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Partial differential equations / *Équations aux dérivées partielles*

Smooth traveling-wave solutions to the inviscid surface quasi-geostrophic equations

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Abstract. In a recent article by Gravejat and Smets [7], it is built smooth solutions to the inviscid surface quasi-geostrophic equation that have the form of a traveling wave. In this article we work back on their construction to provide similar solutions to a more general class of quasi-geostrophic equation where the half-laplacian is replaced by any fractional laplacian.

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1. Presentation of the problem

1.1. *The quasi-geostrophic equations*

We consider the general transport equation for the vorticity of an incompressible fluid in dimension 2.

$$\frac{\partial \theta}{\partial t} + v \cdot \nabla \theta = 0, \tag{1}$$

where $\theta : \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}$ is the called the *active scalar* and $v : \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}$ is the *velocity* of the fluid This equation tells that the active scalar is transported by the induced velocity. Since this velocity v is divergence free (incompressibility condition), it is convenient to relate v and θ through a stream function $\psi : \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}$. The generalized inviscid surface quasi-geostrophic equation corresponds to a stream function that verifies

$$v = \nabla^{\perp} \psi \qquad \text{and} \qquad (-\Delta)^s \psi = \theta, \tag{2}$$

with $s \in \,]0,1[$ and $\perp$ denotes the rotation in the plane of angle $\frac{\pi}{2}$. The three equations formed by (1) and (2) are the generalized inviscid surface quasi-geostrophic equations. If we consider formally the particular case $s = 1$, we obtain the well-known 2D Euler equation written in term of vorticity and stream function. Another important case is $s = \frac{1}{2}$ which correspond to the work made in [7] that we generalize here. This case is the standard surface quasi-geostrofic equation that first appeared as a limit model in the context of geophysical flows [9, 12]. These equations are used to model a fluid in a rotating frame with stratified density and velocity and that is submitted to Brunt–Väisälä thermal oscillations. This models leads to (1)–(2) using the Cafferelli–Silverstre theory for fractional Laplace operator [2]. The case of the exponent $s = \frac{1}{2}$ corresponds to the case of a Brunt–Väisälä frequency N that does not depend on the height. Other exponents for the fractional Laplace operator corresponds to different vertical profiles for the frequency N [6, §1]. These equations has been intensely investigated since the work of Constantin, Majda and Tabak [4] on the case $s = \frac{1}{2}$ where they pointed out the mathematical links that arises between (SQG-$\frac{1}{2}$) and the Euler equation in dimension 3. Besides stationary solution, given by a radially symmetric rearrangement on the active scalar, the only two known examples of global smooth solutions where built by Castro, Córdoba and Gómez-Serrano [3] on the one hand and by Gravejat and Smets [7] on the other hand with two different techniques. The article of Castro, Córdoba and Gómez-Serrano also provides a wide bibliography related on SQG and its Cauchy problem. In this work we generalize the result and the construction provided by [7] to the more general equations (1)–(2) with a fixed $s \in]0,1[$. The idea consists in looking for solutions that have the form of traveling waves with a positive speed c in direction x_2. In short, solutions of the form

$$\theta(x_1,x_2,t) = \Theta(x_1,x_2-ct), \qquad v(x_1,x_2,t) = V(x_1,x_2-ct), \qquad \psi(x_1,x_2,t) = \Psi(x_1,x_2-ct), \tag{3}$$

with $\Theta : \mathbb{R}^2 \to \mathbb{R}$, $V : \mathbb{R}^2 \to \mathbb{R}^2$, and $\Psi : \mathbb{R}^2 \to \mathbb{R}$. We inject this form of solution in (1)–(2) and we get

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}(\Theta(x_1,x_2-ct)) + V(x_1,x_2-ct).\nabla\Theta(x_1,x_2-ct) \\ &= -ce_2.\nabla\Theta(x_1,x_2-ct) + V(x_1,x_2-ct).\nabla\Theta(x_1,x_2-ct) \\ &= -ce_2.\nabla\Theta(x_1,x_2-ct) + \nabla^{\perp}\Psi(x_1,x_2-ct).\nabla\Theta(x_1,x_2-ct), \end{aligned} \tag{4}$$

where (e_1,e_2) denotes the canonical basis of $\mathbb{R}^2$. This leads to the orthogonality condition

$$(\nabla\psi - ce_1)^{\perp} \cdot \nabla\Theta = 0, \tag{5}$$

with the remark that $e_1^{\perp} = e_2$. In other words, the two vectors $\nabla\Theta$ and $\nabla\Psi - ce_1$ must be collinear. Following an idea from Arnold [1], Condition (5) is immediately verified if Θ has the form

$$\Theta(x) = f(\Psi(x) - cx_1 - k). \tag{6}$$

Indeed, in this case

$$\nabla\Theta(x) = f'(\Psi(x) - cx_1 - k) \cdot (\nabla\Psi(x) - ce_1) \tag{7}$$

which does give (5). We now consider the ansatz of a symmetry relatively to the x_2-axis that takes the form

$$\Psi(-x_1,x_2) = -\Psi(x_1,x_2). \tag{8}$$

This implies that $\Theta(-x_1,x_2) = -\Theta(x_1,x_2)$ and if we denote $V = (V_1,V_2)$ the two components of the velocity profile, then $V_1(-x_1,x_2) = -V_1(x_1,x_2)$ and $V_2(-x_1,x_2) = V_2(x_1,x_2)$. More precisely, we impose the following ansatz

$$\Theta(x_1,x_2) = \begin{cases} f(\Psi(x_1,x_2) - cx_1 - k) & \text{if } x_1 \geq 0, \\ -f(-\Psi(x_1,x_2) + cx_1 - k) & \text{otherwise,} \end{cases} \tag{9}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a smooth function supported in $\mathbb{R}_+$ (to avoid a singularity at $x = 0$) with the condition $k > 0$. Using the stream equations (2) we obtain

$$(-\Delta)^s\Psi\,(x_1,x_2) = \begin{cases} f(\Psi(x_1,x_2) - cx_1 - k) & \text{if } x_1 \geq 0, \\ -f(-\Psi(x_1,x_2) + cx_1 - k) & \text{otherwise.} \end{cases} \tag{10}$$

1.2. *Variational formulation*

The studied equation is variational and its solutions are the critical points of

$$E(\Psi) := \frac{1}{2}\int_{\mathbb{R}^2} \Psi(-\Delta)^s\Psi - \int_{\mathbb{H}} F(\Psi - cx_1 - k) + \int_{\mathbb{H}^c} F(\Psi + cx_1 - k), \tag{11}$$

where $\mathbb{H} := \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0\}$ and $F(\xi) := \int_0^\xi f(\xi')\mathrm{d}\xi'$. We are going to build a critical point of E using the technique of the Nehari manifold (defined later). For that purpose, since the choice of f is free, we are imposing on this function the following properties

(a) $f \in \mathscr{C}^\infty(\mathbb{R},\mathbb{R}), \quad f_{|\mathbb{R}_-} = 0 \quad \text{and} \quad f_{|\mathbb{R}_+^*} > 0,$
(b) $\exists\, \nu \in \left]1, \frac{1+s}{1-s}\right[, \ \forall\, \xi \geq 0, \quad f(\xi) \leq C\xi^\nu,$
(c) $\exists\, \mu \in]1,\nu[, \ \forall\, \xi \geq 0, \quad \mu f(\xi) \leq \xi f'(\xi).$

This last hypothesis on the variations of f is equivalent to the hypothesis that the function

$$\xi \longmapsto \frac{f(\xi)}{\xi^\mu}. \tag{12}$$

is non-decreasing on $\mathbb{R}_+$. In particular and since $\mu > 1$,

$$\forall\, \xi_0 \geq 0, \quad \xi \in \mathbb{R}_+ \longmapsto \frac{f(\xi)}{\xi + \xi_0} \tag{13}$$

is increasing and diverging at infinity. Examples of functions that satisfies these three hypothesis (a)–(c) are the functions

$$\xi \longmapsto \xi^\nu e^{-\frac{1}{\xi}}\, \mathbb{1}_{\mathbb{R}_+}(\xi), \tag{14}$$

with $\nu \in [\mu,\nu]$. Given the hypothesis (a) and (b), the functional E is well-defined on the Hilbert space

$$X^s := L^{\frac{2}{1-s}} \cap \dot{H}^s(\mathbb{R}^2) \tag{15}$$

with the scalar product induced by $\dot{H}^s$ given by

$$\langle \Phi, \Psi\rangle_{X^s} := p.v.\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\big(\Phi(x) - \Phi(y)\big)\big(\Psi(x) - \Psi(y)\big)}{|x-y|^{2(1+s)}}\,\mathrm{d}x\,\mathrm{d}y \tag{16}$$

where $p.v.$ refers to the principal value of the singularity of the kernel $(x, y) \mapsto 1/|x-y|^{2(1+s)}$. For further work, we make use of the notations $x = (x_1, x_2)$ and $y = (y_1, y_2)$ to distinguish the coordinates of x and the coordinates of y. We recall here that the Gagliardo half-norms defining the spaces $\dot{W}^{s,p}$ are given in general by

$$|\Phi|^p_{W^{s,p}} := p.v.\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\big|\Phi(x) - \Phi(y)\big|^p}{|x-y|^{d+sp}}\,\mathrm{d}x_1\,\mathrm{d}x_2. \tag{17}$$

For the rest of the work we refer E as being the "*energy*" of the problem although this energy does not correspond to a physical energy. We remark that it is invariant by the action of the group of symmetry generated by (8). We denote by X^s_{sym} the subspace of X^s made with the functions that are left invariant by the action of this symmetry group.

$$X^s_{sym} := \{\Psi \in X^s : \forall\, (x_1, x_2) \in \mathbb{R}^2, \Psi(-x_1, x_2) = -\Psi(x_1, x_2)\}. \tag{18}$$

It follows from the Palais principle of symmetric criticality [8] that any critical point of E on X^s actual belongs to X^s_{sym}. We can therefore restrict our investigations to the subspace X^s_{sym}, inside which the energy can be rewritten

$$E(\Psi) = \frac{1}{2}\|\Psi\|^2_{X^s} - 2V(\Psi) \tag{19}$$

with

$$V(\Psi) := \int_{\mathbb{H}} F(\Psi - cx_1 - k). \tag{20}$$

1.3. *Nehari Manifold and presentation of the main result*

The Nehari manifold associated to the energy E is defined by

$$\mathcal{N} = \{\Psi \in X^s_{sym} \setminus \{0\} \, : \, E'(\Psi)(\Psi) = 0\}, \tag{21}$$

so that $\Psi \in \mathcal{N}$ implies

$$\int_{\mathbb{R}^2} \Psi(-\Delta)^s \Psi - 2\int_{\mathbb{H}} f(\Psi - cx_1 - k)\Psi = 0. \tag{22}$$

It is proven after that the Nehari manifold $\mathcal{N}$ is a sub-manifold of X^s_{sym} non empty, of regularity $\mathscr{C}^1$ without boundary. The main result of this article is the following theorem.

Theorem 1. *Let c and k positive. Let $f : \mathbb{R} \to \mathbb{R}$ verifying* (a), (b) *and* (c).

Then the energy E admits a minimizer $\Psi \neq 0$ on $\mathcal{N}$. As a consequence there exist a non-trivial smooth solution Θ to the inviscid quasi-geostrofic equations (1)–(2) *which has the form*

$$\Theta(x_1, x_2, t) = \Theta(x_1, x_2 - ct) = f\big(\Psi(x_1, x_2 - ct) - cx_1 - k\big), \tag{23}$$

for all $(x_1, x_2) \in \mathbb{H}$ and that satisfies the symmetries $\Theta(x_1, x_2) = -\Theta(-x_1, x_2) = \Theta(x_1, -x_2)$, for all $(x_1, x_2) \in \mathbb{R}^2$. Moreover, The restriction of Θ to $\mathbb{H}$ is non-negative, compactly supported and non-increasing relatively to the variable $|x_2|$.

2. Strategy of proof and main lemmas

We regroup in this section the main Lemmas involved in the proof of Theorem 1 and how they follow one another. The detailed proof of these different lemmas are provided in Section 3.

2.1. *Properties of the Nehari Manifold and minimizing sequences*

We are interested in the minimization problem

$$\alpha := \inf\{E(\Psi) : \Psi \in \mathcal{N}\}. \tag{24}$$

Since the function f is worth 0 on $\mathbb{R}_-$ then a given function Ψ cannot belong to $\mathcal{N}$ if $\Psi \leq 0$ on $\mathbb{H}$. Indeed, this would imply that

$$\int_{\mathbb{H}} f(\Psi - cx_1 - k)\Psi = 0 \tag{25}$$

and then $\|\Psi\|_{X^s} = 0$. The only function in X^s_{sym} such that this quantity is worth 0 is the null function which has been excluded from the definition of the Nehari manifold). We have the following description of the Nehari manifold.

Lemma 2. *The set $\mathcal{N}$ is a $\mathscr{C}^1$ non-empty sub-manifold of X^s_{sym}. For every $\Psi \in X^s_{sym}$ such that $\mathscr{L}_2(\mathrm{supp}(\Psi_+) \cap \mathbb{H})$ is non zero*[1]*, there exist a unique $t_\Psi > 0$ such that $t_\Psi \Psi \in \mathcal{N}$. The value of this t_Ψ is characterized by*

$$E(t_\Psi \Psi) = \max\{E(t\Psi) : t > 0\}. \tag{26}$$

Moreover, any local minimizer of E on N is a smooth non-trivial solution of (10). *We also have that*

$$\beta := \inf\left\{\|\Psi\|^2_{X^s} \, : \, \Psi \in \mathcal{N}\right\} > 0. \tag{27}$$

and for every $\Psi \in \mathcal{N}$,

$$\|\Psi\|^2_{X^s} \leq \left(1 + \frac{1}{\mu}\right) E(\Psi). \tag{28}$$

[1] The notation $\mathscr{L}_d$ refers to the d-dimensional Lebesgue measure (the Lebesgue measure on $\mathbb{R}^d$). The function $\Psi_+ := \max\{\Psi, 0\}$ is the positive part of Ψ.

Remark that this last assertion implies that α is positive. This proposition also implies that any minimizing sequence of E on $\mathcal{N}$ is a bounded sequence.

Definition 3 (Polarization). *We now define the* polarization *of a function* $\Psi \in X^s$ *by*

$$\forall\ x = (x_1, x_2) \in \mathbb{R}^2, \qquad \Psi^\dagger(X) := \begin{cases} \max\left\{\Psi(x), \Psi\big(\sigma(x)\big)\right\} & \text{if } x_1 > 0, \\ \min\left\{\Psi(x), \Psi\big(\sigma(x)\big)\right\} & \text{if } x_1 < 0, \end{cases} \tag{29}$$

where σ denotes the linear map $(x_1, x_2) \in \mathbb{R}^2 \mapsto (-x_1, x_2)$. In the particular case $\Psi \in X^s_{sym}$, we obtain $\Psi^\dagger_{|\mathbb{H}} \geq 0$ and $\Psi^\dagger_{|\mathbb{H}^c} \leq 0$.

For more details about polarization, see for instance [10].

Lemma 4 (Polarization inequality). *For all $\Psi \in \mathcal{N}$,*

$$E(t_{\Psi^\dagger}\Psi^\dagger) \leq E(\Psi) \tag{30}$$

and this inequality is strict when $\Psi \neq \Psi^\dagger$.

Denote with a † the image of a given set by the polarization. This lemma tells that if (Ψ_n) is a minimizing sequence for E on $\mathcal{N}$ then so is $t_{\Psi^\dagger}\Psi^\dagger$ because by definition of $\Psi \mapsto t_\Psi$ the function $t_{\Psi^\dagger}\Psi^\dagger$ belongs to $\mathcal{N}$. Thus, the minimizer, if it exists, belongs to $\mathcal{N}^\dagger$. It is then possible to restrict the investigations to $X^{s,\dagger}_{sym}$.

Definition 5 (Steiner rearrangement). *We define the* Steiner rearrangement *of $\Psi \in X^{s,\dagger}_{sym}$, noted $\Psi^\sharp$, as being the function of $X^{s,\dagger}_{sym}$ which super-level sets on $\mathbb{H}$ are given for all $\nu > 0$ by*

$$\{\Psi^\sharp \geq \nu\} := \bigcup_{x_1 \in \mathbb{R}+} \{x_1\} \times \left[-\frac{\zeta_\Psi(x_1)}{2}, +\frac{\zeta_\Psi(x_1)}{2}\right] \tag{31}$$

with

$$\zeta_\Psi(x_1) := \mathcal{L}_1\{x_2 \in \mathbb{R} : \Psi(x_1, x_2) \geq \nu\}. \tag{32}$$

We extend this definition on $\mathbb{H}^c$ by symmetry to ensure that $\Psi^\sharp \in X^{s,\dagger}_{sym}$.

Lemma 6 (Steiner inequality). *For all $\Psi \in \mathcal{N}^\dagger$,*

$$E(t_{\Psi^\sharp}\Psi^\sharp) \leq E(\Psi) \tag{33}$$

and the equality holds if and only if $\Psi = \Psi^\sharp$ up to a translation on the x_2 axis.

Then, if (Ψ_n) is a minimizing sequence for E on $\mathcal{N}^\dagger$ then so is $t_{\Psi^\sharp}\Psi^\sharp$. Thus, similarly as before it is possible to restrict the investigations to $X^{s,\sharp}_{sym}$.

2.2. *Existence of the solution for the minimizing problem*

Let $(\Psi_n) \in \mathcal{N}^\sharp$ a minimizing sequence. We already know that such a sequence in bounded as a consequence of Lemma 2. To start with, we establish the following compactness result.

Lemma 7 (compactness). *Let c and k be positive. Define the non-linear map*

$$T : \Psi \in X^s \longmapsto \begin{cases} (\Psi - cx_1 - k)_+ & \text{on } \mathbb{H}, \\ -(\Psi - cx_1 + k)_- & \text{on } \mathbb{H}^c. \end{cases} \tag{34}$$

Then T maps X^s_{sym} into himself and maps bounded sets into bounded sets. Moreover, the map $T \circ \sharp \circ \dagger$ is a compact map from X^s_{sym} into $L^p_{sym}(\mathbb{R}^2)$, with $1 \leq p < \frac{2}{1-s}$.

Up to an extraction we can suppose that the minimizing sequence $\Psi_n \to \Psi^\star$ weakly in $X^{s,\sharp}_{sym}$ and that $(\Psi_n - cx_1 - k)_+ \to (\Psi^\sharp - cx_1 - k)_+$ strongly in $L^p(\mathbb{H})$ for all $p < \frac{2}{1-s}$.

Lemma 8 (convergence). *The convergence of Ψ_n towards $\Psi^\star$ in $X^{s,\sharp}$ is a strong convergence.*

This implies that $\Psi^\star$ is solution to the studied minimization problem.

2.3. *Properties of the solution*

We finally define $\Theta^\star$ from $\Psi^\star$ according to formula (9). Since $T(\Psi^\star) \in L^p(\mathbb{R}^2)$ for all $p \in \left[1, \frac{2}{1-s}\right]$ then $\Theta^\star \in L^q$ for all $q \in \left[1, \frac{2}{\nu(1-s)}\right]$ as a consequence of (b). We have the following regularity result

Lemma 9 (regularity). *The functions $\Psi^\star$ and $\Theta^\star$ are $\mathscr{C}^\infty$.*

We can also establish a result on the decay of $\Psi^\star$ at infinity.

Lemma 10 (decay estimate). *There exists a constant $C > 0$ such that for all $x \in \mathbb{R}^2$,*

$$|\Psi^\star(x)| \leq \frac{C}{1 + |x|^{2(1-s)}}, \tag{35}$$

With the positive cut-off level $k > 0$ appearing in the definition of T, this proposition implies in particular that $\Theta^\star$ is compactly supported.

3. Proofs of the lemmas

3.1. *Proof of Lemma 2*

Let $\Psi \in X^s_{sym}$ with $\mathscr{L}_2\big(\mathrm{supp}(\Psi_+) \cap \mathbb{H}\big) \neq 0$. For any $t > 0$, we define

$$g(t) := \frac{E'(t\Psi)(t\Psi)}{t^2} = \frac{1}{2}\|\Psi\|^2_{X_s} - 2t\int_{\mathbb{H}} f(t\Psi_+ - cx_1 - k)\Psi_+. \tag{36}$$

We observe that the integral above can we rewritten

$$g(t) = \frac{1}{2}\|\Psi\|^2_{X^s} - 2\int_{\mathbb{H}} \frac{f(t\Psi_+(x_1,x_2) - cx_1 - k)}{t\Psi_+(x_1,x_2) - cx_1 - k + (cx_1 + k)}\,(\Psi+)^2(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2. \tag{37}$$

Since we have $\mathscr{L}_2\big(\mathrm{supp}(\Psi_+)\cap\mathbb{H}\big) \neq 0$, our remark on the variations $\xi \mapsto f(\xi)/(\xi+\xi_0)$, consequence of (c), indicates that $t \mapsto g(t)$ is decreasing and $g(t) \to -\infty$ as $t \to +\infty$. Indeed, one have to apply this property of f to (37) with $\xi = t\Psi + (x_1,x_2) - cx_1 - k$ and $\xi_0 = cx_1 + k$ and then integrate on $\mathbb{H}$ against the non-negative weight $(\Psi_+)^2$. Now, we use Hypothesis (b) to write on $\mathbb{H}$

$$0 \leq \frac{1}{t} f(t\Psi_+ - cx_1 - k)\Psi_+ \leq \frac{1}{t} f(t\Psi_+)\Psi_+ \leq Ct^{\nu-1}\big(\Psi_+\big)^{\nu+1}. \tag{38}$$

Since $\nu > 1$, then as $t \to 0^+$, we have

$$g(t) \longrightarrow \frac{1}{2}\|\Psi\|^2_{X^2} > 0. \tag{39}$$

Since f is smooth then g is continuous, and then the function g admits a unique root on $\mathbb{R}^*_+$. The characterization (26) comes from the fact that

$$tg(t) = \frac{\mathrm{d}}{\mathrm{d}t}E(t\Psi). \tag{40}$$

The estimate (28) is obtained, for $\Psi \in \mathscr{N}$, as follows

$$\begin{aligned} E(\Psi) &= E(\Psi) - \frac{1}{\mu+1}E'(\Psi)(\Psi) \\ &= \frac{\mu}{2(\mu+1)}\|\Psi\|^2_{X^s} + \frac{2}{\mu+1}\int_{\mathbb{H}}\big[f(\Psi - cx_1 - k)\Psi(x_1,x_2) - (\mu+1)F(\Psi - cx_1 - k)\big]\,\mathrm{d}x_1\,\mathrm{d}x_2 \\ &\geq \frac{\mu}{2(\mu+1)}\|\Psi\|^2_{X^s}, \end{aligned} \tag{41}$$

where the last inequality comes from the integration on $[0, x_1]$ of hypothesis (c) that gives $\mu F(t) \leq tf(t) - F(t)$. The fact that β is not zero is obtained using (b) and the Sobolev embedding

$$\|\Psi\|^2_{X^s} = \int_{\mathbb{H}} f(\Psi_+ - cx_1 - k)\Psi_+ \leq 4K\int_{\mathbb{H}} (\Psi+)^{\frac{2}{1-s}} = 4K\|\Psi_+\|^{\frac{2}{1-s}}_{L^{\frac{2}{1-s}}} \leq C\|\Psi\|^{\frac{2}{1-s}}_{X^s}. \tag{42}$$

Concerning the regularity of $\mathcal{N}$, it is a consequence of the implicit functions theorem applied to $\Xi : (s,\Psi) \mapsto E'(s\Psi)(\Psi)$ defined on the open set $\mathbb{R}_+^* \times X^s_{sym} \setminus \{0\}$. The hypothesis of the theorem are verified because for $\Psi \in \mathcal{N}$ we have:

$$\partial_1 \Xi(1,\Psi) = t_\Psi^2 g'(t\Psi) < 0. \tag{43}$$

It remains to prove that any minimizer of E on $\mathcal{N}$ is a critical point for E defined on the whole space. We first remark that a minimizer of E on $\mathcal{N}$ is a minimizer of $\Psi \mapsto E(t_\Psi \Psi)$ on X^s_{sym}. Then, using the definition of the Nehari manifold and the fact that we have $\Psi \in \mathcal{N}$ implies $t_\Psi = 1$, we conclude

$$\forall\ h \in X^s_{sym}, E'(\Psi)(h) = E'(t_\Psi \Psi)[t'_\Psi(h)\Psi + t_\Psi h] = 0. \tag{44}$$

□

3.2. *Proof of Lemma 4*

We first recall that $\Psi \in \mathcal{N}$ implies that $\mathcal{L}_2\big(\operatorname{supp}(\Psi_+) \cap \mathbb{H}\big) \neq 0$. Using the characterization (26) we get $E(\Psi) \geq E(t_{\Psi^\dagger}\Psi)$. Using the fact that $\Psi(x_1,x_2) = -\Psi(-x_1,x_2)$, we conclude that here the polarization consists in switching the two values of $\Psi(x_1,x_2)$ and $\Psi(-x_1,x_2)$ if and only if we have $\Psi(x_1,x_2) \leq 0 \leq \Psi(-x_1,x_2)$. Therefore since F is worth 0 on $\mathbb{R}_-$ and is positive on $\mathbb{R}_+$, we obtain

$$V(t_{\Psi^\dagger}\Psi) \leq V(t_{\Psi^\dagger}\Psi^\dagger). \tag{45}$$

To finish the proof of this lemma, we have to establish

$$\|\Psi^\dagger\|_{X^s} \leq \|\Psi\|_{X^s} \tag{46}$$

and that this inequality is strict if and only if $\Psi^\dagger \neq \Psi$. Actually the fact that the polarization decreases the $\dot W^{s,p}(\mathbb{R}^d)$ half-norms (17) is a general result so that we can establish it in the general case. By definition of the principal values, we have

$$\iint_{|x-y|\geq\varepsilon} \frac{|\Psi(x)-\Psi(y)|^p}{|x-y|^{d+sp}}\,\mathrm{d}x\,\mathrm{d}y \longrightarrow |u|^p_{W^{s,p}} \qquad \text{as } \varepsilon \to 0. \tag{47}$$

We then establish the inequality for any fixed $\varepsilon > 0$. First, the integral is split as follows,

$$\begin{aligned}\iint_{|x-y|\geq\varepsilon} &\frac{|\Psi(x)-\Psi(y)|^p}{|x-y|^{d+sp}}\,\mathrm{d}x\,\mathrm{d}y \\ &= \iint_{\mathbb{H}^2\setminus\{|x-y|<\varepsilon\}} \bigg(\frac{1}{|x-y|^{d+sp}}\big(|\Psi(x)-\Psi(y)|^p + |\Psi\circ\sigma(x)-\Psi\circ\sigma(y)|^p\big) \\ &\qquad + \frac{1}{|x-\sigma(y)|^{d+sp}}\big(|\Psi(x)-\Psi\circ\sigma(y)|^p + |\Psi\circ\sigma(x)-\Psi(y)|^p\big)\bigg)\,\mathrm{d}x\,\mathrm{d}y\end{aligned} \tag{48}$$

Let $x,y \in \mathbb{H}$. Observe that

$$|x-y|^{d+sp} < |x-\sigma(y)|^{d+sp}. \tag{49}$$

Case 1: $\Psi(x) \geq \Psi\circ\sigma(x)$ and $\Psi(y) \geq \Psi\circ\sigma(y)$. In this case, with the definition of the polarization, $\Psi(x) = \Psi^\dagger(x)$ and $\Psi(y) = \Psi\dagger(y)$. Then when we integrate on the couples (x,y) that belongs to Case 1, the associated term in the integral (48) is not modified by the polarization.

Case 2: $\Psi(x) \geq \Psi\circ\sigma(x)$ and $\Psi(y) < \Psi\circ\sigma(y)$. By computing its derivative, we obtain that the function

$$u_{\beta,\gamma} : \alpha \in \mathbb{R} \mapsto |\alpha+\beta|^p - |\alpha+\gamma|^p \tag{50}$$

is non-decreasing when $\beta > \gamma$. Indeed we have (with $p \geq 1$)

$$u'_{\beta,s}(\alpha) = p(\alpha+\beta)|\alpha+\beta|^{p-2} - p(\alpha+\gamma)|\alpha+\gamma|^{p-2}(31) \tag{51}$$

which is non-negative because $y \mapsto y|y|^{p-2}$ is an non-decreasing function. We now use this property of $u_{\beta,\gamma}$ with $\alpha_1 := \Psi(x) \geq \Psi\circ\sigma(x) =: \alpha_2$ and with $\beta := -u(y) > \gamma := -u\circ\sigma(y)$. We obtain

$$|\Psi(x)-\Psi(y)|^p + |\Psi\circ\sigma(x)-\Psi\circ\sigma(y)|^p > |\Psi\circ\sigma(x)-\Psi(y)|^p + |\Psi(x)-\Psi\circ\sigma(y)|^p. \tag{52}$$

If we combine this with (49) we get

$$\begin{aligned}
&\frac{1}{|x-y|^{d+sp}}\big(|\Psi(x)-\Psi(y)|^p + |\Psi\circ\sigma(x)-\Psi\circ\sigma(y)|^p\big)\\
&\quad+\frac{1}{|x-\sigma(y)|^{d+sp}}\big(|\Psi\circ\sigma(x)-\Psi(y)|^p + |\Psi(x)-\Psi\circ\sigma(y)|^p\big)\\
&>\frac{1}{|x-y|^{d+sp}}\big(|\Psi\circ\sigma(x)-\Psi(y)|^p + |\Psi(x)-\Psi\circ\sigma(y)|^p\big)\\
&\quad+\frac{1}{|x-\sigma(y)|^{d+sp}}\big(|\Psi(x)-\Psi(y)|^p + \Psi\circ\sigma(x)-\Psi\circ\sigma(y)|^p\big)\\
&=\frac{1}{|x-y|^{d+sp}}\big(|\Psi^\dagger(x)-\Psi^\dagger(y)|^p + |\Psi^\dagger\circ\sigma(x)-\Psi^\dagger\circ\sigma(y)|^p\big)\\
&\quad+\frac{1}{|x-\sigma(y)|^{d+sp}}\big(|\Psi^\dagger\circ\sigma(x)-\Psi^\dagger(y)|^p + \Psi^\dagger(x)-\Psi^\dagger\circ\sigma(y)|^p\big).
\end{aligned} \tag{53}$$

Case 3: $\Psi(x) < \Psi\circ\sigma(x)$ **and** $\Psi(y) < \Psi\circ\sigma(y)$**.** In this case we have both $\Psi(x)$ and $\Psi(y)$ that are swapped with respectively $\Psi\circ\sigma(x)$ and $\Psi\circ\sigma(y)$. Then this case is the same as Case 1 and the term associated to Case 3 in the integral (48) is not modified by the polarization.

Case 4: $\Psi(x) < \Psi\circ\sigma(x)$ **and** $\Psi(y) \geq \Psi\circ\sigma(y)$**.** This case is the same as Case 2.

Gathering these four cases, we obtain that for any $\varepsilon > 0$,

$$\iint_{|x-y|\geq\varepsilon} \frac{|\Psi(x)-\Psi(y)|^p}{|x-y|^{d+sp}}\,\mathrm{d}x\,\mathrm{d}y \geq \iint_{|x-y|\geq\varepsilon} \frac{|\Psi^\dagger(x)-\Psi^\dagger(y)|^p}{|x-y|^{d+sp}}\,\mathrm{d}x\,\mathrm{d}y. \tag{54}$$

Concerning the cases of equality, we obtained from Cases 2 and 4 that if

$$\mathscr{L}_2\Big(\big\{(x,y)\in\mathbb{H}^2 : \Psi(x)=\Psi^\dagger(x) \quad\text{and}\quad \Psi(y)\neq\Psi^\dagger(y)\big\}\cap|x-y|\geq\varepsilon\Big) > 0 \tag{55}$$

then the inequality (54) is actually strict. We now observe that the above set is of measure zero for every $\varepsilon > 0$ if and only if we have either $\Psi = \Psi^\dagger$ or $\Psi = \Psi^\dagger\circ\sigma$. But this last case is not possible when $\Psi\in\mathscr{N}$ and then the only case of equality in our case is $\Psi = \Psi^\dagger$. □

3.3. *Proof of Lemma 6*

Arguing similarly as the previous proof, we only have to prove that

$$\forall\,\Psi\in X^{s,\dagger}_{sym},\quad E(\Psi^\sharp)\leq E(\Psi). \tag{56}$$

Since the Steiner rearrangement only involves rearrangements of the super-level sets perpendicularly to the x_1-axis, we get

$$V(\Psi^\sharp) = V(\Psi). \tag{57}$$

To conclude we have to establish that

$$\|\Psi^\sharp\|_{X^s} \leq \|\Psi\|_{X^s}. \tag{58}$$

To start with, we suppose that Ψ is smooth and compactly supported. In this case

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|\Psi(x)-\Psi(y)|^2}{(|x-y|^2+\varepsilon^2)^{1+s}}\,\mathrm{d}x\,\mathrm{d}y \longrightarrow \|\Psi\|_{X^s}, \tag{59}$$

as $\varepsilon \to 0^+$. Since the considered functions are $\mathscr{C}^\infty$, it is possible to develop the square above and write

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|\Psi(x)-\Psi(y)|^2}{(|x-y|^2+\varepsilon^2)^{1+s}}\,\mathrm{d}x\,\mathrm{d}y = 2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\Psi(x)^2}{(|x-y|^2+\varepsilon^2)^{1+s}}\,\mathrm{d}x\,\mathrm{d}y - 2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\Psi(x)\Psi(y)}{(|x-y|^2+\varepsilon^2)^{1+s}}\,\mathrm{d}x\,\mathrm{d}y. \tag{60}$$

The first integral in the right-hand side of the above inequality is not modified by rearrangement of the super-level sets of the function Ψ. Concerning the second integral, using the fact that $\Psi(x_1,x_2) = -\Psi(-x_1,x_2)$ we get

$$\begin{aligned}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} &\frac{\Psi(x_1,x_2)\Psi(y_1,y_2)}{((x_1-y_1)^2+(x_2-y_2)^2+\varepsilon^2)^{1+s}}\,\mathrm{d}x_2\,\mathrm{d}y_2\,\mathrm{d}x_1\,\mathrm{d}y_1 \\ &= 2\int_0^{+\infty}\int_0^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\Psi(x_1,x_2)\Psi(y_1,y_2)}{((x_1-y_1)^2+(x_2-y_2)^2+\varepsilon^2)^{1+s}}\,\mathrm{d}x_2\,\mathrm{d}y_2\,\mathrm{d}x_1\,\mathrm{d}y_1 \\ &\quad - 2\int_0^{+\infty}\int_0^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\Psi(x_1,x_2)\Psi(y_1,y_2)}{((x_1+y_1)^2+(x_2-y_2)^2+\varepsilon^2)^{1+s}}\,\mathrm{d}x_2\,\mathrm{d}y_2\,\mathrm{d}x_1\,\mathrm{d}y_1.\end{aligned} \tag{61}$$

We now observe that the function

$$\Upsilon_{x_1,y_1} : u \longmapsto \frac{1}{\big((x_1-y_1)^2+u^2+\varepsilon^2\big)^{1+s}} - \frac{1}{\big((x_1+y_1)^2+u^2+\varepsilon^2\big)^{1+s}} \tag{62}$$

is non-negative and radially decreasing on $\mathbb{R}$. Moreover, for $x_1, y_1 \geq 0$ the functions $x_2 \mapsto \Psi(x_1,x_2)$ and $y_2 \mapsto \Psi(y_1,y_2)$ are both non-negative $\mathbb{R}$. Thus, using the Riesz rearrangement inequality, we obtain

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Psi(x_1,x_2)\Psi(y_1,y_2)\Upsilon_{x_1,y_1}(x_2-y_2)\,\mathrm{d}x_2\,\mathrm{d}y_2 \leq \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Psi^\sharp(x_1,x_2)\Psi^\sharp(y_1,y_2)\Upsilon_{x_1,y_1}(x_2-y_2)\,\mathrm{d}x_2\,\mathrm{d}y_2. \tag{63}$$

We now inject this inequality back into (61) and get

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\Psi(x)\Psi(y)}{(|x-y|^2+\varepsilon^2)^{1+s}}\,\mathrm{d}x\,\mathrm{d}y \leq \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\Psi^\natural(x)\Psi^\natural(y)}{(|x-y|^2+\varepsilon^2)^{1+s}}\,\mathrm{d}x\,\mathrm{d}y. \tag{64}$$

We use this estimate in (60), we take the limit $\varepsilon \to 0$ and we conclude by density of the smooth compactly supported functions. □

Remark. It was not possible to use directly the Riesz rearrangement inequality to the second integral appearing in (60) because this inequality in only true for non-negative functions.

3.4. *Proof of Lemma 7*

Step 1 : T maps $X^{s,\dagger}_{sym}$ into itself and maps bounded sets into bounded sets. First, if Ψ satisfies the symmetry property then so does $T(\Psi)$. Define the set[2]

$$\Omega(\Psi) := \{(x_1,x_2) \in \mathbb{H} : T(\Psi)(x_1,x_2) > 0\}, \tag{65}$$

where $T(\Psi)(x_1,x_2) := (\Psi(x_1,x_2) - cx_1 - k)_+$. By definition of T and of Ω

$$\mathscr{L}_2(\Omega) = \int_\Omega 1 \leq \int_\Omega \left(\frac{\Psi(x_1,x_2)}{cx_1+k}\right)^{\frac{2}{1-s}} \mathrm{d}x_1\,\mathrm{d}x_2 \leq \int_\Omega \left(\frac{\Psi(x_1,x_2)}{k}\right)^{\frac{2}{1-s}} \mathrm{d}x_1\,\mathrm{d}x_2 \leq \frac{1}{k^{\frac{2}{1-s}}}\int_{\mathbb{H}} \Psi^{\frac{2}{1-s}}. \tag{66}$$

[2] The adherence of this set is the support of the function Θ. This corresponds physically speaking to the vorticity zone.

Using a Sobolev inequality above leads to

$$\mathscr{L}_2(\Omega) \leq \frac{C_s}{k^{\frac{2}{1-s}}} \|\Psi\|_{X^s}^{\frac{2}{1-s}}. \tag{67}$$

The computation of the double integral defining the $\dot{H}^s$ half-norm (17) is done separating the integrals on $\mathbb{R}^2$ on two between Ω and Ω^c. On Ω^c the quantity $\Psi(x_1, x_2) - cx_1 - k$ is non-positive and then $T(\Psi)(x_1, x_2) = 0$. Therefore,

$$\int_{\Omega^c}\int_{\Omega^c} \frac{|T(\Psi)(x) - T(\Psi)(y)|^2}{|x-y|^2(1+s)} \,\mathrm{d}x\,\mathrm{d}y = 0. \tag{68}$$

Concerning the integral on $\Omega \times \Omega$, using the notation $x = (x_1, x_2)$ and $y = (y_1, y_2)$,

$$\begin{aligned}\int_{\Omega}\int_{\Omega} \frac{|T(\Psi)(x) - T(\Psi)(y)|^2}{|x-y|^{2(1+s)}} \,\mathrm{d}x\,\mathrm{d}y &= \int_{\Omega}\int_{\Omega} \frac{|(\Psi(x) - cx_1 - k) - (\Psi(y) - cy_1 - k)|^2}{|x-y|^{2(1+s)}} \,\mathrm{d}x_1\,\mathrm{d}x_2 \\ &\leq \int_{\Omega}\int_{\Omega} \frac{|\Psi(x) - \Psi(y)|^2 + c^2|x_1 - y_1|^2}{|x-y|^{2(1+s)}} \mathrm{d}x_1 \mathrm{d}x_2.\end{aligned} \tag{69}$$

Denote with an $*$ the radially decreasing rearrangement and $R_\Omega > 0$ the radius such that

$$\mathscr{L}_2(\Omega) = \mathscr{L}_2(\mathscr{B}(0, R_\Omega)). \tag{70}$$

To simplify the notations we simply note this ball $B(\Omega)$. By the Riesz rearrangement inequality we have

$$\begin{aligned}\int_{\Omega}\int_{\Omega} \frac{|x_1 - y_1|^2}{|x-y|^{2(1+s)}} \,\mathrm{d}x\,\mathrm{d}y \leq \int_{\Omega}\int_{\Omega} \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^{2s}} &\leq \int_{\mathscr{B}(\Omega)}\int_{\mathscr{B}(\Omega)} \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^{2s}} \\ &\leq \int_{\mathscr{B}(\Omega)}\int_{\mathscr{B}(\Omega)} \frac{\mathrm{d}x}{|x|^{2s}} \mathrm{d}y = \frac{\pi^s}{1-s} \mathscr{L}_2(\Omega)^{2-s}.\end{aligned} \tag{71}$$

Using now (67) we get

$$\int_{\Omega}\int_{\Omega} \frac{|x_1 - y_1|^2}{|x_1 - x_2|^{2(1+s)}} \,\mathrm{d}x_1\,\mathrm{d}x_2 \leq C_s \|\Psi\|_{X^s}^{2\frac{2-s}{1-s}}. \tag{72}$$

Concerning the last term,

$$\int_{\Omega}\int_{\Omega^c} \frac{|T(\Psi)(x) - T(\Psi)(y)|^2}{|x-y|^{2(1+s)}} \,\mathrm{d}x\,\mathrm{d}y = \int_{\Omega} |T(\Psi)(x)|^2 \int_{\Omega^c} \frac{\mathrm{d}y}{|x-y|^{2(1+s)}} \mathrm{d}x. \tag{73}$$

For all $x \in \Omega$ we define $\Lambda(x) := \frac{\Psi(x) - cx_1 - k}{2c}$ and $\mathscr{O}_x := \{y \in \Omega^c : y_1 \geq x_1 + \Lambda(x)\}$. Then,

$$\int_{\mathscr{O}_x} \frac{\mathrm{d}y}{|x-y|^{2(1+s)}} \leq \int_{\mathscr{O}_x} \frac{\mathrm{d}y}{|x_1 - y_1|^{2(1+s)}} \leq \int_{\mathscr{B}(x,\Lambda(x))^c} \frac{\mathrm{d}y}{|x-y|^{2(1+s)}} = \frac{\pi^s}{\Lambda(x)^{2s}}. \tag{74}$$

Using the fact that $x \in \Omega$, that $y \in \Omega^c$ and that $y \in \mathscr{O}_x$ in this order, we obtain

$$\begin{aligned}0 \leq T(\Psi)(x) = \Psi(x) - cx_1 - k &\leq \Psi(x) - \Psi(y) + c(x_1 - y_1) \\ &\leq \Psi(x) - \Psi(y) + \Psi(x) - \frac{cx_1 - k}{2} = \Psi(x) - \Psi(y) + \frac{1}{2} T(\Psi)(x).\end{aligned} \tag{75}$$

Therefore

$$|T(\Psi)(x)| \leq 2|\Psi(x) - \Psi(y)| \tag{76}$$

Combing (73), (74) and (76) leads to

$$\begin{aligned}&\int_{\Omega}\int_{\Omega^c} \frac{|T(\Psi)(x) - T(\Psi)(y)|^2}{|x-y|^{2(1+s)}} \mathrm{d}x\mathrm{d}y \\ &\quad\leq 4\int_{\Omega}\int_{\Omega^c\setminus\mathscr{O}_x} \frac{|\Psi(x) - \Psi(y)|^2}{|x-y|^{2(1+s)}} \,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega} |T(\Psi)(x)|^2 \frac{\pi}{s\Lambda(x)^{2s}} \mathrm{d}x \\ &\quad\leq 4\int_{\Omega}\int_{\Omega^c} \frac{|\Psi(x) - \Psi(y)|^2}{|x-y|^{2(1+s)}} \,\mathrm{d}x\,\mathrm{d}y + \frac{\pi}{s}\int_{\Omega} |T(\Psi)(x)|^{2(1-s)} \mathrm{d}x.\end{aligned} \tag{77}$$

Now, to estimate the last term of the above inequality, we use the fact that $T(\Psi) \le \Psi$ and then the Hölder inequality gives

$$\int_\Omega |T(\Psi)(x)|^{2(1-s)}\mathrm{d}x \le \int_\Omega |\Psi(x)|^{2(1-s)} \le \mathscr{L}_2(\Omega)^{s(2-s)}\,\|\Psi\|^{2(1-s)}_{L^{\frac{2}{1-s}}}. \tag{78}$$

We continue this estimate using (67) and a Sobolev embedding,

$$\le C\|\Psi\|^{2s(1+\frac{1}{1-s})}_{X^s}\,\|\Psi\|^{2(1-s)}_{L^{\frac{2}{1-s}}} \le C\|\Psi\|^{\frac{2}{1-s}}_{X^s}. \tag{79}$$

Thus, gathering all these estimates we obtain.

$$\|T(\Psi)\|^2_{X^s} := \int_\Omega\int_{\Omega^c} \frac{|T(\Psi)(x)-T(\Psi)(y)|^2}{|x-y|^{2(1+s)}}\,\mathrm{d}x\,\mathrm{d}y \le C\|\Psi\|^2_{X^s}\Big(1+\|\Psi\|^{\frac{2s}{1-s}}_{X^s}\Big). \tag{80}$$

Therefore, T does map $X^{s,\dagger}_{sym}$ into itself and maps bounded subsets of $X^{s,\dagger}_{sym}$ into bounded subsets.

Step 2 : $T\circ\sharp\circ\dagger$ defined on X^s_{sym} is a compact operator for the L^p topology. Set the convention that $\{|x_2|\ge R\}$ designates the set $\{(x_1,x_2)\in\mathbb{H} : |x_2|\ge R\}$. Let $\kappa>0$ and $R\ge 0$. For all $x_1\in\Omega^\sharp$ we define

$$\mathscr{U}_x := \mathscr{B}(x,\kappa)\cap\big(\Omega^\sharp\big)^c. \tag{81}$$

Then,

$$\begin{aligned}\int_{\{|x_2|\ge R\}} |T(\Psi^\sharp)(x)|^2\mathrm{d}x &= \int_{\{|x_2|\ge R\}} \frac{1}{\mathscr{L}_2(\mathscr{U}_x)}\int_{\mathscr{U}_x} |T(\Psi^\sharp)(x)|^2\,\mathrm{d}y\,\mathrm{d}x\\ &\le \int_{\{|x_2|\ge R\}} \frac{1}{\mathscr{L}_2(\mathscr{U}_x)}\int_{\mathscr{U}_x} |T(\Psi^\sharp)(x)-T(\Psi^\sharp)(y)|^2\,\mathrm{d}y\,\mathrm{d}x\\ &\le \int_{\{|x_2|\ge R\}} \frac{\kappa^{2(1+s)}}{\mathscr{L}_2(\mathscr{U}_x)}\int_{\mathscr{U}_x} \frac{|T(\Psi^\sharp)(x)-T(\Psi^\sharp)(y)|^2}{|x-y|^{2(1+s)}}\,\mathrm{d}y\,\mathrm{d}x.\end{aligned} \tag{82}$$

Denote by $P_{\mathbb{R}}$ the projection on $\mathbb{R}\times\{0\}$ (that is identified to $\mathbb{R}$). As a consequence of the Steiner symmetrization, with (66),

$$2R\,\mathscr{L}_1\Big(P_{\mathbb{R}}\big((\Omega^\sharp)^c\cap\{|x_2|\ge R\}\big)\Big) \le \mathscr{L}_2\big(\Omega^\sharp\big) \le \frac{1}{k^{\frac{2}{1-s}}}\|\Psi^\sharp\|^{\frac{2}{1-s}}_{L^{\frac{2}{1-s}}}. \tag{83}$$

Since $|x_2|\ge R-\kappa$ then using again the Steiner symmetry of $\Omega^\sharp$, gives that $\mathscr{U}_x$ contains the ball $B(x_1,\kappa)$ minus the rectangle centered at x, of width

$$\mathscr{L}_1\Big(P_{\mathbb{R}}\big((\Omega^\sharp)^c\cap\{|x_2|\ge R-\kappa\}\big)\Big)$$

and height 2κ. Then, with (83),

$$\mathscr{L}_2(\mathscr{U}_x) \ge \pi\kappa^2 - \frac{\kappa}{(R-\kappa)k^{\frac{2}{1-s}}}\|\Psi^\sharp\|^{\frac{2}{1-s}}_{L^{\frac{2}{1-s}}}. \tag{84}$$

The choice of κ is free and then we choose to fix it equal to C/R with

$$C := \frac{4}{\pi\,k^{\frac{2}{1-s}}}\|\Psi^\sharp\|^{\frac{2}{1-s}}_{L^{\frac{2}{1-s}}}.$$

Choose now R such that $R\ge\sqrt{2C}$. Then in this case the inequality (84) becomes

$$\mathscr{L}_2(\mathscr{U}_x) \ge \frac{\pi C^2}{2R^2}. \tag{85}$$

Combining the estimate above with (82), leads to the following estimate

$$\int_{\{|x_2|\ge R\}} |T(\Psi^\sharp)(x)|^2\mathrm{d}x \le \left(\frac{4}{\pi R}\right)^{2s}\left(\frac{\|\Psi^\sharp\|_{L^{\frac{2}{1-s}}}}{k}\right)^{\frac{4s}{1-s}}\|T(\Psi^\sharp)\|^2_{X^s}. \tag{86}$$

On the other hand, using the Hölder inequality,

$$\int_{\{x_1\geq R\}}|T(\Psi^\sharp)|^2=\int_{\{x_1\geq R\}}|T(\Psi^\sharp)|^2\mathbb{1}_{(\Omega^\sharp)^c}\leq\left(\int_{\{x_1\geq R\}}|T(\Psi^\sharp)|^{\frac{2}{1-s}}\right)^{1-s}\left(\int_{\{x_1\geq R\}}\mathbb{1}_{(\Omega^\sharp)^c}\right)^s$$
$$=cL^2\big((\Omega^\sharp)^c\cap\{x_1\geq R\}\big)^s\|\Psi^\sharp\|^2_{L^{\frac{2}{1-s}}},\tag{87}$$

where by convention $\{x_1\geq R\}$ designates the set $\{(x_1,x_2)\in\mathbb{H}:x_1\geq R\}$. Moreover

$$\mathscr{L}^2\big((\Omega^\sharp)^c\cap\{x_1\geq R\}\big)=\int_{(\Omega^\sharp)^c\cap\{x_1\geq R\}}1\leq\int_{(\Omega^\sharp)^c\cap\{x_1\geq R\}}\left(\frac{\Psi^\sharp}{cx_1+k}\right)^{\frac{2}{1-s}}\leq\left(\frac{1}{cR+k}\right)^{\frac{2}{1-s}}\|\Psi^\sharp\|^{\frac{2}{1-s}}_{L^{\frac{2}{1-s}}}.\tag{88}$$

Combining (87) and (88) leads to

$$\int_{\{x_1\geq R\}}|T(\Psi^\sharp)|^2\leq\left(\frac{1}{cR+k}\right)^{\frac{2s}{1-s}}\|\Psi^\sharp\|^{\frac{2}{1-s}}_{L^{\frac{2}{1-s}}}\tag{89}$$

The two decay estimates (86) and (89) and the Rellich–Kondrachov compactness theorem (applied at the local level) give the result. □

3.5. *Proof of Lemma 8*

It follows from the definition of $\mathscr{N}$ and of Lemma 2 that

$$\int_{\mathbb{H}}f(\Psi_n-cx_1-k)\Psi_n=\frac{1}{2}\int_{\mathbb{R}}\Psi_n(-\Delta)^s\Psi_n=\frac{1}{2}\|\Psi_n\|^2_{X^s}\geq\frac{\beta}{2}>0.\tag{90}$$

By the previous lemma, up to a sub-sequence when $n\to+\infty$,

$$\int_{\mathbb{H}}f(\Psi_n-cx_1-k)\Psi_n\longrightarrow\int_{\mathbb{H}}f(\Psi^\star-cx_1-k)\Psi\star\geq\frac{\beta}{2}.\tag{91}$$

In particular $(\Psi^\star-cx_1-k)+\not\equiv0$ on $\mathbb{H}$. By Lemma 2, there exists $t^\star>0$ such that $t^\star\Psi^\star\in\mathscr{N}$. With the characterization of $t^\star$ and since $\Psi_n\in\mathscr{N}$,

$$E(\Psi_n)=E(t_{\Psi_n}\Psi_n)\geq E(t^\star\Psi_n).\tag{92}$$

Thus,

$$\alpha=\lim_{n\to+\infty}E(\Psi_n)\geq\liminf_{n\to+\infty}E(t^\star\Psi_n)\geq E(t^\star\Psi^\star)\geq\alpha.\tag{93}$$

Therefore all these inequalities are equalities and $\|\Psi_n\|^2_{X^s}\to\|\Psi^\star\|^2_{X^s}$. Since the space X^s is strictly convex, this gives that Ψ_n converges towards $\Psi^\star$ strongly in X^s. □

3.6. *Proof of Lemma 9*

We already know that $T(\Psi^\star)\in L^{\frac{2}{1-s}}(\mathbb{R}2)$. Since the support of $T(\Psi\star)$ has a finite measure, then $T(\Psi^\star)\in L^1(\mathbb{R}^2)$. Define $\Theta^\star$ from $\Psi^\star$ using formula (23). Hypothesis (b) implies

$$\forall\,q\in\left[1,\frac{2}{\nu(1-s)}\right],\quad\Theta\star\in L^q(\mathbb{R}^2).\tag{94}$$

Define now the function $\widetilde{\Psi}$ given by the following representation formula,

$$\widetilde{\Psi}(x)=K_s\int_{\mathbb{R}^2}\frac{\Theta^\star(y)}{|x-y|^{2(1-s)}}\,\mathrm{d}y.\tag{95}$$

where K_s is some renormalization constant. It follows from the weighted inequalities for singular integrals [11, §5] that $\widetilde{\Psi}\in\dot{W}^{2s,q}(\mathbb{R}^2)$, for all $q\in\big[1,\frac{2}{\nu(1-s)}\big]$. Moreover, by the Hardy–Littlewood–Sobolev convolution inequality, $\widetilde{\Psi}\in L^q(\mathbb{R}^2)$, for all $q\in\big[\frac{1}{1-s},\frac{2}{\nu-s(2+\nu)}\big]$. By standard interpolation,

$\widetilde{\Psi} \in X^s_{sym}$. Now, let $\varphi \in X^s_{sym}$ be a test function. Using the spectral properties of the Sobolev spaces [5] gives (up to multiplicative renormalization constants),

$$\begin{aligned}\langle \widetilde{\Psi}, \varphi\rangle_{X^s} &= \int_{\mathbb{R}^2} |\xi|^{2s} \mathscr{F}[\widetilde{\Psi}](\xi)\, \mathscr{F}[\varphi](\xi)\, \mathrm{d}\xi \\ &= \int_{\mathbb{R}^2} |\xi|^{2s} \mathscr{F}\left[\Theta^\star * \frac{1}{|\cdot|^{2(1-s}}\right](\xi)\, \mathscr{F}[\varphi](\xi)\, \mathrm{d}\xi \\ &= \int_{\mathbb{R}^2} \mathscr{F}[\Theta^\star](\xi)\, \mathscr{F}[\varphi](\xi)\, \mathrm{d}\xi = \langle \Theta^\star, \varphi\rangle_{L^2},\end{aligned} \tag{96}$$

where $\mathscr{F}[\cdot]$ designates the Fourier transform. Moreover, since $\Psi^\star$ is a critical point of E, then $\langle \Psi^\star, \varphi\rangle_{X^s} = \langle \Theta^\star, \varphi\rangle_{L^2}$, which implies $\widetilde{\Psi} = \Psi^\star$. The regularity known for $\Theta^\star$ allows to conclude that $\Psi^\star$ is bounded and uniformly continuous. Seen the definitions, to conclude that $\Psi^\star$ is smooth by a bootstrap argument there remain to study possible discontinuities on $x_1 = 0$. Nevertheless, it follows from the symmetry property of $\Psi^\star$ and its uniform continuity that $T(\Psi\star)$ is worth 0 at a distance uniformly positive from $x_1 = 0$, meaning on a strip $]-\delta, \delta[\times \mathbb{R}_+$. Therefore so is the case for $\Theta^\star$ and then the smoothness of $\Psi^\star$ is proved. □

3.7. *Proof of Lemma 10*

Let $x \in \mathbb{R}^2$ such that $|x| \geq 1$. We separate the integral (95) into two,

$$\Psi^\star(x) = K_s \int_{|x-y|\leq \frac{|x|}{2}} \frac{\Theta^\star(y)}{|x-y|^{2(1-s)}}\, \mathrm{d}y + K_s \int_{|x-y|> \frac{|x|}{2}} \frac{\Theta^\star(y)}{|x-y|^{2(1-s)}}\, \mathrm{d}y \tag{97}$$

Concerning the first integral, we choose $\eta \in]1-s, \frac{1}{s+1}[$. This interval is non-empty and included in $]0,1[$. We use the Hölder inequality and Hypothesis (b) and then we are led to

$$\int_{|x-y|\leq \frac{|x|}{2}} \frac{\Theta^\star(y)}{|x-y|^{2(1-s)}}\, \mathrm{d}y \leq \left(\int_{|\zeta|\leq \frac{|x|}{2}} \frac{\mathrm{d}\zeta}{|\zeta|^{\frac{2(1-s)}{\eta}}}\right)^{\eta} \left(\int_{|x-y|\leq \frac{|x|}{2}} |T(\Psi^\star)|^{\frac{\nu}{1-\eta}}\right)^{1-\eta}. \tag{98}$$

Using again the estimates (86) and (89),

$$\int_{|x-y|\leq \frac{|x|}{2}} |T(\Psi^\star)|^2(y) \mathrm{d}y \leq \frac{C}{(1+|x|)^{-2s}}. \tag{99}$$

Knowing that $\frac{\nu}{1-\eta} \geq 2$, the above estimate used in (98) leads to (the constant that depends only on η)

$$\int_{|x-y|\leq \frac{|x|}{2}} \frac{\Theta^\star(y)}{|x-y|^{2(1-s)}}\, \mathrm{d}y \leq C(\eta)\big(|x|^2+1\big)^{\eta(s+1)-1.} \tag{100}$$

The second integral in (97) can be estimated using directly the hypothesis on the function f,

$$\int_{|x-y|> \frac{|x|}{2}} \frac{\Theta^\star(y)}{|x-y|^{2(1-s)}}\, \mathrm{d}y \leq \left(\frac{2}{|x|}\right)^{2(1-s)} \int_{\mathbb{R}^2} |T(\Psi^\star)|^{\nu} \leq \frac{C}{|x|^{2(1-s)}}. \tag{101}$$

By choosing $\eta \in \left]1-s, \frac{1}{s+1}\right[$ such that $\eta \geq \frac{s}{s+1}$, the estimates (100) and (101) give

$$\Psi^\star(x) \leq \frac{C}{1+|x|^{2(1-s)}}. \tag{102}$$

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