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Linear dependence of quasi-periods over the rationals

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Abstract. In this note we shall show that a lattice $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ in $\mathbb{C}$ has $\mathbb{Q}$-linearly dependent quasi-periods if and only if ω_2/ω_1 is equivalent to a zero of the Eisenstein series E_2 under the action of $\mathrm{SL}_2(\mathbb{Z})$ on the upper half plane of $\mathbb{C}$.

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1. Introduction

Let $\mathcal{L} = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$ with $\omega_2/\omega_1 \in \mathbb{H}$, the upper half plane of $\mathbb{C}$. Let $\sigma(z;\omega_1,\omega_2)$ and $\zeta(z;\omega_1,\omega_2)$ respectively be the Weierstrass sigma and zeta functions associated to $\mathcal{L}$. Let g_2 and g_3 be the invariants of $\mathcal{L}$. The numbers $\eta_1(\mathcal{L}) = \eta(\omega_1) = 2\zeta(\omega_1/2;\omega_1,\omega_2)$, $\eta_2(\mathcal{L}) = \eta(\omega_2) = 2\zeta(\omega_2/2;\omega_1,\omega_2)$ are called *the quasi-periods* associated to $\mathcal{L}$. When $\mathcal{L}$ is clear from the context, we simply write η_1,η_2 instead of $\eta_1(\mathcal{L})$ and $\eta_2(\mathcal{L})$ respectively. One of the long standing open problem in transcendental number theory is to find the dimension of the vector space $V_{\mathcal{L}}$ generated by

$$1, \omega_1, \omega_2, \eta_1, \eta_2, \pi \tag{1}$$

over $\overline{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$. Starting from the work of Siegel [10], Schneider [9], Baker [1], Coates [3,4] and finally by Masser [8], it is now known that for a lattice $\mathcal{L}$ with algebraic invariants g_2, g_3, the vector space $V_{\mathcal{L}}$ has dimension 4 in the CM case and 6 in the non-CM case. This is because in the CM case, there are two linear relations among the numbers in (1). The first one is

$$\tau\omega_1 - \omega_2 = 0$$

where $\tau = \omega_2/\omega_1 \in \overline{\mathbb{Q}}$ and the other one is given by

$$C\eta_1 - \tau\eta_2 - \kappa\omega_2 = 0, \tag{2}$$

where C is the constant term of the minimal polynomial of τ over $\mathbb{Q}$ and $\kappa \in \mathbb{Q}(\tau, g_2, g_3)$ (see [8, Lemma 3.1] or [2, Theorem 8] for more details). Masser also proved that the number κ in (2) vanishes if and only if τ is congruent to $i = \sqrt{-1}$ or $\rho = e^{2\pi i/3}$ under $SL_2(\mathbb{Z})$; and in that case, η_1 and η_2 are linearly dependent over $\mathbb{Q}(\tau)$.

Apart from lattices with algebraic invariants, there are two more cases for which we know the dimension of $V_{\mathcal{L}}$. For example, if $\omega_1 = 1$ and $\omega_2 = i$ then by Siegel [10] at least one of the g_2, g_3 is not algebraic. And by (2), the quotient $\eta_2/\eta_1 = -i$ in this case. (Note that we used (2) to find the ratio η_2/η_1; because, as we shall see later that, η_2/η_1 depends only on ω_2/ω_1 and not on g_2, g_3; this ratio can also be obtained from (4) and (9) below by choosing an appropriate γ). Hence by the Legendre's relation [7, p. 241] the vector space $V_{\mathcal{L}}$ has dimension two. Similarly, if $\omega_1 = 1$ and $\omega_2 = \rho$ then in this case also at least one of the g_2, g_3 is not algebraic and by (2) we have $\eta_2/\eta_1 = \rho^{-1}$. Hence in this case also the vector space $V_{\mathcal{L}}$ has dimension two. Except for these cases the author is not aware of any other lattices $\mathcal{L}$ for which the dimension of $V_{\mathcal{L}}$ is known. In [6] Heins shown that a pair of complex numbers (z_1, z_2) occur as quasi-periods of some lattice $\mathcal{L}$ if and only if $|z_1|+|z_2| > 0$. Thus there are lattices with $\mathbb{Q}$-linearly dependent quasi-periods, and therefore, for such lattices $\mathcal{L}$ the vector space $V_{\mathcal{L}}$ has dimension at most five. Unfortunately, Heins method does not allow us to determine the lattices with $\mathbb{Q}$-linearly dependent quasi-periods. The purpose of this note is to classify all such lattices. For $\tau \in \mathbb{H}$, the *generalised Eisenstein series of weight* 2 is defined by

$$G_2(\tau) = \sum_c \sum_d (c\tau + d)^{-2} \tag{3}$$

where the sum is over all integers c and d with $|c| + |d| > 0$; while the *normalised Eisenstein series of weight* 2 is defined by

$$E_2(\tau) = 3G_2(\tau)/\pi^2 = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n) q^n \tag{4}$$

where $\sigma_1(n)$ is the sum of all positive divisors of n, and $q = e^{2\pi i\tau}$. Our main result is the following.

Main Theorem. *Let $\mathcal{L} = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$ with $\tau = \frac{\omega_2}{\omega_1} \in \mathbb{H}$. Then η_1 and η_2 are $\mathbb{Q}$-linearly dependent if and only if τ is congruent to a zero of $E_2(z)$ under $SL_2(\mathbb{Z})$.*

The following corollary is immediate.

Corollary 1. *Let $\mathcal{L} = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$ with $\tau = \frac{\omega_2}{\omega_1} \in \mathbb{H}$ is equivalent to a zero of $E_2(z)$ under $SL_2(\mathbb{Z})$. Then $V_{\mathcal{L}}$ has dimension at most 4 in the CM case and at most five in the non-CM case.*

We shall prove the Main Theorem in the next section. The proof relies on the formula expressing the quasi-periods in-terms of G_2 (see Lemma 3) and the transformation formula of E_2 given by

$$E_2(\gamma.\tau) = (c\tau + d)^2 E_2(\tau) + \frac{6c}{\pi i}(c\tau + d) \tag{5}$$

where $\tau \in \mathbb{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$.

2. Quasi-periods and Laurent's expansions

Let $\sigma(z;\tau) = \sigma(z;1,\tau)$ and $\zeta(z;\tau) = \zeta(z;1,\tau)$ respectively be the Weierstrass sigma and zeta functions associated to the lattice $\mathcal{L}_\tau = \mathbb{Z} + \mathbb{Z}\tau$ with $\tau \in \mathbb{H}$. These two functions are connected by the relation $\zeta(z;\tau) = \frac{\sigma'(z;\tau)}{\sigma(z;\tau)}$.

For $\omega \in \mathcal{L}_\tau \setminus \{0\}$, we write

$$\frac{1}{z-\omega} = -\frac{1}{\omega} - \frac{z}{\omega^2} - \frac{z^2}{\omega^3} - \frac{z^3}{\omega^4} - \cdots$$

for z near the origin. Thus, we have

$$\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} = -\frac{z^2}{\omega^3} - \frac{z^3}{\omega^4} - \cdots.$$

Now summing over all non-zero periods of $\mathcal{L}_\tau$ and adding the term $1/z$, we obtain

$$\zeta(z;\tau) = \frac{1}{z} - \sum_{k=2}^{\infty} \mathrm{G}_{2k} z^{2k-1} \tag{6}$$

where $\mathrm{G}_{2k} = \mathrm{G}_{2k}(\tau) = \sum_{\omega \in \mathcal{L}_\tau \setminus \{0\}} \omega^{-2k}$ for $k \geq 2$ (the coefficients of even powers of z in (6) are zero, since $\zeta(z;\tau)$ is an odd function).

The next lemma gives a connection between quasi-periods and the values of generalized Eisenstein series G_2.

Lemma 2. *Let η_1 be the quasi-period associated to the period* 1 *of the lattice $\mathcal{L}_\tau = \mathbb{Z} + \mathbb{Z}\tau$ with $\tau \in \mathbb{H}$. Then $\eta_1 = G_2(\tau)$.*

Proof. We follow the strategy as given in [7, Chapter 18]. Accordingly, we express the Laurent's expansion of $\zeta(z;\tau)$ near the origin into two different ways and then comparing the corresponding coefficients we obtain the required representation for η_1. The first one is given by (6). For obtaining the second representation, let $q_z = e^{2\pi i z}$. Consider the function

$$\phi_1(z) = (2\pi i)^{-1}\left(q_z - 1\right) \prod_{n=1}^{\infty} \frac{\left(1 - q_{z+n\tau}\right)\left(1 - q_{n\tau - z}\right)}{\left(1 - q_{n\tau}\right)^2}. \tag{7}$$

Since $\tau \in \mathbb{H}$, we have $|q_{n\tau}| < 1/2^n$ for large values of n, and hence, for such values

$$\left|\frac{q_{n\tau}}{(1-q_{n\tau})^2}\right| < \frac{1}{(2^n-1)^2}.$$

It follows that the series

$$\sum_{n=1}^{\infty} \left(\frac{\left(1 - q_{z+n\tau}\right)\left(1 - q_{n\tau - z}\right)}{\left(1 - q_{n\tau}\right)^2} - 1 \right) \tag{8}$$

converges absolutely and uniformly on compact subsets of $\mathbb{C}$. Thus, the function ϕ_1 is entire. Moreover, it satisfying the following transformation formulas (see [7, p. 247] for more details):

$$\phi_1(z+1) = \phi_1(z) \quad \text{and} \quad \phi_1(z+\tau) = -\frac{1}{q_z}\phi_1(z).$$

On the other hand, the entire function

$$\phi_2(z) = e^{-\frac{1}{2}\eta_1 z^2} q_z^{1/2} \sigma(z;\tau)$$

also satisfies

$$\phi_2(z+1) = \phi_2(z) \quad \text{and} \quad \phi_2(z+\tau) = -\frac{1}{q_z}\phi_2(z).$$

Therefore, the quotient $\phi_1(z)/\phi_2(z)$ is elliptic. The product in (7) shows that both ϕ_1 and ϕ_2 have a simple zero at each point of $\mathbb{Z} + \mathbb{Z}\tau$ and no other zeros. Hence $\phi_1(z)/\phi_2(z)$ must be constant. Taking limit $z \to 0$ we see that the constant is 1, and therefore $\phi_1(z) = \phi_2(z)$. We thus have

$$\sigma(z;\tau) = (2\pi i)^{-1} e^{\frac{1}{2}\eta_1 z^2} \left(q_z^{1/2} - q_z^{-1/2}\right) \prod_{n=1}^{\infty} \frac{\left(1 - q_{z+n\tau}\right)\left(1 - q_{n\tau - z}\right)}{\left(1 - q_{n\tau}\right)^2}.$$

Since the series in (8) converges absolutely and uniformly on compact subsets of $\mathbb{C}$, taking logarithmic derivative term by term on the right side of the above equation we obtain

$$\zeta(z;\tau) = \eta_1 z + \pi i \left(\frac{q_z + 1}{q_z - 1}\right) + 2\pi i \sum_{n=1}^{\infty} \left(\frac{q_{n\tau - z}}{1 - q_{n\tau - z}} - \frac{q_{z+n\tau}}{1 - q_{z+n\tau}}\right).$$

If we restrict the values of z such that $|q_\tau| < |q_z| < |q_\tau^{-1}|$, then we have

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{q_{n\tau-z}}{1-q_{n\tau-z}}-\frac{q_{n\tau+z}}{1-q_{n\tau+z}}\right) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(q_{n\tau-z}^m - q_{n\tau+z}^m\right)\\ &= \sum_{m=1}^{\infty}\left(q_z^{-m}-q_z^m\right)\left(\sum_{n=1}^{\infty} q_{n\tau}^m\right)\\ &= \sum_{m=1}^{\infty}\left(\frac{q_{m\tau}}{1-q_{m\tau}}\right)\left(q_z^{-m}-q_z^m\right).\end{aligned}$$

Near the origin, we have

$$i\left(\frac{q_z+1}{q_z-1}\right) = \cot \pi z = \sum_{k=0}^{\infty}\frac{(-1)^k 2^{2k} B_{2k}(\pi z)^{2k-1}}{(2k)!},$$

and

$$q_z^{-m} - q_z^m = -2i\sum_{k=0}^{\infty}(-1)^k\frac{(2\pi m z)^{2k+1}}{(2k+1)!}$$

where B_r is the r^{th} Bernoulli's number. Thus we have,

$$\zeta(z;\tau) = \eta_1 z + \pi\sum_{k=0}^{\infty}\frac{(-1)^k 2^{2k} B_{2k}(\pi z)^{2k-1}}{(2k)!} - 4\pi\sum_{m=1}^{\infty}\sum_{k=0}^{\infty}(-1)^k\left(\frac{q_{m\tau}}{1-q_{m\tau}}\right)\frac{(2\pi m z)^{2k+1}}{(2k+1)!}.$$

Now comparing the coefficients of z on the above equation with that of (6) we get

$$\begin{aligned}\eta_1 &= \frac{\pi^2 2^2 B_2}{2} - 8\pi^2\sum_{m=1}^{\infty}\frac{m q_{m\tau}}{1-q_{m\tau}}\\ &= \frac{\pi^2}{3}\left(1-24\sum_{m=1}^{\infty}\frac{m q_{m\tau}}{1-q_{m\tau}}\right)\\ &= \frac{\pi^2}{3}\left(1-24\sum_{m=1}^{\infty}\sum_{\ell=1}^{\infty} m q_\tau^{\ell m}\right)\\ &= \frac{\pi^2}{3}\left(1-24\sum_{n=1}^{\infty}\sigma_1(n) q^n\right) = \mathrm{G}_2(\tau),\end{aligned}$$

by (4). This completes the proof of the Lemma 2. □

There is a slight change in the notations used in the above lemma from that of [7, Chapter 18]. In [7], lattices in $\mathbb{C}$ are written in the form $\mathbb{Z}\omega_1+\mathbb{Z}\omega_2$ with the assumption $\omega_1/\omega_2 \in \mathbb{H}$. This implies that the quasi-period associated to the period 1 of the lattice $\omega_2^{-1}(\mathbb{Z}\omega_1+\mathbb{Z}\omega_2)$ is denoted by η_2 in [7, Chapter 18]. Whereas, in our notation lattices in $\mathbb{C}$ are written in the form $\mathbb{Z}\omega_1+\mathbb{Z}\omega_2$ with the assumption $\omega_2/\omega_1 \in \mathbb{H}$. This implies that the quasi-period associated to the period 1 of the lattice $\omega_1^{-1}(\mathbb{Z}\omega_1+\mathbb{Z}\omega_2)$ is denoted by η_1.

The following lemma is the homogeneous version of Lemma 2.

Lemma 3. *Let $\mathcal{L} = \mathbb{Z}\omega_1+\mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$ with $\tau = \frac{\omega_2}{\omega_1} \in \mathbb{H}$. We have*

$$\eta_1 = \frac{G_2(\tau)}{\omega_1} \quad \text{and} \quad \eta_2 = \frac{\tau G_2(\tau) - 2\pi i}{\omega_1}. \tag{9}$$

Proof. By the Legendre's relation

$$\omega_2\eta_1(\mathcal{L}) - \omega_1\eta_2(\mathcal{L}) = 2\pi i,$$

hence it is sufficient to show that $\eta_1(\mathcal{L}) = \frac{G_2(\tau)}{\omega_1}$. Since $\eta_1(\mathcal{L})$ is homogeneous of degree -1, it is enough to prove this lemma when $\mathcal{L} = \mathbb{Z}+\mathbb{Z}\tau$ with $\tau \in \mathbb{H}$. We are thus reduced to show that for $\mathcal{L} = \mathbb{Z}+\mathbb{Z}\tau$ with $\tau \in \mathbb{H}$, we have $\eta_1(\mathcal{L}) = G_2(\tau)$; but, this is a consequence of Lemma 2. This completes the proof. □

3. Proof of the Main Theorem

Let $\mathcal{L} = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$ with $\tau = \frac{\omega_2}{\omega_1} \in \mathbb{H}$. By Lemma 3, the quotient $\eta_2(\mathcal{L})/\eta_1(\mathcal{L})$ is a function of τ and we denote it by $F(\tau)$ (this function was first introduced and studied by Heins [5]). Hence by (4) and (9) we have

$$F(\tau) = \frac{\tau E_2(\tau) + 6/\pi i}{E_2(\tau)}. \tag{10}$$

It follows from this identity that $\eta_1(\mathcal{L})$ and $\eta_2(\mathcal{L})$ are $\mathbb{Q}$-linearly dependent if and only if $F(\tau)$ is a rational number (it is convenient here to assume ∞ is a rational). Hence we are reduced to show that $F(\tau)$ is a rational number if and only if there exists a zero τ' of $E_2(z)$ and a matrix $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\tau = \gamma.\tau'$.

If $F(\tau) = \infty$, then we have $E_2(\tau) = 0$. If $F(\tau) = 0$, then we have $\tau E_2(\tau) + 6/\pi i = 0$; and hence $E_2\left(\frac{-1}{\tau}\right) = \tau^2 E_2(\tau) + 6\tau/\pi i = 0$. Suppose that $F(\tau)$ is a rational number which is neither 0 nor ∞, say q/p, with $(p, q) = 1$. Then, by (10) we have

$$\left(-p\tau + q\right) E_2(\tau) = \frac{6p}{\pi i}. \tag{11}$$

Choose $r, s \in \mathbb{Z}$ such that $pr - qs = -1$. Then the matrix

$$\gamma = \begin{pmatrix} s & -r \\ -p & q \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

We set $\tau' = \gamma.\tau$. Then by (5),

$$E_2(\tau') = (-p\tau + q)\left((-p\tau + q)E_2(\tau) - \frac{6p}{\pi i}\right),$$

which is equal to zero by (11).

Conversely, let τ' be a zero of $E_2(z)$, and let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be an element of $\mathrm{SL}_2(\mathbb{Z})$. We shall show that $F(\gamma.\tau')$ is a rational number. If $c = 0$, then $\gamma = T^b$ where $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Thus $E_2(\gamma.\tau') = 0$, and hence $F(\gamma.\tau') = \infty$. If $a = 0$, then $\gamma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & d \end{smallmatrix}\right)$, and hence $\gamma.\tau' = \frac{-1}{\tau' + d}$. It follows from (5) that $F(\gamma.\tau') = 0$. Now let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be an element of $\mathrm{SL}_2(\mathbb{Z})$ such that $ac \neq 0$. Then, by (5) we have

$$0 = E_2(\tau') = E_2\left(\gamma^{-1}(\gamma.\tau')\right) = \left(-c(\gamma.\tau') + a\right)^2 E_2(\gamma.\tau') - \frac{6c}{\pi i}\left(-c(\gamma.\tau') + a\right).$$

Since τ' is not a rational number we must have

$$\left(\gamma.\tau' - a/c\right) E_2(\gamma.\tau') + \frac{6}{\pi i} = 0.$$

Again by (5), we have $E_2(\gamma.\tau') \neq 0$, from this we conclude that $F(\gamma.\tau') = a/c$ is a rational number, and this completes the proof of the Main Theorem.

4. Concluding remarks

It is expected that the zeros of E_2 are transcendental; but so far none of them is known to be transcendental. One may ask whether transcendence of ω_2/ω_1 is a necessary condition for $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ to have $\overline{\mathbb{Q}}$-linearly dependent quasi-periods? The answer is no. For example, the quasi-periods associated to $\mathbb{Z} + \mathbb{Z}i$ are $\overline{\mathbb{Q}}$-linearly dependent. It is interesting to classify all lattices with $\overline{\mathbb{Q}}$-linearly dependent quasi-periods.

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