\]

(2)

where \(B\) is Connes’ operator. Here, \(\overline{C}^*(A, B, ε)\) is the normalization of the co-simplicial module \(C^*(A, B, ε)\) introduced by Laubacher, Staic and Stancu [8], which is used to compute the secondary Hochschild cohomology associated to the triple \((A, B, ε)\). It should be noted (see [8, Remark 4.7]) that despite similar names, the complex \(\overline{C}^*(A, B, ε)\) cannot be expressed as a secondary Hochschild complex with coefficients in some \(A\)-bimodule. The vertical morphisms in (2) are induced by composing the canonical morphisms \((A ⊗ B^{⊗n})^* → A^*\) for each \(n ≥ 0\), the isomorphism \(A^* \cong A\) as well as the inclusion of the quasi-isomorphic subcomplex \(\overline{C}^*(A, B, ε) → \overline{C}^*(A, B, ε)\).

2. Main Result: BV-operator on homotopy \(G\)-algebra

We begin by recalling the notion of a homotopy \(G\)-algebra from [5]. A brace algebra (see [5, Definition 1]) is a graded vector space \(V = \bigoplus_{n≥0} V^n\) with a collection of multilinear operators (braces) \(x[x_1, \ldots, x_n]\) satisfying the following conditions (with \(|x|\) understood to be \(|x|\)):

1. \(\text{deg}(x(x_1, \ldots, x_n)) = \text{deg}(x) + \sum_{i=1}^{n} \text{deg}(x_i) - n\)
2. For homogeneous elements \(x, x_i, \ldots, x_m, y_1, \ldots, y_n\), we have

\[
x[x_1, \ldots, x_m][y_1, \ldots, y_n] = \sum_{0 ≤ i_1 ≤ j_1 ≤ i_2 ≤ j_2 ≤ \cdots ≤ i_m ≤ j_m ≤ n} (-1)^{ε} x\{y_1, \ldots, y_{i_1}, x_1[y_{i_1+1}, \ldots, y_{j_1}], y_{j_1+1}, \ldots, y_m\}
\]

\[
x\{y_{i_m+1}, \ldots, y_{j_m}, y_{j_m+1}, \ldots, y_n\}\]

where \(ε = \sum_{p=1}^{m} |x_p|(|\sum_{q=1}^{p} |y_q||)\) and \(|x| := \text{deg}(x) - 1\).

**Definition 1 (see [5, Definition 2]).** A homotopy \(G\)-algebra consists of the following data:

1. A brace algebra \(V = \bigoplus_{n≥0} V^n\).
2. A dot product of degree zero

\[
V^m ⊗ V^n → V^{m+n} \quad x ⊗ y → x \cdot y
\]

for all \(m, n ≥ 0\).
3. A differential \(d : V^* → V^{*-1}\) of degree one making \(V\) into a DG-algebra with respect to the dot product.
Let us put \( z \) descends to a bracket on the cohomology, it follows that 
\[
[x_1, x_2, \ldots, x_{n+1}] = (d(x_1) \cdot x_2, \ldots, x_{n+1}) + (-1)^{|x_1| \cdot |x_2|} x_1 \cdot (x_2, \ldots, x_{n+1}) - x_1 \cdot x_2, \ldots, x_{n+1}
\]

In particular, a homotopy \( G \)-algebra is equipped with a graded Lie bracket which descends to the cohomology of the corresponding cochain complex \((V^*, d)\) (see [5])
\[
[\cdot, \cdot] : H^m(V^*) \otimes H^n(V^*) \rightarrow H^{m+n-1}(V^*)
\]

The dot product also descends to the cohomology and the bracket with an element becomes a BV-operator, it follows from Definition 2 that
\[
\Delta^n : H^n(V^*) \rightarrow H^{n+1}(V^*)
\]

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\Delta^n : H^n(V^*) \rightarrow H^{n+1}(V^*)
\]

We now introduce the notion of a BV-operator on a homotopy \( G \)-algebra.

**Definition 2.** Let \( V^* = \bigoplus_{n \geq 0} V^n \) be a homotopy \( G \)-algebra, let \( d : V^* \rightarrow V^{*+1} \) be its differential and let \([\cdot, \cdot] : V^n \otimes V^m \rightarrow V^{m+n-1}\) be its Lie bracket. We will say that a family \( \Delta = (\Delta^n : V^n \rightarrow V^{n+1})_{n \geq 0} \) is a BV-operator on \( V^* \) if it satisfies
\[
[f, g] - (-1)^{(|f|+1)m} \Delta(f) \cdot g + (-1)^n f \cdot \Delta(g) - \Delta(f \cdot g)) \in d(V^{m+n-2})
\]

for any cocycles \( f \in Z^n(V^*), g \in Z^m(V^*) \).

If \( V^* \) is a homotopy \( G \)-algebra equipped with a BV-operator \( \Delta \), we now show that the bracket on the Gerstenhaber algebra \( H(V^*) \) is determined by \( \Delta \) in a manner similar to the BV-formalism.

**Theorem 3.** Let \( V^* = \bigoplus_{n \geq 0} V^n \) be a homotopy \( G \)-algebra equipped with a BV-operator \( \Delta = (\Delta^n : V^n \rightarrow V^{n+1})_{n \geq 0} \). Consider \( \overline{f} \in H^n(V^*) \) and \( \overline{g} \in H^m(V^*) \) and choose cocycles \( f \in Z^n(V^*) \) and \( g \in Z^m(V^*) \) corresponding respectively to \( \overline{f} \) and \( \overline{g} \). Then, we have
\[
(\Delta(f) \cdot g + (-1)^n f \cdot \Delta(g) - \Delta(f \cdot g)) \in Z^{m+n-1}(V^*)
\]

The Gerstenhaber bracket on the cohomology of \( V^* \) is now determined by
\[
[\overline{f}, \overline{g}] = (-1)^{(n-1)m} (\Delta(f) \cdot g + (-1)^n f \cdot \Delta(g) - \Delta(f \cdot g)) \in H^{m+n-1}(V^*)
\]

In particular, the right hand side does not depend on the choice of representatives \( f \) and \( g \).

**Proof.** We know that \( f \in Z^n(V^*) \) and \( g \in Z^m(V^*) \). Since the bracket \([\cdot, \cdot] : V^n \otimes V^m \rightarrow V^{m+n-1}\) descends to a bracket on the cohomology, it follows that \([f, g] \in Z^{m+n-1}(V^*) \). Since \( \Delta = (\Delta^n : V^n \rightarrow V^{n+1})_{n \geq 0} \) is a BV-operator, it follows from Definition 2 that
\[
[f, g] - (-1)^{|f|+1|m} \Delta(f) \cdot g + (-1)^n f \cdot \Delta(g) - \Delta(f \cdot g)) \in d(V^{m+n-2})
\]

Let us put \( z_1 = [f, g] \) and \( z_2 = (-1)^{|f|+1|m} \Delta(f) \cdot g + (-1)^n f \cdot \Delta(g) - \Delta(f \cdot g)) \). Since \( z_1 - z_2 \in d(V^{m+n-2}) \), we must have \( z_1 - z_2 \in Z^{m+n-1}(V^*) \). We have already seen that \( z_1 \in Z^{m+n-1}(V^*) \). Hence, \( z_2 \in Z^{m+n-1}(V^*) \). By (4), we know that \( z_1 - z_2 \) is a coboundary and hence the cohomology classes \( \overline{z}_1 = \overline{z}_2 \). The result is now clear. \( \square \)
3. Application: BV-operator on secondary Hochschild cohomology

Let $k$ be a field and $A$ be an algebra over $k$. Let $B$ be a commutative $k$-algebra and $\epsilon : B \to A$ be a morphism of $k$-algebras such that $\epsilon(B) \subseteq Z(A)$, where $Z(A)$ denotes the center of $A$. Let $M$ be an $A$-bimodule such that $\epsilon(b)m = m\epsilon(b)$ for all $b \in B$ and $m \in M$. Following [9, §4.2], we consider the complex $(C^*((A, B, \epsilon); M), \delta')$ whose terms are given by

$$C^n((A, B, \epsilon); M) = \operatorname{Hom}_k\left(A^{\otimes n} \otimes B^{\otimes \frac{n(n-1)}{2}}, M\right)$$

An element in $A^{\otimes n} \otimes B^{\otimes \frac{n(n-1)}{2}}$ will be expressed as a “tensor matrix” of the form

$$\begin{pmatrix}
\begin{array}{ccccccc}
a_1 & b_{1,2} & b_{1,3} & \cdots & b_{1,n-1} & b_{1,n} & b_{1,n+1} \\
1 & a_2 & b_{2,3} & \cdots & b_{2,n-1} & b_{2,n} & b_{2,n+1} \\
1 & 1 & a_3 & \cdots & b_{3,n-1} & b_{3,n} & b_{3,n+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
1 & 1 & 1 & \cdots & 1 & a_n & b_{n,n+1} \\
1 & 1 & 1 & \cdots & 1 & 1 & a_{n+1}
\end{array}
\end{pmatrix} \otimes \begin{pmatrix}
\begin{array}{ccccccc}
a_2 & b_{2,2} & b_{2,3} & \cdots & b_{2,n-1} & b_{2,n} & b_{2,n+1} \\
1 & a_3 & b_{3,3} & \cdots & b_{3,n-1} & b_{3,n} & b_{3,n+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
1 & 1 & 1 & \cdots & 1 & a_n & b_{n,n+1} \\
1 & 1 & 1 & \cdots & 1 & 1 & a_{n+1}
\end{array}
\end{pmatrix} \otimes \cdots \otimes \begin{pmatrix}
\begin{array}{ccccccc}
a_n & b_{n,2} & b_{n,3} & \cdots & b_{n,n-1} & b_{n,n} & b_{n,n+1} \\
1 & a_{n+1} & b_{n+1,3} & \cdots & b_{n+1,n-1} & b_{n+1,n} & b_{n+1,n+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
1 & 1 & 1 & \cdots & 1 & a_n & b_{n,n+1} \\
1 & 1 & 1 & \cdots & 1 & 1 & a_{n+1}
\end{array}
\end{pmatrix}
\right)
$$

where $a_i \in A$ and $b_{i,j} \in B$. The differentials

$$\delta^n : C^n((A, B, \epsilon); M) \to C^{n+1}((A, B, \epsilon); M)$$

may be described as follows

$$\delta^n(f) = a_1 \epsilon(b_{1,2}b_{1,3} \cdots b_{1,n+1})f + \sum_{i=1}^n (-1)^i f$$

$$+ (-1)^{n+1} f$$

$$= \epsilon(b_{1,n+1})a_{n+1}$$

$$+ (-1)^{n+1} f$$

$$= \epsilon(b_{1,n+1})a_{n+1}$$
for \( f \in C^n((A, B, \varepsilon); M) \), \( a_i \in A \), \( b_{i,j} \in B \). The cohomology groups of \((C^*((A, B, \varepsilon); M), \delta^*)\) are known as the secondary Hochschild cohomologies \(H^n((A, B, \varepsilon); M)\) of the triple \((A, B, \varepsilon)\) with coefficients in \(M\) (see [9]).

From [10, Proposition 3.1], we know that the secondary Hochschild complex \(C^*((A, B, \varepsilon) := C^*((A, B, \varepsilon); A)\) carries the structure of a homotopy \(G\)-algebra. This induces a graded Lie bracket

\[
\langle \cdot, \cdot \rangle : H^m(A, B, \varepsilon) \otimes H^n(A, B, \varepsilon) \longrightarrow H^{m+n-1}(A, B, \varepsilon)
\]

on the secondary cohomology. It follows (see [10, Corollary 3.2]) that the secondary cohomology \(H^*(A, B, \varepsilon)\) carries the structure of a Gerstenhaber algebra in the sense of [3].

From now onwards, we always let \( A \) be a finite dimensional \(k\)-algebra equipped with a symmetric, non-degenerate, invariant bilinear form \(\langle \cdot, \cdot \rangle : A \times A \to k\). In particular, \(\langle a_1, a_2 \rangle = \langle a_2, a_1 \rangle\), \(\langle a_1 a_2, a_3 \rangle = \langle a_1, a_2 a_3 \rangle\) for any \(a_1, a_2, a_3 \in A\). For \(i \in \{1, \ldots, n + 1\}\), we define the maps \(\Delta_i : C^{n+1}(A, B, \varepsilon) \to C^n(A, B, \varepsilon)\) as follows:

\[
\langle \Delta_i f \otimes (a_1 b_{1,2} b_{1,3} \ldots b_{1,n}) \rangle, a_{n+1} = \left\langle f \otimes \left( \begin{array}{cccc}
  a_1 & b_{i,1+i} & b_{i,1+i+2} & \ldots b_{i,n} \\
  1 & a_{i+1} & b_{i,i+1} & \ldots b_{i+1,n} \\
  \vdots & \vdots & \vdots & \vdots \\
  1 & 1 & \ldots & a_n \\
  1 & 1 & \ldots & b_{n-1,n} \\
  1 & 1 & \ldots & a_n
\end{array} \right) \right), 1
\]

To clarify the above operator, let us express

\[
\otimes \left( \begin{array}{cccc}
  a_1 & b_{1,2} & b_{1,3} & \ldots b_{1,n} \\
  1 & a_2 & b_{2,3} & \ldots b_{2,n} \\
  \vdots & \vdots & \vdots & \vdots \\
  1 & 1 & \ldots & b_{n-1,n} \\
  1 & 1 & \ldots & a_n
\end{array} \right) = \left( \begin{array}{cc}
  (U(i-1)) & X_{12} \\
  1 & U(n-i-1)
\end{array} \right)
\]

where \(U(k)\) is a square matrix of dimension \(k\). Then, we have

\[
\left\langle \Delta_i f \left( \begin{array}{cc}
  U(i-1) & X_{12} \\
  1 & U(n-i-1)
\end{array} \right), a_{n+1} \right\rangle = \left\langle f \left( \begin{array}{cc}
  U(n-i-1) & 1 \\
  1 & a_{n+1}
\end{array} \right) \right), 1
\]

where \(X_{12}^t\) denotes the transpose of \(X_{12}\). The operator \(\Delta : C^{n+1}(A, B, \varepsilon) \to C^n(A, B, \varepsilon)\) is then defined as

\[
\Delta := \sum_{i=1}^{n+1} (-1)^i \Delta_i.
\]

Following [10, §3], we know that the complex \(C^*((A, B, \varepsilon)\) carries a dot product of degree 0, i.e., for \(f \in C^n(A, B, \varepsilon)\), \(g \in C^m(A, B, \varepsilon)\), we have \(f \cdot g \in C^{n+m}(A, B, \varepsilon)\). We also consider the operations

\[
\circ : C^n(A, B, \varepsilon) \otimes C^m(A, B, \varepsilon) \longrightarrow C^{m+n-1}(A, B, \varepsilon)
\]
and set \( f \circ g := \sum_{i=1}^{n} (-1)^{(i-1)(m-1)} f \circ_i g \) as in [10, §3]. We also set

\[
\rho^1, \rho^2 : C^n(A, B, \varepsilon) \otimes C^m(A, B, \varepsilon) \rightarrow C^{n+m-1}(A, B, \varepsilon)
\]

\[
\rho^1(f \otimes g) := \sum_{i=1}^{m} (-1)^{(i+m-1)} \Delta_i(f \cdot g) \quad \rho^2(f \otimes g) := \sum_{i=m+1}^{m+n} (-1)^{(i+m-1)} \Delta_i(f \cdot g)
\]

for \( f \in C^n(A, B, \varepsilon) \), \( g \in C^m(A, B, \varepsilon) \). It is clear that \( \rho^1(f \otimes g) + \rho^2(f \otimes g) = \Delta(f \cdot g) \).

**Lemma 4.** \( \rho^1(f \otimes g) = (-1)^{nm} \rho^2(g \otimes f) \) for all \( f \in C^n(A, B, \varepsilon) \) and \( g \in C^m(A, B, \varepsilon) \).

**Proof.** This may be verified by direct computation. \( \square \)

**Lemma 5.** Let \( f \in Z^n(A, B, \varepsilon) \), \( g \in Z^m(A, B, \varepsilon) \). Then \( f \circ g - (-1)^{(n-1)m} \Delta(f) \cdot g + (-1)^{(n-1)m} \rho^2(f \otimes g) \) is a coboundary. In fact, if we define \( H \)

\[
H = \sum_{i,j \geq 0, i+j \leq n} (-1)^{(j-1)(m-1)+i(n+m)+1} \Delta_i(f \circ_j g),
\]

then,

\[
\delta H = f \circ g - (-1)^{(n-1)m} \Delta(f) \cdot g + (-1)^{(n-1)m} \rho^2(f \otimes g).
\]

**Proof.** We set, for \( k \geq 0, p \geq 0 \):

\[
T_{k+p}^k = \bigotimes \begin{pmatrix} a_{k+1} \ldots b_{k+1,k+p} \\ \vdots \ldots \vdots \\ 1 \ldots a_{k+p} \end{pmatrix}
\]

We see that

\[
\left\langle \delta \langle f \circ_j g \rangle \right\rangle \bigotimes \begin{pmatrix} a_{1,2} b_{1,3} \ldots b_{1,n+m-1} \\ 1 a_{2,3} \ldots b_{2,n+m-1} \\ \vdots \ldots \vdots \\ 1 1 1 \ldots b_{n+m-2,n+m-1} \\ 1 1 1 \ldots a_{n+m-1} \end{pmatrix}, a_{n+m} \right\rangle
\]

\[
= \left\langle f \right\rangle \begin{pmatrix} a_{i+1} \ldots b_{i+1,i+j} \Pi_{k=0}^{m-1} b_{i+1,i+j+k} \ldots 1 b_{2,i+1} \ldots b_{i-1,i+1} \\ \vdots \ldots \vdots \\ 1 \ldots a_{i+j-1} \Pi_{k=0}^{m-3} b_{i+j-1,i+j+k} \ldots 1 b_{2,i+j-1} \ldots b_{i-1,i+j-1} \\ \vdots \ldots \vdots \\ 1 \ldots 1 \ldots 1 \ldots a_{1} \ldots 1 \\ 1 \ldots 1 \ldots 1 \ldots a_{2} \ldots b_{2,i} \\ 1 \ldots 1 \ldots 1 \ldots 1 \ldots a_{i} \end{pmatrix}, 1 \right\rangle
\]

\[
+ \sum_{\lambda=1}^{n} (-1)^{\lambda} \left\langle f \right\rangle \begin{pmatrix} a_{i+1} \ldots \Pi_{k=0}^{m-1} b_{i+1,i+j+k} \ldots 1 b_{i+1,i+j+k} \ldots b_{i+1,i+1} \\ \vdots \ldots \vdots \\ 1 \ldots \Pi_{k=0}^{m-1} b_{i,j+1,i+j+k} \ldots 1 b_{i,j+1,i+j+k} \ldots b_{i,j+1,i+1} \\ \vdots \ldots \vdots \\ 1 \ldots a_{n+m} \ldots 1 \ldots 1 \\ 1 \ldots 1 \ldots a_{i} \\ \vdots \ldots \vdots \\ 1 \ldots 1 \ldots 1 \ldots \beta_{i} \ldots \beta_{i} \\ 1 \ldots 1 \ldots 1 \ldots 1 \\ 1 \ldots 1 \ldots 1 \ldots a_{i} \end{pmatrix}, 1 \right\rangle
\]
\[ \sum_{\lambda=i+j-2}^{i+j-m-2} (-1)^{\lambda} \left( f^{(1)} \right) = \begin{pmatrix} a_i b_{i,j} & b_{i,j+1} & b_{i,j+2} & \cdots & b_{i,j+m-2} & b_{i,j+m-1} \\ a_{i+1} b_{i,j+1} & b_{i,j+2} & \cdots & b_{i,j+m-2} & b_{i,j+m-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{i+n-1} b_{i,j+n-2} & b_{i,j+n-1} & \cdots & b_{i,j+m-2} & b_{i,j+m-1} \end{pmatrix} \]

\[ \sum_{\lambda=i+j-2}^{i+j-m-2} (-1)^{\lambda} \left( f^{(1)} \right) = \begin{pmatrix} a_i b_{i,j} & b_{i,j+1} & \cdots & b_{i,j+m-2} & b_{i,j+m-1} \\ a_{i+1} b_{i,j+1} & b_{i,j+2} & \cdots & b_{i,j+m-2} & b_{i,j+m-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{i+n-1} b_{i,j+n-1} & \cdots & b_{i,j+m-2} & b_{i,j+m-1} \end{pmatrix} \]

\[ \sum_{\lambda=i+j-1}^{n+m-2} (-1)^{\lambda} \left( f^{(1)} \right) = \begin{pmatrix} a_i b_{i,j} & b_{i,j+1} & \cdots & b_{i,j+m-2} & b_{i,j+m-1} \\ a_{i+1} b_{i,j+1} & b_{i,j+2} & \cdots & b_{i,j+m-2} & b_{i,j+m-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{i+n-1} b_{i,j+n-1} & \cdots & b_{i,j+m-2} & b_{i,j+m-1} \end{pmatrix} \]

\[ (-1)^{n+m-1} \left( f^{(1)} \right) = \begin{pmatrix} a_i b_{i,j} & b_{i,j+1} & \cdots & b_{i,j+m-2} & b_{i,j+m-1} \\ a_{i+1} b_{i,j+1} & b_{i,j+2} & \cdots & b_{i,j+m-2} & b_{i,j+m-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{i+n-1} b_{i,j+n-1} & \cdots & b_{i,j+m-2} & b_{i,j+m-1} \end{pmatrix} \]

where \( \alpha := \varepsilon(b_{1,2} \ldots b_{1,n+m-1})a_{n+m}a_1, \gamma := \varepsilon(b_{1,n+m-1} \ldots b_{n+m-2,n+m-1})a_{n+m}a_{n+m-1}, \beta_\lambda := \varepsilon(b_{\lambda,\lambda+1})a_\lambda a_{\lambda+1} \) for \( 1 \leq \lambda \leq n + m - 2 \) and

\[ \frac{\prod_{i+j-2}^{i+j-m-2}}{i+j-m-2} := \begin{pmatrix} a_{i+j-1} & b_{i+j-1,i+j} & \cdots & b_{i+j-1,i+1} & b_{i+j-1,i+j+2} \\ a_{i+j} & b_{i+j,i+1} & \cdots & b_{i+j,i+1} & b_{i+j,i+j+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{i+j-1} & \cdots & \cdots & \cdots & a_{i+j-2} \\ 1 & \cdots & \cdots & \cdots & a_{i+j-1} \end{pmatrix} \]
We write the entire expression of (6) as
\[
\left\langle \delta(\Delta_i(f \circ g)) \otimes \begin{pmatrix}
\begin{array}{cccc}
1 & b_{1,2} & b_{1,3} & \ldots & b_{1,n+m-1} \\
1 & a_2 & b_{2,3} & \ldots & b_{2,n+m-1} \\
1 & 1 & 1 & \ldots & b_{n+m-2,n+m-1} \\
1 & 1 & 1 & \ldots & a_{n+m-1}
\end{array}
\end{pmatrix}, a_{n+m} \right\rangle = E_1 + E_2 + E_3 + E_4 + E_5 + E_6,
\]

where \( E_k \) denotes the \( k \)-th term in the expression.

We set for \( i, j \geq 1 \) and \( i + j \leq n \),
\[
A_{i,j} := (-1)^{i+1} \left\langle a'_i f, 1 \right\rangle
\]
\[
+ E_3 + (-1)^{i+j-1} \left\langle f, 1 \right\rangle
\]
\[
B_{i,j} := (-1)^{i+j+m-2} \left\langle f, 1 \right\rangle + E_5 + E_6
\]

where \( a'_i = a_i \varepsilon(b_{i,i+1} \ldots b_{i,n+m-1} b_{1,i} \ldots b_{1,i-1}) \). We also set
\[
\eta = a_{i+j-1} \varepsilon(b_{i+j-1,i+j} \ldots b_{i+j-1,n+m-1}) g(T_{i+j-1}^{i+j-1})
\]
\[
\zeta = g(T_{i+j-2}^{i+j-2}) \varepsilon(\prod_{k=1}^{m-2} b_{i+j+k,i+j+m-1})
\]

where

\( C. R. Mathématique, 2020, 358, n° 11-12, 1239-1258 \)
and

\[
C_{i,j} := (-1)^i \left\langle f \left( \begin{array}{cccc}
  a_i & b_{i+1,1,i+1} & \cdots & b_{1,n+m-1} \\
  a_i & b_{i+1,1,i+1} & \cdots & b_{1,n+m-1} \\
  : & : & \cdots & : \\
  1 & 1 & \cdots & 1
\end{array} \right) \right\rangle, a_{n+m} \right\rangle = A_{i,j} + B_{i,j} + C_{i,j}
\]

The first term of \( A_{i,j} \) and that of \( C_{i,j} \) are the same modulo a sign. Using the fact that \( \delta g = 0 \),
the third term of \( A_{i,j} \) and the first term of \( B_{i,j} \) add up to give \( E_4 \). Thus, we have

\[
\left\langle \delta (\Delta_i(f \circ j g)) \left( \begin{array}{cccc}
  a_1 & b_{1,2} & b_{1,3} & \cdots & b_{1,n+m-1} \\
  1 & a_2 & b_{2,3} & \cdots & b_{2,n+m-1} \\
  : & : & \cdots & : & : \\
  1 & 1 & \cdots & b_{n+m-2,n+m-1} & a_{n+m-1}
\end{array} \right) \right\rangle, a_{n+m} \right\rangle = 0
\]

(7)

It may be verified that

\[
(-1)^{i+1} A_{i,j-1} + (-1)^{i+m} B_{i,j} + (-1)^{i+n} C_{i-1,j}
\]

\[
= \left\langle \Delta_i((\delta f) \circ j g)) \left( \begin{array}{cccc}
  a_1 & b_{1,2} & b_{1,3} & \cdots & b_{1,n+m-1} \\
  1 & a_2 & b_{2,3} & \cdots & b_{2,n+m-1} \\
  : & : & \cdots & : & : \\
  1 & 1 & \cdots & a_{n+m} & 1
\end{array} \right) \right\rangle, a_{n+m} \right\rangle = 0
\]

(8)

for \( 2 \leq i \leq n, 2 \leq j \leq n-1 \) and \( i+j \leq n \). The second equality in (8) uses the fact that \( \delta f = 0 \).

For \( i, j \in \{1, \ldots, n+1\} \), we define

\[
A_{i,0} := (-1)^{i+1} \left\langle g \left( T_{i+1}^{i-1} \right) f \right\rangle, 1 \right\rangle, a_{n+m} \right\rangle
\]

\[
C_{0,j} = \left\langle f \left( \begin{array}{cccc}
  a_1 & b_{j+1,1} & \cdots & b_{1,n+m-1} \\
  : & : & \cdots & : \\
  1 & a_j & \cdots & b_{j-1,n+m-1} \\
  1 & 1 & \cdots & a_{n+m}
\end{array} \right) \right\rangle, a_{n+m}, 1 \right\rangle
\]

\[
= \left\langle f \left( \begin{array}{cccc}
  a_1 & b_{j+1} & \cdots & b_{1,n+m-1} \\
  : & : & \cdots & : \\
  1 & a_j & \cdots & b_{j-1,n+m-1} \\
  1 & 1 & \cdots & a_{n+m}
\end{array} \right) \right\rangle, a_{n+m}, 1 \right\rangle
\]
and for \( i \in \{1, \ldots, n\} \), define

\[
B_{i,n-i+1} := (-1)^{n+m+1} \left\langle f \left( \begin{array}{cccc}
\cdots & \cdots & \cdots & \cdots \\
1 & \cdots & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots \\
1 & \cdots & 1 & \cdots \\
\end{array} \right) \right\rangle, 1 \right\rangle.
\]

Thus, \( A_{i,j}, B_{i,j+1}, C_{i-1,j} \) are defined for all the values of \( i, j \) with \( i, j \geq 1 \) and \( i + j \leq n + 1 \). Moreover, it may be verified that

\[
A_{1,j-1} + (-1)^{m+1} B_{1,j} + (-1)^{n+1} C_{0,j} = 0
\]

and

\[
A_{i,n-i} + (-1)^{i+m} B_{i,n-i+1} + (-1)^{i+n} C_{i-1,n-i+1} = 0
\]
Thus, we obtain
\[
0 = \sum_{1 \leq i \leq n, 1 \leq j \leq n-1, i+j \leq n+1} (-1)^{(j-1)(n-1)+i(n+m-1)} \left( (-1)^{i+1} A_{i,j-1} + (-1)^{i+m} B_{i,j} + (-1)^{i+n} C_{i,j} \right) (9)
\]

Rearranging the terms in the above sum, and using equation (7), we get,
\[
0 = \sum_{1 \leq i \leq n, 1 \leq j \leq n, i+j \leq n} (-1)^{(j-1)(n-1)+(i+m-1)} (A_{i,j} + B_{i,j} + C_{i,j})
\]
\[
- \sum_{i=1}^{n} (-1)^{m-1+i(n+m)} A_{i,0} - \sum_{i=1}^{n} (-1)^{n(m+1)+i(n+1)} B_{i,n-i+1} - \sum_{j=1}^{n} (-1)^{(j-1)(m-1)} C_{0,j}
\]
\[
= \left\langle \delta(H) \left( \begin{array}{c}
  a_1 b_{1,2} b_{1,3} \cdots b_{1,n,m-1} \\
  1 a_2 b_{2,3} \cdots b_{2,n,m-1} \\
  \vdots \vdots \vdots \\
  1 1 1 \cdots b_{n+m-2,n+m-1} \\
  1 1 1 \cdots a_{n+m-1}
\end{array} \right), a_{n+m} \right\rangle
\]
\[
- (-1)^{m(n+1)} \left\langle (\partial^2 (f \otimes g)) \left( \begin{array}{c}
  a_1 b_{1,2} b_{1,3} \cdots b_{1,n,m-1} \\
  1 a_2 b_{2,3} \cdots b_{2,n,m-1} \\
  \vdots \vdots \vdots \\
  1 1 1 \cdots b_{n+m-2,n+m-1} \\
  1 1 1 \cdots a_{n+m-1}
\end{array} \right), a_{n+m} \right\rangle
\]
\[
- (-1)^{m(n+1)} \left\langle (\Delta(f)) \cdot g \left( \begin{array}{c}
  a_1 b_{1,2} b_{1,3} \cdots b_{1,n,m-1} \\
  1 a_2 b_{2,3} \cdots b_{2,n,m-1} \\
  \vdots \vdots \vdots \\
  1 1 1 \cdots b_{n+m-2,n+m-1} \\
  1 1 1 \cdots a_{n+m-1}
\end{array} \right), a_{n+m} \right\rangle
\]
\[
- \left\langle (f \circ g) \left( \begin{array}{c}
  a_1 b_{1,2} b_{1,3} \cdots b_{1,n,m-1} \\
  1 a_2 b_{2,3} \cdots b_{2,n,m-1} \\
  \vdots \vdots \vdots \\
  1 1 1 \cdots b_{n+m-2,n+m-1} \\
  1 1 1 \cdots a_{n+m-1}
\end{array} \right), a_{n+m} \right\rangle.
\]

**Proposition 6.** The family \( \Delta = \{ \Delta^* : C^*(A,B,\varepsilon) \to C^{*+1}(A,B,\varepsilon) \} \) determines a BV-operator on the homotopy G-algebra \( C^*(A,B,\varepsilon) \).

**Proof.** We consider \( f \in Z^k(A,B,\varepsilon) \) and \( g \in Z^m(A,B,\varepsilon) \). By definition (see [10, §3]), we know that
\[
[f,g] = f \circ g - (-1)^{(n-1)(m-1)} g \circ f \in C^{m+n-1}(A,B,\varepsilon)
\]
(10)

Applying Lemma 5, we know that the cochains
\[
f \circ g - (-1)^{(n-1)m} \Delta(f) \cdot g + (-1)^{(n-1)m} \rho^2(f \otimes g)
\]
\[
g \circ f - (-1)^{(m-1)n} \Delta(g) \cdot f + (-1)^{(m-1)n} \rho^2(g \otimes f)
\]
are coboundaries. From (10), it now follows that
\[
[f,g] - (-1)^{(n-1)m} \Delta(f) \cdot g + (-1)^{(n-1)m} \rho^2(f \otimes g) + (-1)^{(m-1)n} \Delta(g) \cdot f + (-1)^{(m-1)n} \rho^2(g \otimes f)
\]
(11)
is a coboundary. Applying Lemma 4, it follows from (11) that
\[
[f,g] - (-1)^{(n-1)m} \Delta(f) \cdot g + (-1)^{(n-1)m} \rho^2(f \otimes g) = \Delta(f \cdot g)
\]
is a coboundary. Since \( \rho^1(f \otimes g) + \rho^2(f \otimes g) = \Delta(f \cdot g) \), we get
\[
[f,g] - (-1)^{(n-1)m} \Delta(f) \cdot g + (-1)^{(n-1)m} \Delta(f \cdot g) + (-1)^{(m-1)n} \Delta(g) \cdot f
\]
is a coboundary. Using the fact that the dot product is graded commutative, we can put \( \Delta(g) \cdot f = (-1)^{(m-1)}f \cdot \Delta(g) \). The result is now clear. \( \square \)

**Theorem 7.** For secondary cohomology classes \( \overline{f} \in H^r(A, B, \varepsilon) \) and \( \overline{g} \in H^m(A, B, \varepsilon) \), the Gerstenhaber bracket is determined by

\[
[\overline{f}, \overline{g}] = (-1)^{(n-1)m}(\Delta(f) \cdot g + (-1)^nf \cdot \Delta(g) - \Delta(f) \cdot g) \in H^{m+n-1}(A, B, \varepsilon)
\]

Here \( f \) and \( g \) are any cocycles representing the classes \( \overline{f} \) and \( \overline{g} \) respectively.

**Proof.** This follows directly by applying Theorem 3 and Proposition 6. \( \square \)

It is natural to ask whether the BV-operator determined by \( \Delta = |\Delta^*| : C^*(A, B, \varepsilon) \to C^{*+1}(A, B, \varepsilon) \) induces a BV-algebra structure on the secondary Hochschild cohomology \( H^*(A, B, \varepsilon) \). In the special case of \( B = k \), we are reduced to ordinary Hochschild cohomology and hence \( \Delta \) determines a BV-algebra structure on \( H^*(A, A) \). However, this is not true in general because \( \Delta^* : C^*(A, B, \varepsilon) \to C^{*+1}(A, B, \varepsilon) \) does not commute with the differentials. For instance, we take \( n = 2 \) and ask when the following diagram commutes:

\[
\begin{array}{ccc}
C^2(A, B, \varepsilon) & \xrightarrow{\delta^2} & C^3(A, B, \varepsilon) \\
\Delta^2 & & -\Delta^3 \\
C^1(A, B, \varepsilon) & \xrightarrow{\delta^1} & C^2(A, B, \varepsilon)
\end{array}
\]

For any \( a_1, a_2, a_3 \in A \), \( b_{1,2} \in B \) and \( f \in C^2(A, B, \varepsilon) \), we have

\[
\left\langle \left(-\Delta^3 \delta^2 f\right) \left[ a_1 b_{1,2} a_2, a_3 \right] \right\rangle = -\left\langle \left(\delta^2 f\right) \left[ a_1 b_{1,2} 1 a_2, a_3 \right], 1 \right\rangle - \left\langle \left(\delta^2 f\right) \left[ a_2 b_{1,2} 1 a_3, 1 \right], a_1 \right\rangle - \left\langle \left(\delta^2 f\right) \left[ a_3 1 b_{1,2}, 1 \right], a_1 a_2 \right\rangle
\]

\[
= -\left\langle a_1 \varepsilon(b_{1,2}) f \left[ a_2 1 a_3, 1 \right], 1 \right\rangle + \left\langle f \left[ a_1 a_2 \varepsilon(b_{1,2}) 1 a_3, a_1 \right], 1 \right\rangle - \left\langle f \left[ a_1 b_{1,2} a_2, a_3 \right], 1 \right\rangle + \left\langle f \left[ a_1 b_{1,2} a_3, a_2 \right], 1 \right\rangle
\]

\[
- \left\langle a_2 \varepsilon(b_{1,2}) f \left[ a_3 1 a_1, 1 \right], 1 \right\rangle + \left\langle f \left[ a_2 a_3 b_{1,2} 1 a_1, a_2 \right], 1 \right\rangle - \left\langle f \left[ a_2 b_{1,2} a_3 a_1, 1 \right], 1 \right\rangle + \left\langle f \left[ a_2 1 a_3 \varepsilon(b_{1,2}), 1 \right], a_1 a_2 \right\rangle
\]

\[
- \left\langle a_3 f \left[ a_1 b_{1,2} 1 a_2, 1 \right], 1 \right\rangle + \left\langle f \left[ a_3 a_1 b_{1,2} 1 a_2, 1 \right], a_1 a_2 \varepsilon(b_{1,2}) \right\rangle - \left\langle f \left[ a_3 1 a_1 a_2 \varepsilon(b_{1,2}), 1 \right], a_2 \right\rangle + \left\langle f \left[ a_3 a_1 b_{1,2} 1 a_2, 1 \right], a_1 a_2 \varepsilon(b_{1,2}) \right\rangle
\]

Now, using the properties of the inner product \( \left\langle \cdot, \cdot \right\rangle \) on \( A \) we see that the first term cancels with the eighth, the fourth term cancels with the ninth, the fifth term cancels with the twelfth. Thus, the above expression reduces to

\[
\left\langle f \left[ a_1 a_2 \varepsilon(b_{1,2}) 1 a_3, 1 \right], 1 \right\rangle - \left\langle f \left[ a_1 b_{1,2} a_2 a_3, 1 \right], 1 \right\rangle + \left\langle f \left[ a_2 a_3 b_{1,2} 1 a_1, a_2 \right], 1 \right\rangle - \left\langle f \left[ a_2 b_{1,2} a_3 a_1, 1 \right], a_1 \right\rangle
\]

\[
+ \left\langle f \left[ a_3 a_1 b_{1,2} 1 a_2, 1 \right], a_1 a_2 \varepsilon(b_{1,2}) \right\rangle - \left\langle f \left[ a_3 1 a_1 a_2 \varepsilon(b_{1,2}), 1 \right], a_2 \right\rangle \quad (13)
\]
On the other hand, we have
\[
\langle (\delta^1 \Delta^2 f) \left( \begin{array}{c}
1 \\
a_1 \\
da_2 \\
3 \\
a_3
\end{array} \right), a_3 \rangle \\
= \langle a_1 \varepsilon (b_{12}) (\Delta^2 f) (a_3), a_3 \rangle - \langle (\Delta^2 f) (a_1 a_2 \varepsilon (b_{12})), a_3 \rangle + \langle (\Delta^2 f) (a_1) a_2 \varepsilon (b_{12}), a_3 \rangle \\
= \langle (\Delta^2 f) (a_2), a_3 \varepsilon (b_{12}) \rangle - \langle (\Delta^2 f) (a_1 a_2 \varepsilon (b_{12})), a_3 \rangle + \langle (\Delta^2 f) (a_1), a_2 a_3 \varepsilon (b_{12}) \rangle \\
= -\left( f \left( \begin{array}{c}
a_2 \\
a_3 \\
a_2 a_3 \varepsilon (b_{12}) \end{array} \right), 1 \right) + \left( f \left( \begin{array}{c}
a_2 a_3 \varepsilon (b_{12}) \\
a_2 \\
a_3 \end{array} \right), 1 \right) + \left( f \left( \begin{array}{c}
a_1 \\
a_3 \\
a_2 a_3 \varepsilon (b_{12}) \end{array} \right), 1 \right) \\
- \left( f \left( \begin{array}{c}
a_3 \\
a_1 a_3 \varepsilon (b_{12}) \\
a_2 a_3 \varepsilon (b_{12}) \end{array} \right), 1 \right) - \left( f \left( \begin{array}{c}
a_1 \\
a_3 \\
a_2 a_3 \varepsilon (b_{12}) \end{array} \right), 1 \right) + \left( f \left( \begin{array}{c}
a_2 a_3 \varepsilon (b_{12}) \\
a_1 \\
a_3 \end{array} \right), 1 \right) \tag{14}
\]
From the expressions in (13) and (14), it is clear that the diagram (12) does not commute in general, even if we take A to be commutative and B = A.

4. Relation with extra degeneracy and norm operator

We continue with A being a finite dimensional k-algebra equipped with a symmetric, non-degenerate and invariant bilinear form \(\langle \cdot, \cdot \rangle : A \times A \to k\) and B a commutative k-algebra with a morphism of k-algebras \(\varepsilon : B \to A\) such that \(\varepsilon (B) \subseteq Z (A)\). In particular, the non-degenerate pairing on A induces mutually inverse isomorphisms
\[
\phi : A^* \xrightarrow{\cong} A \quad \phi^{-1} : A \xrightarrow{\cong} A^*
\tag{15}
\]
We let \(C^* (A, M)\) denote the ordinary Hochschild complex of A with coefficients in an A-bimodule M and its cohomology by \(H^* (A, M)\). In particular, we may set \(M = A^*\) equipped with the A-bimodule structure \((a' f a'') (a) := f (a'' a a')\) for \(f \in A^* = \text{Hom} (A, k)\) and \(a, a', a'' \in A\). In that case, the terms \(C^n (A, A^*)\) of the Hochschild complex \(C^* (A, A^*)\) may also be written as \(C^n (A, A^*) \cong \text{Hom} (A^{\otimes n+1}, k)\). We denote by \(\tilde{C}^* (A)\) the corresponding complex defined by setting \(\tilde{C}^n (A) := \text{Hom} (A^{\otimes (n+1)}, k)\) for \(n \geq 0\).

From Tradler [11], we know that the operator \(\Delta^* : C^* (A, A) \to C^{*-1} (A, A)\) on Hochschild cochains inducing the BV-structure on \(H^* (A, A)\) fits into the following commutative diagram with the duals of Hochschild chains
\[
\begin{array}{ccc}
\tilde{C}^{n+1} (A) & \xrightarrow{Ns} & \tilde{C}^n (A) \\
\cong & & \cong \\
C^* (A, A^*) & & C^{*-1} (A, A^*) \\
\phi^* & \equiv & \phi^{-1} \\
C^* (A, A) & \xrightarrow{\Delta^*} & C^{*-1} (A, A) \\
\end{array}
\tag{16}
\]
Here, each \(\phi^* : C^* (A, A^*) \to C^* (A, A)\) is the isomorphism induced by \(\phi : A^* \xrightarrow{\cong} A\), while s and N respectively are the usual extra degeneracy and norm operators given by
\[
s : \tilde{C}^{n+1} (A) \longrightarrow \tilde{C}^n (A) \quad \text{such that} \quad (sf) (a_1, \ldots, a_n) := f (a_1, \ldots, a_n, 1) \\
N := 1 + \lambda + \cdots + \lambda^n : \tilde{C}^n (A) \longrightarrow \tilde{C}^n (A) \quad \text{such that} \quad (\lambda \cdot f) (a_0, \ldots, a_n) := (-1)^n f (a_n, a_0, \ldots, a_{n-1})
\tag{17}
\]
If we pass to the normalized Hochschild complex which is a quasi-isomorphic subcomplex of \(C^* (A, A^*)\), then (16) induces the following commutative diagram
\[
\begin{array}{ccc}
H^* (A, A^*) & \xrightarrow{B^*} & H^{*-1} (A, A^*) \\
\phi^* & \equiv & \phi^{-1} \\
H^* (A, A) & \xrightarrow{\Delta^*} & H^{*-1} (A, A) \\
\end{array}
\tag{18}
\]
where \( B^* : H^*(A, A^*) \to H^{*-1}(A, A^*) \) is the standard Connes operator.

In the case of secondary Hochschild cohomology, we have shown in Section 3 that \( \Delta^* \) is not in general a morphism of complexes, i.e., it does not descend to cohomology. We will now show that the operator \( \Delta^* \) on secondary Hochschild cohomology \( H^*(A, B, \varepsilon) \) fits into a commutative diagram similar to (18).

In [8], Laubacher, Staic and Stancu have introduced a co-simplicial module \( \tilde{C}^*(A, B, \varepsilon) \) which is used to compute the secondary Hochschild cohomology associated to the triple \((A, B, \varepsilon)\). The terms of this co-simplicial module are given by

\[
\tilde{C}^*(A, B, \varepsilon) := \left\{ \text{Hom}\left( A^{\otimes n} \otimes B^{\otimes \frac{n(n-1)}{2}}, \text{Hom}(A \otimes B^n, k) \right) \right\}_{n \geq 0} \quad (19)
\]

It is important to note (see [8, Remark 4.7]) that despite the similar names, the complex \( \tilde{C}^*(A, B, \varepsilon) \) cannot be expressed as the secondary Hochschild complex of \((A, B, \varepsilon)\) with coefficients in some \( A \)-bimodule. This is because the “coefficient module” \( \text{Hom}(A \otimes B^n, k) \) appearing in (19) varies with \( n \).

In addition, the cosimplicial module \( \tilde{C}^*(A, B, \varepsilon) \) is equipped with a cyclic operator, which can be used to compute the secondary cyclic cohomology associated to the triple \((A, B, \varepsilon)\). Using the isomorphisms

\[
\Psi^n : \text{Hom}\left( A^{\otimes n} \otimes B^{\otimes \frac{n(n-1)}{2}}, \text{Hom}(A \otimes B^n, k) \right) \xrightarrow{\cong} \text{Hom}\left( A^{\otimes (n+1)} \otimes B^{\otimes \frac{(n+1)(n)}{2}}, k \right) \quad (20)
\]

given by

\[
(\Psi^n g) \equiv \left( \begin{array}{cccc}
  a_0 & b_{0,1} & b_{0,2} & \ldots & b_{0,n-1} & b_{0,n} \\
  1 & a_1 & b_{1,2} & \ldots & b_{1,n-1} & b_{1,n} \\
  1 & 1 & a_2 & \ldots & b_{2,n-1} & b_{2,n} \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  1 & 1 & 1 & \ldots & a_{n-1} & b_{n-1,n} \\
  1 & 1 & 1 & \ldots & 1 & a_n
\end{array} \right)
\]

we first transfer the cyclic operator from [8] to a complex \( \tilde{C}^*(A, B, \varepsilon) \) whose terms are given by

\[
\{ \tilde{C}^n(A, B, \varepsilon) := \text{Hom}(A^{\otimes (n+1)} \otimes B^{\otimes \frac{(n+1)(n)}{2}}, k) \}_{n \geq 0} \quad (22)
\]

**Lemma 8.** For each \( n \geq 0 \), there is an action of the cyclic group \( \mathbb{Z}_{n+1} = \langle \lambda \rangle \) on the \( k \)-space \( \tilde{C}^n(A, B, \varepsilon) = \text{Hom}(A^{\otimes (n+1)} \otimes B^{\otimes \frac{(n+1)(n)}{2}}, k) \) given by

\[
(\lambda \cdot f) \equiv \left( \begin{array}{cccc}
  a_0 & b_{0,1} & b_{0,2} & \ldots & b_{0,n-1} & b_{0,n} \\
  1 & a_1 & b_{1,2} & \ldots & b_{1,n-1} & b_{1,n} \\
  1 & 1 & a_2 & \ldots & b_{2,n-1} & b_{2,n} \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  1 & 1 & 1 & \ldots & a_{n-1} & b_{n-1,n} \\
  1 & 1 & 1 & \ldots & 1 & a_n
\end{array} \right) = (-1)^n f \equiv \left( \begin{array}{cccc}
  a_n & b_{0,n} & b_{1,n} & \ldots & b_{n-2,n} & b_{n-1,n} \\
  1 & a_0 & b_{0,1} & \ldots & b_{0,n-2} & b_{0,n-1} \\
  1 & 1 & a_1 & \ldots & b_{1,n-2} & b_{1,n-1} \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  1 & 1 & 1 & \ldots & a_{n-2} & b_{n-2,n-1} \\
  1 & 1 & 1 & \ldots & 1 & a_{n-1}
\end{array} \right)
\]

for any \( f \in \tilde{C}^n(A, B, \varepsilon) \), \( a_i \in A \) and \( b_{i,j} \in B \).

**Proof.** This is clear from the definition in [8, §4.2] and the isomorphisms in (20). \( \square \)
The norm operator $N: \tilde{C}^n(A,B,\varepsilon) \to \tilde{C}^n(A,B,\varepsilon)$ is then defined as $N = 1 + \lambda + \cdots + \lambda^n$. The extra degeneracy $s: \tilde{C}^n(A,B,\varepsilon) \to \tilde{C}^{n-1}(A,B,\varepsilon)$ is given by

$$ (sf) \begin{pmatrix} a_0 & b_{0,1} & b_{0,2} & \cdots & b_{0,n-2} & b_{0,n-1} \\ 1 & a_1 & b_{1,2} & \cdots & b_{1,n-2} & b_{1,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & a_{n-2} & b_{n-2,n-1} \\ 1 & 1 & 1 & \cdots & 1 & a_{n-1} \end{pmatrix} = f \begin{pmatrix} a_0 & b_{0,1} & b_{0,2} & \cdots & b_{0,n-2} & b_{0,n-1} \\ 1 & a_1 & b_{1,2} & \cdots & b_{1,n-2} & b_{1,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & a_{n-2} & b_{n-2,n-1} \\ 1 & 1 & 1 & \cdots & 1 & a_{n-1} \end{pmatrix} $$

for any $f \in \tilde{C}^n(A,B,\varepsilon), a_i \in A$ and $b_{i,j} \in B$.

For $n \geq 0$, let $a^n : (A \otimes B^{\otimes n})^* \to A^*$ be the map defined by $p \mapsto \overline{p} := a^n(p)$, where $\overline{p}(a) := p(a \otimes 1_B \otimes \cdots \otimes 1_B)$. We denote by $a^* : \tilde{C}^*(A,B,\varepsilon) \to C^*((A,B,\varepsilon); A^*)$ the induced map.

We also let $\Phi^* : C^*((A,B,\varepsilon); A^*) \to C^*((A,B,\varepsilon); A)$ be the map induced by the isomorphism $\phi : A^* \to A$ and $\Phi^{**} : C^*((A,B,\varepsilon); A^*) \to C^*((A,B,\varepsilon); A^*)$ be the map induced by $\phi^{-1}$. It may also be verified that the inverse $\Psi^* : \text{Hom}(A^{\otimes(n+1)} \otimes B^{\otimes \frac{n(n+1)}{2}}, k) \to \text{Hom}(A^{\otimes n} \otimes B^{\otimes \frac{n(n-1)}{2}}, \text{Hom}(A \otimes B^n, k))$ of the map in (21) is given by

$$(\Psi^n f) \begin{pmatrix} a_1 & b_{1,2} & \ldots & b_{1,n-1} & b_{1,n} \\ 1 & a_2 & \ldots & b_{2,n} & b_{2,n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & a_{n-1} & b_{n-1,n} \\ 1 & 1 & \cdots & 1 & a_{n} \end{pmatrix} \begin{pmatrix} a_{n+1} \\ b_{1,n+1} \\ \vdots \\ b_{n-1,n+1} \end{pmatrix} = f \begin{pmatrix} a_1 & b_{1,2} & \ldots & b_{1,n-1} & b_{1,n} \\ 1 & a_2 & \ldots & b_{2,n} & b_{2,n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & a_{n-1} & b_{n-1,n} \\ 1 & 1 & \cdots & 1 & a_{n} \end{pmatrix} \begin{pmatrix} a_{n+1} \\ b_{1,n+1} \\ \vdots \\ b_{n-1,n+1} \end{pmatrix} $$

Proposition 9. Let $A$ be a finite dimensional $k$-algebra equipped with a symmetric, non-degenerate and invariant bilinear form $\langle \cdot, \cdot \rangle : A \times A \to k$, and $B$ be a commutative $k$-algebra with a morphism of $k$-algebras $\varepsilon : B \to A$ such that $\varepsilon(B) \subseteq Z(A)$. Then, the following diagram commutes:

$$ \begin{array}{ccc} \tilde{C}^*(A,B,\varepsilon) & \xrightarrow{N} & \tilde{C}^{*1}(A,B,\varepsilon) \\ \Phi^{n*} \downarrow \quad \scriptstyle \phi^{*^{-1}} a^n \downarrow \quad \scriptstyle \phi^{*^{-1}} a^n \downarrow & \quad & \scriptstyle \phi^{*^{-1}} a^n \downarrow \quad \scriptstyle \phi^{*^{-1}} a^n \downarrow \\ C^*((A,B,\varepsilon); A) & \xrightarrow{\Delta^*} & C^{*1}((A,B,\varepsilon); A) \end{array} $$

(24)

Proof. We will show that for any $n \geq 0$, the following diagram is commutative:

$$ \begin{array}{ccc} \text{Hom}(A^{\otimes(n+1)} \otimes B^{\otimes \frac{n(n+1)}{2}}, k) & \xrightarrow{N} & \text{Hom}(A^{\otimes n} \otimes B^{\otimes \frac{n(n-1)}{2}}, k) \\ \Psi^{*n} \downarrow \quad \scriptstyle \psi^{*n-1} \downarrow & \quad & \scriptstyle \psi^{*n-1} \downarrow \quad \scriptstyle \psi^{*n-1} \downarrow \\ \text{Hom}(A^{\otimes n} \otimes B^{\otimes \frac{n(n-1)}{2}}, (A \otimes B^{\otimes n})^*) & \xrightarrow{a^n} & \text{Hom}(A^{\otimes(n-1)} \otimes B^{\otimes \frac{(n-1)(n-2)}{2}}, (A \otimes B^{\otimes(n-1)})^*) \\ \phi^n \downarrow & \quad & \phi^n \downarrow \\ \text{Hom}(A^{\otimes n} \otimes B^{\otimes \frac{n(n-1)}{2}}, A^*) & \xrightarrow{\Delta} & \text{Hom}(A^{\otimes(n-1)} \otimes B^{\otimes \frac{(n-1)(n-2)}{2}}, A^*) \end{array} $$

(25)
Since $\Phi^{n-1}$ is an isomorphism, it suffices to check that this diagram is commutative when composed with $\Phi^{n-1}: \text{Hom}\left(A^{\otimes (n-1)} \otimes B^{\otimes \frac{n(n-1)}{2}}, A\right) \to \text{Hom}\left(A^{\otimes (n-1)} \otimes B^{\otimes \frac{n(n-1)}{2}}, A^*\right)$. Let $f \in \text{Hom}(A^{\otimes (n+1)} \otimes B^{\otimes \frac{n(n+1)}{2}}, k)$. Then, for $a_i \in A$ and $b_{i,j} \in B$, we have

$$\langle (\Delta \circ \Phi^i \circ \alpha^j \circ \Psi^m(f)) \otimes \left(\begin{array}{cccc} a_1 & b_{1,2} & b_{1,3} & \ldots & b_{1,n-2} & b_{1,n-1} \\ 1 & a_2 & b_{2,3} & \ldots & b_{2,n-2} & b_{2,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \ldots & a_{n-2} & b_{n-2,n-1} \\ 1 & 1 & 1 & \ldots & 1 & a_{n-1} \end{array}\right), a_n \rangle$$

$$= \sum_{i=1}^{n} (-1)^{(n-1)i} \left(\begin{array}{cccc} a_i & b_{i,1} & b_{i,2} & \ldots & b_{i,n-1} & 1 \\ 1 & a_{i+1} & b_{i+1,2} & \ldots & b_{i+1,n-2} & b_{i+1,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \ldots & a_{n-1} & 1 \\ 1 & 1 & 1 & \ldots & 1 & a_{n-1} \end{array}\right) \left(\begin{array}{c} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{array}\right)$$

$$= \sum_{i=1}^{n} (-1)^{(n-1)i} \left(\begin{array}{cccc} a_i & b_{i,1} & b_{i,2} & \ldots & b_{i,n-1} & 1 \\ 1 & a_{i+1} & b_{i+1,2} & \ldots & b_{i+1,n-2} & b_{i+1,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \ldots & a_{n-1} & 1 \\ 1 & 1 & 1 & \ldots & 1 & a_{n-1} \end{array}\right) \left(\begin{array}{c} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{array}\right)$$

$$= \sum_{i=1}^{n} (-1)^{(n-1)i} \left(\begin{array}{cccc} a_i & b_{i,1} & b_{i,2} & \ldots & b_{i,n-1} & 1 \\ 1 & a_{i+1} & b_{i+1,2} & \ldots & b_{i+1,n-2} & b_{i+1,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \ldots & a_{n-1} & 1 \\ 1 & 1 & 1 & \ldots & 1 & a_{n-1} \end{array}\right) \left(\begin{array}{c} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{array}\right)$$
This proves the result. □
We now let $\overline{C}^\ast(A, B, \varepsilon)$ denote the normalized complex associated to the cosimplicial module $\overline{C}^\ast(A, B, \varepsilon)$ given in (19). Again, using the isomorphisms in (20), the complex $\overline{C}^\ast(A, B, \varepsilon)$ becomes isomorphic to the normalized complex $\overline{C}^\ast(A, B, \varepsilon)$ whose terms are given by

\[
\overline{C}^n(A, B, \varepsilon) := \text{Ker} \left( \overline{C}^n(A, B, \varepsilon) \to \bigoplus_{j=0}^{n-1} \overline{C}^{n-1}(A, B, \varepsilon) \right)
\]

where $s_j : \overline{C}^n(A, B, \varepsilon) \to \overline{C}^{n-1}(A, B, \varepsilon)$ for $0 \leq j \leq n - 1$ is the degeneracy

\[
(s_j) = \left( \begin{array}{c} a_1 b_{1,2} \ldots b_{1,n} \\ 1 a_2 \ldots b_{2,n} \\ \vdots \vdots \vdots \\ 1 1 \ldots a_{n-1} b_{n-1,n} \\ 1 1 \ldots 1 \end{array} \right) = \left( \begin{array}{c} a_1 b_{1,2} \ldots b_{1,j} 1 b_{1,j+1} \ldots b_{1,n} \\ 1 a_2 \ldots b_{2,j} 1 b_{2,j+1} \ldots b_{2,n} \\ \vdots \vdots \vdots \\ 1 1 \ldots a_{j-1} b_{j,n-1} 1 a_{j+1} \ldots a_{n-1} b_{n-1,n} \\ 1 1 \ldots 1 \end{array} \right)
\]

**Theorem 10.** Let $A$ be a finite dimensional $k$-algebra equipped with a symmetric, non-degenerate and invariant bilinear form $(\cdot, \cdot) : A \times A \to k$ and $B$ be a commutative $k$-algebra with a morphism of $k$-algebras $\varepsilon : B \to A$ such that $\varepsilon(B) \subseteq Z(A)$. Then, the following diagram commutes:

\[
\begin{array}{c}
\overline{C}^\ast(A, B, \varepsilon) \xrightarrow{B} \overline{C}^{-1}(A, B, \varepsilon) \\
\overline{C}^\ast(A, B, \varepsilon) \xrightarrow{\phi^* \circ \alpha^*} \overline{C}^{-1}(A, B, \varepsilon) \\
C^\ast((A, B, \varepsilon); A) \xrightarrow{\Delta^*} C^{-1}((A, B, \varepsilon); A)
\end{array}
\]

where $B$ is Connes’ operator.

**Proof.** The commutativity of the lower square has already been shown in Proposition 9. By definition, Connes’ operator on the complex $\overline{C}^\ast(A, B, \varepsilon)$ is given by $B = Ns(1 - \lambda)$ which reduces to $Ns$ on the normalized complex $\overline{C}^\ast(A, B, \varepsilon)$. Hence, it may be directly verified that the upper diagram commutes. \hfill \square

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**References**


