INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus
Mathématique

Sarah Al-Sheikh, Raneah Al-Maalwi and H. A. Ashi

A Mathematical Model of Unemployment with the Effect of Limited Jobs

Volume 359, issue 3 (2021), p. 283-290.

<https://doi.org/10.5802/crmath.164>

Comptes Rendus
Mathématique
2021, 359, nº 3, p. 283-290
https://doi.org/10.5802/crmath.164

Numerical Analysis, Dynamical systems / *Analyse numérique, Systèmes dynamiques*

A Mathematical Model of Unemployment with the Effect of Limited Jobs

Sarah Al-Sheikh [a], **Raneah Al-Maalwi** [a] **and H. A. Ashi** [a]

[a] Department of Mathematics, King AbdulAziz University, Jeddah 21551, Saudi Arabia
E-mail: haashi@kau.edu.sa

Abstract. In this paper, we have proposed and analyzed a non-linear mathematical model of the issue of unemployment by considering three main variables, namely the numbers of unemployed, employed and available vacancies. The model resembles the situation in some countries where the support of the government reaches a certain limited level where the rate of creating new jobs becomes constant and can no longer be proportional to the number of unemployed due to limited financial and economics resources. The qualitative results for the mathematical model are obtained utilizing the stability theory of nonlinear differential equations. Furthermore, some numerical simulations are illustrated to support the qualitative results.

Manuscript received 12th May 2020, revised 27th October 2020, accepted 1st December 2020.

1. Introduction

Unemployment is an economic situation in which individuals seeking jobs remain out of work. According to the International Labour Organization, the unemployment rate is defined as the number of unemployed people as a percentage of the labor force. The labor force includes people who are employed as well as unemployed, but it does not include people who retired, are not looking for work and children.

Attention to analytical and theoretical studies of this problem is due to the negative impact of this problem not only on the unemployed individuals but also the impact could extend to affect family members and the society as a whole. Research studies have found that unemployment is related to the increase in crime rates, isolation from society, that could lead to depression and consequently results in committing suicide, poverty, and inability to afford educational and medical expenses [6, 7].

Recently, mathematical models have been used to understand the dynamic of unemployment and to investigate the effects of some various factors on reducing the unemployment rate and the number of unemployed people [4, 5, 8–16]. Misra and Singh [8] presented a non-linear mathematical model of unemployment using three dynamic variables: the numbers of unemployed, temporarily employed and regularly employed people. Later, this mathematical model was refined by the authors themselves [9]. They extended the nonlinear dynamic model to take four variables into account: number of unemployed people, temporary/self-employed people, regularly employed people and avenues for the skill development among unemployed. Galindro and Torres [5] studied a simple mathematical model that describes more accurately the real data of unemployment from Portugal in the period 2004-2016. Munoli et al [11] aimed to analyze the model of unemployment by using five variables namely; a number of unemployed, temporarily employed, regularly employed people and the number of temporary and regular vacancies. In 2018 [2], our reaserch found the unemployment problem can be tackled if the following strategies are implemented: governments should increase the employment rate while decreasing the diminution rate of available vacancies. In 2020 [1], we introduced a new model by dividing the unemployed people into two subclasses: unemployed and skilled unemployed people. We studied the impact of these two subclasses on the existence and stability of the equilibrium points. Also, we emphasized on the importance of the training programs for aligning the skills of unemployed people with labor market demands.

The aim of this paper is to use mathematical modeling, analysis, and simulation to study the effect of the limited availability of jobs on reducing the unemployment rate. The paper is organized as follows: in Section 2, we explain the formulation of the model. The equilibrium points of the model are found in Section 3. The local and global stability of the equilibrium points is investigated in Section 4 and Section 5, respectively. In Section 6, a comparison between the present model and the model in [3] is done. In Section 7, we end the paper with our conclusions.

2. Mathematical Model

Job creation by the government is an important strategy to control the problem of unemployment, and it is an indispensable factor for ensuring that unemployed people have to get jobs. This is also confirmed by our previous reaserch [3].

In [3], we examined the effect of the government's support on reducing the unemployment rate by assuming that the rate of creating jobs by the government is proportional to the number of unemployed people. We built the model taking into account three main variables: the number of unemployed people $U(t)$, the number of employed people $E(t)$ and the number of available vacancies $V(t)$. The model was formulated as:

$$\begin{aligned} \frac{dU(t)}{dt} &= A - kU(t)V(t) + \beta E(t) - \mu U(t), \\ \frac{dE(t)}{dt} &= kU(t)V(t) - \beta E(t) - \alpha E(t), \\ \frac{dV(t)}{dt} &= \sigma U(t) - \delta V(t), \end{aligned} \tag{1}$$

where $A, k, \beta, \mu, \alpha, \sigma$ and δ are positive constants that are defined as follows:

- A: Rate of increase in the number of unemployed people.
- k: Rate of change of the number of unemployed people becoming employed.
- β: Rate of employed people resigned, being fired or dismissed from their jobs.
- μ: Rate of migration as well as death of unemployed people.
- α: Rate of migration, retirement or death of employed people.
- σ: Rate of creating new vacancies.
- δ: Diminution rate of available vacancies due to lack of government funds.

From the previous analysis [3], it was shown that model (1) has only one positive equilibrium point $Q^* = (U^*, E^*, V^*)$, where

$$\begin{aligned} V^* &= \frac{\sigma}{\delta}U^*, \\ E^* &= \frac{k\sigma}{(\alpha+\beta)\delta}U^{*2}, \\ U^* &= \frac{-(\alpha+\beta)\mu\delta + \sqrt{((\alpha+\beta)\mu\delta)^2 + 4(\alpha+\beta)\delta Ak\alpha\sigma}}{2k\alpha\sigma}. \end{aligned} \tag{2}$$

It was proved that the positive equilibrium point Q^* is always locally asymptotically stable. Also, it was proved, by using Lyapunov function, that the positive equilibrium Q^* is globally asymptotically stable whenever $U^*V^* \le \frac{A}{2k}$.

In this paper, we assume that the government support declines due to its limited financial and economic resources.

Based on the above motivation, we present the dynamic of unemployment by the following system

$$\begin{aligned} \frac{dU(t)}{dt} &= A - kU(t)V(t) + \beta E(t) - \mu U(t), \\ \frac{dE(t)}{dt} &= kU(t)V(t) - \beta E(t) - \alpha E(t), \\ \frac{dV(t)}{dt} &= C(U(t)) - \delta V(t). \end{aligned} \tag{3}$$

The function $C(U(t))$ is defined by[1]

$$C(U(t)) = \begin{cases} \sigma U(t), & \text{if } 0 < U(t) \le U_m \\ M, & \text{if } U(t) > U_m, \end{cases}$$

where $M = \sigma U_m$. This means that the rate of creating jobs is proportional to the number of unemployed people as long as the number of unemployed people has not reached its maximum limit U_m where the rate of creating jobs becomes constant.

The following Theorem 1 states the region of attraction of system (3).

Theorem 1. *If* $(U(t), E(t), V(t)) \in R_3^+$*, then the set defined by*

$$\Omega = \left\{ (U(t), E(t), V(t)) : 0 \le U(t) + E(t) \le \frac{A}{\gamma},\ 0 \le V(t) \le \frac{\sigma A}{\gamma\delta} \right\},$$

where $\gamma = \min(\mu, \alpha)$*, is positively invariant.*

3. Equilibria Analysis

To find the equilibrium points of system (3), we set the rates in (3) to zero

$$\begin{aligned} A - k\overline{U}\,\overline{V} + \beta\overline{E} - \mu\overline{U} &= 0, \\ k\overline{U}\,\overline{V} - \beta\overline{E} - \alpha\overline{E} &= 0, \\ C(\overline{U}) - \delta\overline{V} &= 0. \end{aligned} \tag{4}$$

[1] The function is defined by Wang to simulate a limited capacity for treating the spread of disease [17].

Direct calculations show that there exists only one positive equilibrium point $\overline{Q} = (\overline{U}, \overline{E}, \overline{V})$, where

$$\begin{aligned}\overline{V} &= \frac{C(\overline{U})}{\delta},\\ \overline{E} &= \frac{k\overline{U}}{(\alpha+\beta)\delta}C(\overline{U}),\\ \overline{U} &= \frac{A\delta(\alpha+\beta)}{\mu\delta(\alpha+\beta)+k\alpha C(\overline{U})}.\end{aligned}$$

When $0 < U(t) \leq U_m$ then, $C(U) = \sigma U(t)$ and the equilibrium point is defined as the same as the equilibrium point (2) of model (1).

On the other hand, when $U(t) > U_m$ then, $C(U) = \sigma U_m$, and we define the positive equilibrium point $Q^{**} = (U^{**}, E^{**}, V^{**})$, where

$$\begin{aligned}V^{**} &= \frac{\sigma}{\delta}U_m,\\ E^{**} &= \frac{k\sigma}{(\alpha+\beta)\delta}U_m U^{**},\\ U^{**} &= \frac{A\delta(\alpha+\beta)}{k\sigma U_m\alpha+\mu\delta(\alpha+\beta)}.\end{aligned} \tag{5}$$

We summarize the above results in the following theorem

Theorem 2. *The unemployment model* (3) *has only one positive equilibrium point that exists depending on the solution of the model:*

- $Q^* = (U^*, E^*, V^*)$, *which exists when* $0 < U(t) \leq U_m$.
- $Q^{**} = (U^{**}, E^{**}, V^{**})$ (5), *which exists when* $U(t) > U_m$.

4. Local Stability Analysis

This section investigates the local stability behavior of the equilibrium point $Q^{**} = (U^{**}, E^{**}, V^{**})$ (5) using the *linearization method*. The analysis concerning the equilibrium point $Q^* = (U^*, E^*, V^*)$ had been already investigated in [3]. The outcome showed that this point is always locally asymptotically stable. Note that this equilibrium exists when $0 < U(t) \leq U_m$ and when $U(t) > U_m$ the positive equilibrium Q^{**} (5) exists.

To determine the stability of the equilibrium point Q^{**}, we state the following Theorem 3

Theorem 3. *The positive equilibrium $Q^{**} = (U^{**}, E^{**}, V^{**})$ is locally asymptotically stable.*

Proof. Computing the Jacobian matrix of system (3) at the positive equilibrium Q^{**} (5) gives

$$J(Q^{**}) = \begin{pmatrix} -\mu - kV^{**} & \beta & -kU^{**} \\ kV^{**} & -(\alpha+\beta) & kU^{**} \\ 0 & 0 & -\delta \end{pmatrix}.$$

Note that the first eigenvalue $\lambda_1 = -\delta$ of $J(Q^{**})$ is negative. The second and third eigenvalues are determined by characteristic equation

$$\lambda^2 + a_1\lambda + a_2 = 0,$$

where

$$\begin{aligned} a_1 &= kV^{**} + \mu + \alpha + \beta,\\ a_2 &= \alpha k V^{**} + \mu(\alpha+\beta).\end{aligned}$$

Based on *Routh Hurwitz criterion*, the second and third eigenvalues have negative real part because $a_1 > 0$ and $a_2 > 0$. Therefore, the positive equilibrium Q^{**} is always locally asymptotically stable. □

5. Global Stability Analysis

In this section, we discuss in Theorem 4, the global stability of the positive equilibrium Q^{**} (5) of system (3). The analysis concerning the equilibrium point $Q^{*} = (U^{*}, E^{*}, V^{*})$ had been already investigated in [3]. The outcome showed that this point is globally asymptotically stable whenever $U^{*}V^{*} \le \frac{A}{2k}$. Now, we give the global asymptotic stability theorem regarding the positive equilibrium Q^{**} (5) as follows:

Theorem 4. *If $U^{**}V^{**} \le \frac{A}{k}$, then the positive equilibrium Q^{**} is globally asymptotically stable.*

Proof. Consider the following

$$\begin{aligned} L(Q(t)) &= U(t) - U^{**} - U^{**}\ln\frac{U(t)}{U^{**}} + E(t) - E^{**} - E^{**}\ln\frac{E(t)}{E^{**}} \\ &+ \frac{(\alpha+\beta)E^{**}}{\sigma U_m}\left(V(t) - V^{**} - V^{**}\ln\frac{V(t)}{V^{**}}\right) + \frac{\beta}{2(\mu+\alpha)U^{**}}\left(U(t) - U^{**} + E(t) - E^{**}\right)^2. \end{aligned}$$

Differentiating L with respect to time and applying the derivative of the variables in system (3) yields

$$\begin{aligned} \frac{dL}{dt} &= \left(1 - \frac{U^{**}}{U}\right)(A - \mu U - kUV + \beta E) \\ &+ \left(1 - \frac{E^{**}}{E}\right)(kUV - (\alpha+\beta)E) \\ &+ \left(\frac{(\alpha+\beta)E^{**}}{\sigma U_m} - \frac{(\alpha+\beta)E^{**}V^{**}}{\sigma U_m V}\right)(\sigma U_m - \delta V) \\ &+ \frac{\beta}{(\mu+\alpha)U^{**}}\left(U - U^{**} + E - E^{**}\right)(A - \mu U - \alpha E). \end{aligned} \tag{6}$$

Substituting the positive equilibrium $Q^{**} = (U^{**}, E^{**}, V^{**})$ (5) in system (4), the case when $C(U) = \sigma U_m$

$$A = (\alpha+\beta)E^{**} - \beta E^{**} + \mu U^{**},\ kU^{**}V^{**} = (\alpha+\beta)E^{**},\ \text{and}\ \sigma U_m = \delta V^{**}, \tag{7}$$

which can be utilized in (6) as follows

$$\begin{aligned} \frac{dL}{dt} &= \left(1 - \frac{U^{**}}{U}\right)\left(-\mu(U - U^{**}) + (\alpha+\beta)E^{**} - kUV + \beta(E - E^{**})\right) \\ &+ \left(1 - \frac{E^{**}}{E}\right)(kUV - (\alpha+\beta)E) + (\alpha+\beta)E^{**} \\ &- \frac{(\alpha+\beta)E^{**}V}{V^{**}} + (\alpha+\beta)E^{**} - \frac{(\alpha+\beta)E^{**}V^{**}}{V} \\ &+ \frac{\beta}{(\mu+\alpha)U^{**}}\left(U - U^{**} + E - E^{**}\right)\left(-\mu(U - U^{**}) - \alpha(E - E^{**})\right), \\ &= -\frac{\mu}{U}(U - U^{**})^2 + \left(1 - \frac{U^{**}}{U}\right)(\alpha+\beta)E^{**} + \frac{\beta}{U}(E - E^{**})(U - U^{**}) \\ &- (\alpha+\beta)E^{**}\left(\frac{E^{**}UV}{EU^{**}V^{**}}\right) - (\alpha+\beta)E + 3(\alpha+\beta)E^{**} - \frac{(\alpha+\beta)E^{**}V^{**}}{V} \\ &- \frac{\mu\beta}{(\mu+\alpha)U^{**}}(U - U^{**})^2 - \frac{\beta}{U^{**}}(U - U^{**})(E - E^{**}) - \frac{\alpha\beta}{(\mu+\alpha)U^{**}}(E - E^{**})^2, \end{aligned}$$

$$
\begin{aligned}
&= -\frac{\mu}{U}\left(U-U^{**}\right)^2 - \frac{\beta}{UU^{**}}\left(U-U^{**}\right)^2\left(E-E^{**}\right) - \frac{\beta\mu}{\left(\mu+\alpha\right)U^{**}}\left(U-U^{**}\right)^2 \\
&\quad + \left(\alpha+\beta\right)E^{**}\left(4-\frac{U^{**}}{U}-\frac{E^{**}UV}{EU^{**}V^{**}}-\frac{E}{E^{**}}-\frac{V^{**}}{V}\right) - \frac{\alpha\beta}{\left(\mu+\alpha\right)U^{**}}\left(E-E^{**}\right)^2 \\
&= -\frac{\left(U-U^{**}\right)^2}{UU^{**}}\left(\mu U^{**}+\beta\left(E-E^{**}\right)+\frac{\beta\mu U}{\mu+\alpha}\right) - \frac{\alpha\beta}{\left(\mu+\alpha\right)U^{**}}\left(E-E^{**}\right)^2 \\
&\quad + \left(\alpha+\beta\right)E^{**}\left(4-\frac{U^{**}}{U}-\frac{E^{**}UV}{EU^{**}V^{**}}-\frac{E}{E^{**}}-\frac{V^{**}}{V}\right).
\end{aligned}
$$

Since the geometric mean of a data set is less than or equal to its arithmetic mean, it follows that

$$
4-\frac{U^{**}}{U}-\frac{E^{**}UV}{EU^{**}V^{**}}-\frac{E}{E^{**}}-\frac{V^{**}}{V}\le 0
$$

Now, by assuming that $U^{**}V^{**}\le \frac{A}{k}$, we find that $A-kU^{**}V^{**}\ge 0$ which means that $\mu U^{**}-\beta E^{**}\ge 0$. This prove that $\frac{dL}{dt}<0 \;\; \forall \;\; (U(t),E(t),V(t))\neq (U^{**},E^{**},V^{**})$. Therefore, the proof of Theorem 4 has been accomplished, and we conclude that Q^{**} (5) is globally asymptotically stable. □

6. Impact of Limited Jobs

In this section, we investigate the impact of limiting the creation of new jobs to a constant rate on the behavior of the positive equilibrium values related to the number of unemployed people by comparing model (1) with model (3).

From the previous analysis, we note that when $0<U(t)\le U_m$, the positive equilibrium U^* exists

$$
U^* = \frac{A\delta\left(\alpha+\beta\right)}{k\alpha\sigma U^* + \mu\delta\left(\alpha+\beta\right)}, \tag{8}
$$

and is locally asymptotically stable, whereas, when $U(t)>U_m$, the positive equilibrium

$$
U^{**} = \frac{A\delta\left(\alpha+\beta\right)}{k\alpha\sigma U_m + \mu\delta\left(\alpha+\beta\right)}. \tag{9}
$$

exists and is locally asymptotically stable (see Theorem 3). Note that the equilibrium point (8) is the same as the equilibrium point of model (1). Model (1) has only one positive equilibrium, which exists always and also it is locally asymptotically stable (see [3]).

To conduct the investigation, we divide the analysis to three cases:

Case 1. If $U_m = U^* \Rightarrow U^* = U^{**}$.

In this case, the behavior of the unemployed solution of model (3) resembles that of model (1), i.e, the solution approaches the same equilibrium point U^*.

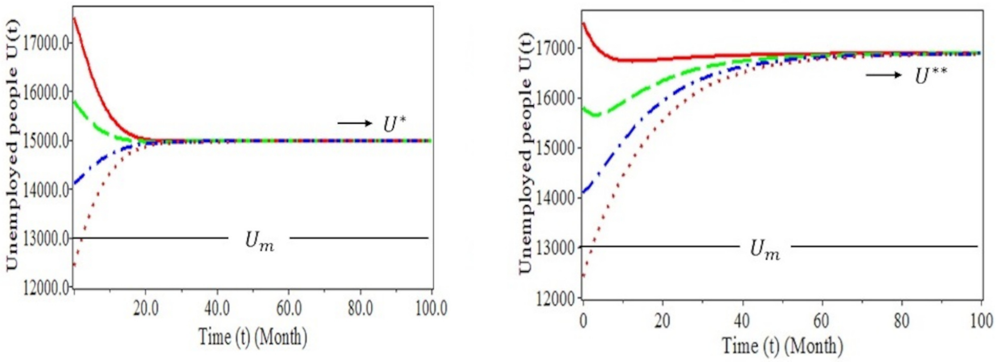
Case 2. If $U_m > U^* \Rightarrow U^* > U^{**}$.

In this case if we start with a solution $U(t)\le U_m$, then the solution approaches U^*. If we start with a solution $U(t)>U_m$, then the solution should approaches U^{**}. But unfortunately, the solution changes its behavior and instead approaches U^* due to the fact that the solution becomes less than U_m on its way to reach U^{**}. Hence, the behavior of the unemployed solution in this case is the same as in case 1.

Case 3. If $U_m < U^* \Rightarrow U^* < U^{**}$.

Here, from model (3), if we start with a solution $U(t)>U_m$, then the solution approaches U^{**}. If we start with a solution $U(t)\le U_m$, then the solution unfortunately approaches U^{**} instead of

U^* due to the fact that the solution becomes larger than U_m on its way to reach U^* (see Figure 1 (b)). On the other hand, the solution of the unemployed in model (1) always approaches U^* (see Figure 1 (a)).



(a) Model (1)

(b) Model (3)

Figure 1. Behavior simulations for the unemployed in model (1) and in model (3).

Our above investigation conclude that the limitation of creating new jobs to be a constant rate when the unemployed solution $U(t)$ reaches a certain level U_m in model (3), has an unsatisfying impact on the unemployment problem. The findings reveal that the problem will be exacerbated by the increase in the number of unemployed people annually and will turn into a disaster threatening the society. Therefore, to avoid this problem, the maximum capacity to create more jobs must be greater than or equal U^*.

7. Conclusion

In this paper, a mathematical model for unemployment is presented to include the impact of limited creating jobs by the government. This model consists of three variables: the number of unemployed people, the number of employees and the number of available vacancies. The existence of the equilibrium point in our model depends on the behavior of the number of the unemployed solution $U(t)$. If $U(t)$ is beyond a certain limited level U_m, the rate of creating jobs becomes a constant and the equilibrium point Q^{**} exists. If not, the rate of creating jobs is proportional to the number of unemployed people and the equilibrium point Q^* exists. Stability analysis of the equilibrium point is proved using stability methods of differential equations. Finally, a comparison between this model and the model in [3] is done; we found that the maximum capacity to create more jobs must be greater than or equal U^* to avoid the increase in the number of unemployed people.

References

[1] R. Al-maalwi, S. Al-Sheikh, H. A. Ashi, S. Asiri, "Mathematical Modeling and Parameter Estimation of Unemployment with the Impact of Training Programs", *Math. Comput. Simulat.* **182** (2021), p. 705-720.

[2] R. Al-maalwi, H. A. Ashi, S. Al-Sheikh, "Unemployment Model", *Appl. Math. Sci.* **12** (2018), no. 21, p. 989-1006.

[3] H. A. Ashi, R. Al-maalwi, S. Al-Sheikh, "Study of the Unemployment Problem by Mathematical Modeling: Predictions and Controls", To appear in the *The Journal of Mathematical Sociology*, 2021.

[4] A. A. M. Daud, A. W. Ghozali, "Stability analysis of a simple mathematical model for unemployment", *Caspian Journal of Applied Sciences Research* **4** (2015), no. 2, p. 15-18.

[5] A. Galindro, D. F. M. Torres, "A simple mathematical model for unemployment: a case study in Portugal with optimal control†", *Stat. Optim. Inf. Comput.* **6** (2018), no. 1, p. 116-129.

[6] A. H. Goldsmith, J. R. Veum, W. Darity Jr, "The psychological impact of unemployment and joblessness", *The Journal of Socio-Economics* **25** (1996), no. 3, p. 333-358.

[7] A. Kidwai, Z. Sarwar, "Psychological Impacts of Unemployment Evidence from the Literature", *Rev. Integr. Bus. Econ.* **4** (2015), no. 3, p. 141-152.

[8] A. K. Misra, A. K. Singh, "A Mathematical Model for Unemployment", *Nonlinear Anal., Real World Appl.* **12** (2011), p. 128-136.

[9] A. K. Misra, A. K. Singh, P. K. Singh, "Modeling the Role of Skill Development to Control Unemployment", *Differ. Equ. Dyn. Syst.* (2017).

[10] S. B. Munoli, S. R. Gani, "Optimal control analysis of a mathematical model for unemployment", *Optim. Control Appl. Meth.* **37** (2016), no. 4, p. 798-806.

[11] ———, "A Mathematical Approach to Employment policies: An Optimal Control Analysis", *International Journal of Statistics and Systems (IJSS)* **12** (2017), no. 3, p. 549-565.

[12] G. N. Pathan, P. H. Bhathawala, "A Mathematical Model for unemployment with effect of self-employment", *IOSR Journal of Mathematics (IOSR-JM)* **11** (2015), p. 37-43.

[13] G. N. Pathan, "A Mathematical Model for Unemployment-Taking an action without delay", *Adv. Dyn. Syst. Appl.* **12** (2017), no. 1, p. 41-48.

[14] G. N. Pathan, P. H. Bhathawala, "Unemployment-Discussion with a Mathematical Model", *International Journal of Business Management and Economic Research (IJBMER)* **3** (2016), no. 1, p. 19-27.

[15] G. N. Pathan, P. H. Bhathawala, "Mathematical Model for Unemployment Control-A Numerical Study", *Journal of Emerging Technologies and Innovative Research (JETIR)* **5** (2018), no. 8, p. 253-259.

[16] S. Sundar, A. Tripathi, R. Naresh, "Does Unemployment Induce Crime in Society? A Mathematical Study", *American Journal of Applied Mathematics and Statistics* **6** (2018), no. 2, p. 44-53.

[17] W. Wang, "Backward Bifurcation of an epidemic model with treatment", *Math. Biosci.* **201** (2006), no. 1-2, p. 58-71.