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Dynamical systems / *Systèmes dynamiques*

Families of polynomials of every degree with no rational preperiodic points

Familles de polynômes de degré arbitraire sans points prépériodiques rationnels

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Abstract. Let K be a number field. Given a polynomial $f(x) \in K[x]$ of degree $d \geq 2$, it is conjectured that the number of preperiodic points of f is bounded by a uniform bound that depends only on d and $[K:\mathbb{Q}]$. However, the only examples of parametric families of polynomials with no preperiodic points are known when d is divisible by either 2 or 3 and $K = \mathbb{Q}$. In this article, given any integer $d \geq 2$, we display infinitely many parametric families of polynomials of the form $f_t(x) = x^d + c(t)$, $c(t) \in K(t)$, with no rational preperiodic points for any $t \in K$.

Résumé. Soit K un corps de nombres. Étant donné un polynôme $f(x) \in K[x]$ de degré $d \geq 2$, il est conjecturé que le nombre de points prépériodiques de f est borné par une constante ne dépendant que de d et $[K:\mathbb{Q}]$. Cependant, les seuls exemples de familles paramétriques de polynômes sans points prépériodiques supposent $2|d$ ou $3|d$ et $K = \mathbb{Q}$. Dans cet article, étant donné un entier $d \geq 2$, nous démontrons qu'il existe une infinité de familles paramétriques de polynômes de la forme $f_t(x) = x^d + c(t)$, $c(t) \in K(t)$, sans points prépériodiques rationnels pour tout $t \in K$.

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1. Introduction

An arithmetic dynamical system over a number field K consists of a rational function $f : \mathbb{P}^n(K) \to \mathbb{P}^n(K)$ of degree at least 2 with coefficients in K where the m^{th} iterate of f is defined recursively by $f^1(x) = f(x)$ and $f^m(x) = f(f^{m-1}(x))$ when $m \geq 2$. A point $P \in \mathbb{P}^n(K)$ is said to be a *periodic* point for f if there exists a positive integer m such that $f^m(P) = P$. If N is the smallest positive integer such that $f^N(P) = P$, then the periodic point P is said to be of *exact period* N. A point $P \in \mathbb{P}^n(K)$ is said to be a *preperiodic* point for f if the orbit $\{f^i(P) : i = 0,1,2,\ldots\}$ of P is finite, i.e., if some iterate $f^i(P)$ is periodic.

The following conjecture was proposed by Morton and Silverman in p. 4 of [6].

Conjecture 1. *There exists a bound $B(D,n,d)$ such that if $K/\mathbb{Q}$ is a number field of degree D, and $f : \mathbb{P}^n(K) \to \mathbb{P}^n(K)$ is a morphism of degree $d \geq 2$ defined over K, then the number of K-rational preperiodic points of f is bounded by $B(D,n,d)$.*

When f is taken to be a quadratic polynomial over $\mathbb{Q}$, the following conjecture was suggested in [2, Conjecture 2] and [9, Conjecture 2].

Conjecture 2. *If $N \geq 4$, then there is no quadratic polynomial $f(x) \in \mathbb{Q}[x]$ with a rational point of exact period N.*

The conjecture has been proved when $N = 4$, see [5, Theorem 4], and $N = 5$, see [2, Theorem 1]. A conditional proof for the case $N = 6$ was given in [10, Theorem 7].

Although polynomials described by an equation of the form $x^d + c$, $d \geq 2$, $c \in K$, with rational preperiodic points are scarce, examples of parametric families of polynomials with no preperiodic points are very few in the literature. In Theorem 4 of [3], families of such polynomials were given when d is even or when d is divisible by 3, and $K = \mathbb{Q}$. The main finding of this article can be described as follows. Let K be a number field. Given an arbitrary integer $d \geq 2$, we prove the existence of infinitely many parametric families of polynomials of degree d with no K-rational preperiodic points. This is achieved using some recent results on the non existence of rational points on certain twisted superelliptic curves.

2. Parametric families of polynomials with no periodic points

In what follows K will denote a number field with ring of integers $\mathcal{O}_K$. The following proposition is [8, Lemma 1].

Proposition 3. *Let $f(x) = x^d + c$, where $d \geq 2$ is an integer and $c \in K \setminus \{0\}$. If f has a K-rational periodic point, then there exist $a, b \in \mathcal{O}_K$ such that $c = a/b^d$, and $(a\mathcal{O}_K, b^d\mathcal{O}_K) = I^d$ for some ideal I in $\mathcal{O}_K$.*

Proposition 3 shows that polynomials described by an equation of the form $x^d + c$, $d \geq 2$, $c \in K$, with K-rational preperiodic points are rare. However, up to the knowledge of the author, the only such family is given as Theorem 4 in [3] where $K = \mathbb{Q}$. The statement of the latter theorem is as follows.

Theorem 4. *Let $2 \mid d$ and $m \geq 4$; or $3 \mid d$ and $m \geq 3$. Then for $t \in \mathbb{Q}$, the polynomial*

$$x^d + \frac{1}{1+t^m}$$

has no $\mathbb{Q}$-rational preperiodic points.

Now we state the main result of this work.

Theorem 5. *Let K be a number field with ring of integers $\mathcal{O}_K$. Let $d \geq 2$ be an integer. Let $P(T) \in \mathcal{O}_K[T]$ be of degree N a multiple of d such that the multiplicity of each of its roots is at most $d-1$. Assume moreover that the Galois group of $P(T)$ over K has an element fixing no root of $P(T)$. Then there exists $w \in \mathcal{O}_K \setminus \{0\}$ such that the polynomial*

$$f(x) = x^d + \frac{1}{w \cdot P(t)}$$

has no K-rational preperiodic points for any $t \in K$.

Proof. According to [4, Theorem 3.1], given that the Galois group of $P(T)$ over K has an element fixing no root of $P(T)$, it follows that there exists $w \in \mathscr{O}_K \setminus \{0\}$ such that the twisted superelliptic curve defined by $y^d = w \cdot P(T)$ has no K-rational points. In other words, there exists no $(y,t,s) \in K^3 \setminus \{(0,0,0)\}$ such that $y^d = w \cdot Q(t,s)$, where $Q(T,S) = S^N \cdot P(T/S)$. In view of Proposition 3, f has no K-rational periodic points of any period, hence no K-rational preperiodic points for any $t \in K$. □

Remark 6. One knows that the proportion of degree N polynomials $P(T) \in \mathscr{O}_K[T]$ with height bounded by H and such that the Galois group of $P(T)$ over K is isomorphic to the symmetric group S_N tends to 1 as H tends to ∞, see for example [1, Theorem 2.1]. Consequently, the proportion of fixed degree polynomials $P(T)$ introduced in Theorem 5 with height bounded by H tends to 1 as H tends to ∞.

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