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Support points of some classes of analytic and univalent functions

Vasudevarao Allu[*, a] **and Abhishek Pandey**[a]

[a] School of Basic Sciences, Indian Institute of Technology Bhubaneswar, Argul, Bhubaneswar, PIN-752050, Odisha (State), India

E-mails: avrao@iitbbs.ac.in (V. Allu), ap57@iitbbs.ac.in (A. Pandey)

Abstract. Let $\mathscr{A}$ denote the class of analytic functions in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ satisfying $f(0) = 0$ and $f'(0) = 1$. Let $\mathscr{U}$ be the class of functions $f \in \mathscr{A}$ satisfying

$$\left| f'(z)\left(\frac{z}{f(z)}\right)^2 - 1\right| < 1 \quad \text{for } z \in \mathbb{D},$$

and $\mathscr{G}$ denote the class of functions $f \in \mathscr{A}$ satisfying

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > -\frac{1}{2} \quad \text{for } z \in \mathbb{D}.$$

In the present paper, we characterize the set of support points of the classes $\mathscr{U}$ and $\mathscr{G}$.

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1. Introduction and Preliminaries

Let $\mathscr{H}$ denote the class of analytic functions in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Here $\mathscr{H}$ is a locally convex topological vector space endowed with the topology of uniform convergence over compact subsets of $\mathbb{D}$. Let $\mathscr{A}$ denote the class of functions $f \in \mathscr{H}$ such that $f(0) = 0$ and $f'(0) = 1$, and $\mathscr{S}$ the subclass of functions $f \in \mathscr{A}$ which are univalent (i.e., one-to-one) in $\mathbb{D}$. Each function $f \in \mathscr{S}$ has the following representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

A set $D \subset \mathbb{C}$ is said to be starlike with respect to a point $z_0 \in D$ if the line segment joining z_0 to every other point $z \in D$ lies entirely in D. A set D is called convex if line segment joining any two points of D lies entirely in D. A function $f \in \mathscr{A}$ is called starlike (convex respectively) if $f(\mathbb{D})$ is starlike with respect to the origin (convex respectively). Denote by $\mathscr{S}^*$ and $\mathscr{C}$ the classes of starlike and convex functions in $\mathscr{S}$ respectively. It is well-known that a function $f \in \mathscr{A}$ belongs to

[*] Corresponding author.

$\mathscr{S}^*$ if, and only if, $\operatorname{Re}(zf'(z)/f(z)) > 0$ for $z \in \mathbb{D}$. Similarly, a function $f \in \mathscr{A}$ belongs to $\mathscr{C}$ if, and only if, $\operatorname{Re}(1 + zf''(z)/f'(z)) > 0$ for $z \in \mathbb{D}$. Alexander [3] proved that $f \in \mathscr{C}$ if, and only if, $zf' \in \mathscr{S}^*$. Given $\alpha \in (-\pi/2, \pi/2)$ and $g \in \mathscr{S}^*$, a function $f \in \mathscr{A}$ is said to be close-to-convex with argument α with respect to g if

$$\operatorname{Re}\left(e^{i\alpha}\frac{zf'(z)}{g(z)}\right) > 0 \quad \text{for } z \in \mathbb{D}.$$

Let $\mathscr{K}_\alpha(g)$ denote the class of all such functions, and

$$\mathscr{K}(g) := \bigcup_{\alpha \in (-\pi/2, \pi/2)} \mathscr{K}_\alpha(g) \text{ and } \mathscr{K}_\alpha := \bigcup_{g \in \mathscr{S}^*} \mathscr{K}_\alpha(g)$$

be the classes of close-to-convex functions with respect to g, and close-to-convex functions with argument α, respectively. Let

$$\mathscr{K} := \bigcup_{\alpha \in (-\pi/2, \pi/2)} \mathscr{K}_\alpha = \bigcup_{g \in \mathscr{S}^*} \mathscr{K}(g)$$

be the class of close-to-convex functions. It is well-known that every close-to-convex function is univalent in $\mathbb{D}$. Geometrically, $f \in \mathscr{K}$ means that the complement of the image domain $f(\mathbb{D})$ is the union of non-intersecting half- lines. These standard classes are related by the proper inclusions $\mathscr{C} \subsetneq \mathscr{S}^* \subsetneq \mathscr{K} \subsetneq \mathscr{S}$.

Suppose X is a linear topological vector space and $V \subseteq X$. A point $x \in V$ is called an extreme point of V if it has no representation of the form $x = ty + (1-t)z, 0 < t < 1$ as a proper convex combination of two distinct points y, $z \in V$. We denote by EV the set of extreme points of V. The convex hull of a set $V \subseteq X$ is the smallest convex set containing V. The closed convex hull, denoted by $\overline{co}V$, is defined as the intersection of all closed convex sets containing V. Therefore the closed convex hull of V is the smallest closed convex set containing V, which is the closure of the convex hull of V. The Krein–Milman theorem [9] asserts that every compact subset of a locally convex topological vector space is contained in the closed convex hull of its extreme points.

A function f is called support point of a compact subset $\mathscr{F}$ of $\mathscr{H}$ if $f \in \mathscr{F}$, and if there is a continuous linear functional J on $\mathscr{H}$ such that $\operatorname{Re} J$ is non-constant on $\mathscr{F}$, and

$$\operatorname{Re} J(f) = \max\left\{\operatorname{Re} J(g) : g \in \mathscr{F}\right\}.$$

The set of all support points of a compact family $\mathscr{F}$ is denoted by $supp\mathscr{F}$

Support points of families of starlike, convex functions have been studied in [5], and support points of close-to-convex functions in [11] and [19]. Support points for starlike and convex function of order α have also been studied in [7]. For some important open problems related to extreme points and support points we refer to [13]. For a general reference, and for many important results on the above topic, we refer the reader to [12]. For a very recent work related to support points and extreme points we refer to [8]. In this paper Deng, Ponnusamy and Qiao have proved a necessary and sufficient condition for harmonic Bloch mapping f to be a support point of the unit ball of the normalized harmonic Bloch spaces in $\mathbb{D}$.

For $0 < \lambda \leq 1$, let $\mathscr{U}(\lambda)$ be the class of functions $f \in \mathscr{A}$ satisfying

$$\left|f'(z)\left(\frac{z}{f(z)}\right)^2 - 1\right| < \lambda \quad \text{for } z \in \mathbb{D}.$$

Since $f'(z)(z/f(z))^2$ has only finite values, each function in $\mathscr{U}(\lambda)$ is non-vanishing in $\mathbb{D}\setminus\{0\}$. Set $\mathscr{U} := \mathscr{U}(1)$. It is also clear that functions in $\mathscr{U}(\lambda)$ are locally univalent. Furthermore, Aksentev [2], and Ozaki and Nunokawa [14] have shown that functions in $\mathscr{U}(\lambda)$ are univalent, i.e., $\mathscr{U}(\lambda) \subseteq \mathscr{S}$ for $0 < \lambda \leq 1$. Functions in $\mathscr{U}$ are therefore univalent, but not all are starlike, which might be expected from the similarity in their analytic representations. This makes them interesting since the class of starlike functions is very large, and in theory of univalent functions it is significant if a class does not entirely lie inside $\mathscr{S}^*$. Properties of meromorphic functions associated with the class $\mathscr{U}(\lambda)$ have been studied by Ali et al. in [10], and the integral mean problem and arc length problem for functions in $\mathscr{U}$ have been studied in [10]. Although neither $\mathscr{U} \subset \mathscr{S}^*$ nor $\mathscr{S}^* \subset \mathscr{U}$, Allu and Pandey [18] have proved that $\overline{co}\,\mathscr{U} = \overline{co}\,\mathscr{S}^*$, their main observation being that $z/(1-xz)^2 \in \mathscr{S}^* \cap \mathscr{U}$ for each x such that $|x| = 1$. For more details of the class $\mathscr{U}(\lambda)$, we refer the reader to [16, Chapter 12].

Let $\mathscr{G}$ denote the class of functions $f \in \mathscr{A}$ satisfying

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > -\frac{1}{2} \quad \text{for } z \in \mathbb{D}.$$

Functions in $\mathscr{G}$ are convex in one direction (see [17]), and hence close-to-convex. For extremal problems on this class we refer to [1].

The importance of the class $\mathscr{G}$ in the case of certain univalent harmonic mappings is explored in [6], and properties of sections of functions in $\mathscr{G}$ have been studied in [4] and [15]. Moreover if $f \in \mathscr{G}$, and is of the form (1), then we have the following coefficient inequality [1]

$$|a_n| \le \frac{n+1}{2} \quad \text{for all } n \ge 2, \tag{2}$$

with equality in (2) only for the function $g_0(z)$ and its rotations, where

$$g_0(z) = \frac{z - \left(\frac{x}{2}\right) z^2}{(1-xz)^2}, \quad |x| = 1.$$

Before proving our main results, we mention some important lemmas which play a vital role in the proof of our main results.

Lemma 1 ([12, Theorem 4.3]). *J is a complex-valued continuous linear functional on $\mathscr{H}$ if, and only if, there is a sequence $\{b_n\}$ of complex numbers satisfying*

$$\overline{\lim_{n\to\infty}} |b_n|^{1/n} < 1$$

and are such that

$$J(f) = \sum_{n=0}^{\infty} b_n a_n,$$

where $f \in \mathscr{H}$, and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $|z| < 1$.

Lemma 2 ([12, Theorem 4.5]). *Let $\mathscr{F}$ be a compact subset of $\mathscr{H}$, and J be a complex-valued continuous linear functional on $\mathscr{H}$. Then*

$$\max\left\{\operatorname{Re} J(f) : f \in \overline{co}\,\mathscr{F}\right\} = \max\left\{\operatorname{Re} J(f) : f \in \mathscr{F}\right\} = \max\left\{\operatorname{Re} J(f) : f \in E\overline{co}\,\mathscr{F}\right\}.$$

Lemma 3 ([18]). *$\overline{co}\,\mathscr{U}$ consists of all functions f with represenation*

$$f(z) = \int_{|x|=1} \frac{z}{(1-xz)^2}\, d\mu(x),$$

where $\mu \in \wedge$, and $\wedge$ denotes the set of probability measures on $\partial\mathbb{D}$. Further, $E\overline{co}\,\mathscr{U}$ consists of functions of the form

$$f(z) = \frac{z}{(1-xz)^2}, \quad |x| = 1.$$

Lemma 4 ([1]). *$\overline{co}\,\mathscr{G}$ consists of all functions f with representation*

$$f(z) = \int_{|x|=1} \frac{z - \left(\frac{x}{2}\right) z^2}{(1-xz)^2}\, d\mu(x),$$

where $\mu \in \wedge$, and $\wedge$ denotes the set of probability measures on $\partial\mathbb{D}$. Further, $E\overline{co}\,\mathscr{G}$ consists of the functions f of the form

$$f(z) = \frac{z - \left(\frac{x}{2}\right) z^2}{(1-xz)^2}, \quad |x| = 1. \tag{3}$$

The main aim of this paper is to characterize the set of support points of the classes $\mathscr{U}$ and $\mathscr{G}$.

2. Support points of the class $\mathscr{U}$

Theorem 5.
$$\operatorname{supp}\mathscr{U} = E\overline{co}\,\mathscr{U} = \left\{\frac{z}{(1-xz)^2} : |x| = 1\right\}.$$

Proof. We first show that the functions of the form $z/(1-xz)^2$, where $|x| = 1$, belong to $\operatorname{supp}\mathscr{U}$.

For $x \in \mathbb{C}$ such that $|x| = 1$, consider the function
$$f_0(z) = \frac{z}{(1-xz)^2} = z + \sum_{n=2}^{\infty} n x^{n-1} z^n = z + 2xz^2 + \sum_{n=3}^{\infty} n x^{n-1} z^n.$$

A simple computation shows that $f_0 \in \mathscr{U}$. Consider now the continuous linear functional ϕ defined by, $\phi(f) = \bar{x} a_2$. Clearly $\operatorname{Re}\phi$ is non-constant on $\mathscr{U}$. We know that $\phi(f_0) = \bar{x}2x = 2$, which gives $\operatorname{Re}\phi(f_0) = 2$, and so
$$\operatorname{Re}\phi(f_0) = \max\left\{\operatorname{Re}\phi(f) : f \in \mathscr{U}\right\}.$$

Hence each function of the form $z/(1-xz)^2$ is a support point of $\mathscr{U}$, i.e.,
$$E\overline{co}\,\mathscr{U} \subseteq \operatorname{supp}\mathscr{U}. \tag{4}$$

We next show that $\operatorname{supp}\mathscr{U} \subseteq E\overline{co}\,\mathscr{U}$. In order to prove this, we first prove that each function in the class $\operatorname{supp}\overline{co}\,\mathscr{U}$ is of the form
$$\sum_{k=1}^{m} \lambda_k \frac{z}{(1-x_k z)^2} \tag{5}$$
where $\lambda_k \ge 0$, $\sum_{k=1}^{m} \lambda_k = 1$, $|x_k| = 1$, and $m = 1, 2, 3, \ldots$.

Let $\widetilde{f}$ be any support point of $\overline{co}\,\mathscr{U}$. Thus there exists a continuous linear functional on $\mathscr{H}$ say $\widetilde{J}$, with $\operatorname{Re}\widetilde{J}$ non-constant on $\overline{co}\,\mathscr{U}$ such that
$$\operatorname{Re}\widetilde{J}(\widetilde{f}) = \max\left\{\operatorname{Re}\widetilde{J}(f) : f \in \overline{co}\,\mathscr{U}\right\}.$$

By Lemma 1, we obtain a sequence $\{b_n\}$ such that $\limsup_{n\to\infty} |b_n|^{1/n} < 1$, and
$$\widetilde{J}(f) = \sum_{n=0}^{\infty} b_n a_n,$$
where $f \in \mathscr{H}$, and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $|z| < 1$. Consider
$$F(z,x) = \frac{z}{(1-xz)^2} = z + \sum_{n=2}^{\infty} a_n(x) z^n,$$
where $a_n(x) = n x^{n-1}$. Then $|a_n(x)| \le n r^{n-1}$ if $1/r < |x| < r$ for $r > 1$.

Let $\operatorname{Re}\widetilde{J}(\widetilde{f}) = \widetilde{M}$. Hence from Lemma 2 we obtain
$$\begin{aligned}\operatorname{Re}\widetilde{J}(\widetilde{f}) &= \max\left\{\operatorname{Re}\widetilde{J}(f) : f \in \overline{co}\,\mathscr{U}\right\} = \max\left\{\operatorname{Re}\widetilde{J}(f) : f \in \mathscr{U}\right\}\\ &= \max\left\{\operatorname{Re}\widetilde{J}(f) : f \in E\overline{co}\,\mathscr{U}\right\} := \widetilde{M}.\end{aligned}$$

Let
$$G(x) = \widetilde{J}(F(z,x)) = b_1 + \sum_{n=2}^{\infty} b_n a_n(x).$$

Since
$$\limsup_{n\to\infty} |b_n|^{1/n} < 1$$
for some α satisfying $0 < \alpha < 1$, and some $N \in \mathbb{N}$, we obtain $|b_n|^{1/n} \le \alpha$ whenever $n > N$. It follows that $|b_n a_n(x)| \le \alpha^n n r^{n-1} = (n/r)(\alpha r)^n$. Choosing r such that $\alpha r < 1$, the series representation of $G(x)$ is convergent if $1/r < |x| < r$. Thus G is analytic on $\{x : 1/r < |x| < r\}$.

Now let H be defined by

$$H(x)=\frac{1}{2}\left(G(x)+\overline{G\left(\frac{1}{\bar{x}}\right)}\right).$$

It is easy to see that H is analytic on $\{x:1/r<|x|<r\}$. Also note that if $|x|=1$, then

$$\begin{aligned}H(x)&=\frac{1}{2}\left(G(x)+\overline{G\left(\frac{1}{\bar{x}}\right)}\right)=\frac{1}{2}\left(G(x)+\overline{G\left(\frac{x}{|x|^2}\right)}\right)\\&=\frac{1}{2}\left(G(x)+\overline{G(x)}\right)=\operatorname{Re}G(x).\end{aligned}$$

Suppose $H(x)=\widetilde{M}$ has infinitely many solutions on $\{x:|x|=1\}$. Since $\{x:|x|=1\}$ is a compact set, and H is analytic on $\{x:1/r<|x|<r\}$, we obtain

$$H(x)=\widetilde{M}\text{ for }x\text{ on }\{x:1/r<|x|<r\}.$$

In particular

$$H(x)=\widetilde{M}\text{ for all }x\in\{x:|x|=1\},$$

and so

$$\operatorname{Re}G(x)=\widetilde{M}\text{ for all }x\in\{x:|x|=1\},$$

which gives

$$\operatorname{Re}\widetilde{J}(F(z,x))=\widetilde{M}\text{ for all }x\in\{x:|x|=1\}.$$

Since $\operatorname{Re}\widetilde{J}$ is constant on $E\overline{co}\mathscr{U}$, it follows that $E\overline{co}\mathscr{U}=\{F(z,x):|x|=1\}$. Thus $\operatorname{Re}\widetilde{J}$ is constant on $\overline{co}\mathscr{U}$ is not possible. Therefore $H(x)=\widetilde{M}$ has only finitely many solutions on $\{x:|x|=1\}$. Hence

$$H(x)=\widetilde{M}\text{ has only finitely many solutions on }\{x:|x|=1\},$$

and so

$$\operatorname{Re}\widetilde{J}(F(z,x))=\widetilde{M}\text{ for finitely many points on }\{x:|x|=1\}.$$

Thus there are only finitely many functions in $E\overline{co}\mathscr{U}$ which maximizes $\operatorname{Re}\widetilde{J}$ over $\overline{co}\mathscr{U}$.

Now consider the set of all functions in $\overline{co}\mathscr{U}$ which maximizes $\operatorname{Re}\widetilde{J}$ over $\overline{co}\mathscr{U}$. So we consider the set

$$\mathscr{C}=\left\{g:g\in\overline{co}\mathscr{U}\text{ and }\operatorname{Re}\widetilde{J}(g)=\max\left\{\operatorname{Re}\widetilde{J}(f):f\in\overline{co}\mathscr{U}\right\}\right\}.$$

It is easy to see that $\mathscr{C}$ is convex and compact. Hence by the Krein–Milman theorem $E\mathscr{C}$ is non-empty. Also suppose there is a function $f^*\in E\mathscr{C}$ with $f^*\notin E\overline{co}\mathscr{U}$, then there are functions f_1^* and f_2^* in $\overline{co}\mathscr{U}$ such that

$$f^*(z)=\beta f_1^*(z)+(1-\beta)f_2^*(z)\quad\text{for some }\ \beta\in(0,1).$$

By the linearity of $\operatorname{Re}\widetilde{J}$ we obtain

$$\operatorname{Re}\widetilde{J}\big(f^*(z)\big)=\beta\operatorname{Re}\widetilde{J}\big(f_1^*(z)\big)+(1-\beta)\operatorname{Re}\widetilde{J}\big(f_2^*(z)\big),$$

and so f_1^* and f_2^* belong to $\mathscr{C}$, which is a contradiction since $f^*\in E\mathscr{C}$. Thus $E\mathscr{C}\subset E\overline{co}\mathscr{U}$. Since $E\overline{co}\mathscr{U}$ contains only a finite number of elements which are also in $E\mathscr{C}$, it follows that $E\mathscr{C}$ is a finite set of distinct functions f_k of the form

$$f_k(z)=\frac{z}{(1-x_kz)^2},\quad\text{where }\ |x_k|=1.$$

Hence

$$\mathscr{C}=\left\{f:f(z)=\sum_{k=1}^{m}\lambda_k\frac{z}{(1-x_kz)^2},\quad\lambda_k\geq0,\ \sum_{k=1}^{m}\lambda_k=1\text{ and }|x_k|=1\right\}.$$

From the definition of $\mathscr{C}$ it is clear that $\widetilde{f}\in\mathscr{C}$, and so $\widetilde{f}$ is of the form (5). Thus we have shown that every support point of $\overline{co}\mathscr{U}$ is of the form (5).

We next show that $\operatorname{supp}\mathscr{U} \subseteq E\overline{co}\mathscr{U}$.

Let $\widehat{f} \in \mathscr{U}$ be any support point of $\mathscr{U}$. Since $\mathscr{U} \subset \overline{co}\mathscr{U}$ we have $\widehat{f} \in \overline{co}\mathscr{U}$, and there exists a continuous linear functional $\widehat{J}$ on $\mathscr{H}$ with $\operatorname{Re}\widehat{J}$ non-constant on $\mathscr{U}$, such that

$$\operatorname{Re}\widehat{J}(\widehat{f}) = \max\left\{\operatorname{Re}\widehat{J}(f) : f \in \mathscr{U}\right\}.$$

Since $\mathscr{U}$ is compact, it follows from Lemma 2 that

$$\operatorname{Re}\widehat{J}(\widehat{f}) = \max\left\{\operatorname{Re}\widehat{J}(f) : f \in \overline{co}\mathscr{U}\right\}.$$

Thus $\widehat{f}$ is a support point of $\overline{co}\mathscr{U}$, and is of the form (5) i.e.,

$$\widehat{f} = \sum_{k=1}^{m} \lambda_k \frac{z}{(1-x_k z)^2}.$$

If $\lambda_k \neq 0$ for at least two values of k, then $\widehat{f}$ has at least two poles at $\bar{x_k}$ on the unit circle, each of order 2. Such a function is not univalent in $\mathbb{D}$. Hence

$$\widehat{f}(z) = \frac{z}{(1-xz)^2} \quad \text{where } |x| = 1,$$

and so $\widehat{f} \in E\overline{co}\mathscr{U}$ which gives

$$\operatorname{supp}\mathscr{U} \subseteq E\overline{co}\mathscr{U}, \tag{6}$$

and thus by (4) and (6), we obtain

$$\operatorname{supp}\mathscr{U} = E\overline{co}\mathscr{U} = \left\{\frac{z}{(1-xz)^2} : |x| = 1\right\}.$$

This completes the proof of Theorem 5. □

3. Support points of the class $\mathscr{G}$

Theorem 6.

$$\operatorname{supp}\mathscr{G} = E\overline{co}\mathscr{G} = \left\{\frac{z - \left(\frac{x}{2}\right)z^2}{(1-xz)^2} : |x| = 1\right\}.$$

Proof. We first show that each function of the form (3) is a support point of $\mathscr{G}$.

For $x \in \mathbb{C}$ such that $|x| = 1$, consider the function

$$g_0(z) = \frac{z - \left(\dfrac{x}{2}\right)z^2}{(1-xz)^2} = z + \sum_{n=2}^{\infty} \frac{n+1}{2} x^{n-1} z^n = z + \frac{3}{2}xz^2 + \sum_{n=3}^{\infty} \frac{n+1}{2} x^{n-1} z^n.$$

A simple computation shows that for each x such that $|x| = 1$, $g_0 \in \mathscr{G}$. Consider the continuous linear functional $\phi : \mathscr{H} \to \mathbb{C}$ defined by $\phi(f) = \bar{x}a_2$. Then $\phi(g_0) = 3/2$, which shows that $\operatorname{Re}\phi(g_0) = 3/2$. Thus from (2) we obtain

$$\operatorname{Re}\phi(g_0) = \max\left\{\operatorname{Re}\phi(g) : g \in \mathscr{G}\right\},$$

and so each function of the form $(z - (x/2)z^2)/(1-xz)^2$ is a support point of $\mathscr{G}$, i.e.,

$$E\overline{co}\mathscr{G} \subseteq \operatorname{supp}\mathscr{G}. \tag{7}$$

We next show that $\operatorname{supp}\mathscr{G} \subseteq E\overline{co}\mathscr{G}$.

In order to prove this, we first prove that each function in the class $\operatorname{supp}\overline{co}\mathscr{G}$ is of the form

$$\sum_{k=1}^{m} \lambda_k \frac{z - \left(\dfrac{x_k}{2}\right)z^2}{(1-x_k z)^2}, \tag{8}$$

where $\lambda_k \geq 0$, $\sum_{k=1}^{m} \lambda_k = 1$, and $|x_k| = 1$ for $m = 1, 2, 3, \ldots$.

Let $\widetilde{g}$ be any support point of $\overline{co}\mathscr{G}$. Thus there exists a continuous linear functional on $\mathscr{H}$ say $\widetilde{L}$, with $\operatorname{Re}\widetilde{L}$ non-constant on $\overline{co}\mathscr{G}$ such that

$$\operatorname{Re}\widetilde{L}(\widetilde{g}) = \max\left\{\operatorname{Re}\widetilde{L}(g) : g \in \overline{co}\mathscr{U}\right\}.$$

By Lemma 1 we obtain a sequence $\{c_n\}$ such that $\limsup_{n\to\infty} |c_n|^{1/n} < 1$, and

$$\widetilde{L}(f) = \sum_{n=0}^{\infty} b_n a_n,$$

where $f \in \mathscr{H}$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $|z| < 1$. Consider

$$\widetilde{F}(z,x) = \frac{z - \left(\frac{x}{2}\right) z^2}{(1-xz)^2} = z + \sum_{n=2}^{\infty} a_n(x) z^n,$$

where

$$a_n(x) = \frac{n+1}{2} x^{n-1},$$

so that

$$|a_n(x)| \le \frac{n+1}{2} r^{n-1} \text{ for } \quad 1/r < |x| < r, \text{ when } r > 1.$$

Let $\operatorname{Re}\widetilde{L}(\widetilde{g}) = \widehat{M}$. Then Lemma 2 gives

$$\begin{aligned}\operatorname{Re}\widetilde{L}(\widetilde{g}) = \max\left\{\operatorname{Re}\widetilde{L}(g) : g \in \overline{co}\mathscr{G}\right\} &= \max\left\{\operatorname{Re}\widetilde{L}(g) : g \in \mathscr{G}\right\} \\ = \max\left\{\operatorname{Re}\widetilde{L}(g) : g \in E\overline{co}\mathscr{G}\right\} &= \widehat{M}.\end{aligned}$$

Now define

$$\widetilde{G}(x) = \widetilde{L}\left(\widetilde{F}(z,x)\right) = c_1 + \sum_{n=2}^{\infty} c_n \frac{n+1}{2} x^{n-1}(x).$$

Since

$$\limsup_{n\to\infty} |c_n|^{1/n} < 1,$$

it can easily be shown that the series representation of $\widetilde{G}(x)$ is convergent if $1/r < |x| < r$ for r sufficiently close to 1. Thus $\widetilde{G}$ is analytic on $\{x : 1/r < |x| < r\}$. Next let $\widetilde{H}$ be defined by

$$\widetilde{H}(x) = \frac{1}{2}\left(\widetilde{G}(x) + \overline{\widetilde{G}\left(\frac{1}{\bar{x}}\right)}\right).$$

Clearly $\widetilde{H}$ is an analytic function on $\{x : 1/r < |x| < r\}$. We note that if $|x| = 1$, then $\widetilde{H}(x) = \operatorname{Re}\widetilde{G}(x)$. By using a similar argument to that in the proof of Theorem 5, one can show that

$$\widetilde{H}(x) = \widehat{M} \text{ has finitely many solutions on } \{x : |x| = 1\}.$$

That is

$$\operatorname{Re}\widetilde{L}\left(\widetilde{F}(z,x)\right) = \widehat{M} \text{ for finitely many points in } \{x : |x| = 1\},$$

and so there are only finitely many functions in $E\overline{co}\mathscr{G}$ which maximizes $\operatorname{Re}\widetilde{L}$ over $\overline{co}\mathscr{G}$.

Consider now the set of all functions in $\overline{co}\mathscr{G}$ which maximizes $\operatorname{Re}\widetilde{L}$ over $\overline{co}\mathscr{G}$, i.e., the set

$$\mathscr{K} = \left\{g : g \in \overline{co}\mathscr{G} \text{ and } \operatorname{Re}\widetilde{L}(g) = \max\left\{\operatorname{Re}\widetilde{L}(f) : f \in \overline{co}\mathscr{G}\right\}\right\}.$$

It is easy to see that $\mathscr{K}$ is convex and compact. Therefore by the Krein–Milman theorem, $E\mathscr{K}$ is non-empty, and also $E\mathscr{K} \subset E\overline{co}\mathscr{G}$. It follows therefore that since there are only finite number of elements in $E\overline{co}\mathscr{G}$ which are also in $E\mathscr{K}$, that $E\mathscr{K}$ is a finite set of distinct functions g_k of the form

$$g_k(z) = \frac{z - \left(\frac{x_k}{2}\right) z^2}{(1-x_k z)^2} \quad \text{where } |x_k| = 1.$$

Thus

$$\mathcal{K}=\left\{f: f(z)=\sum_{k=1}^{m}\lambda_k\frac{z-\left(\dfrac{x}{2}\right)z^2}{(1-x_kz)^2},\quad \lambda_k\geq 0,\quad \sum_{k=1}^{m}\lambda_k=1 \text{ and } |x_k|=1\right\}.$$

From the definition of $\mathcal{K}$ it is clear that $\tilde{g}\in\mathcal{K}$, and so $\tilde{g}$ is of the form (8). Thus we have shown that every support point of $\overline{co}\mathcal{G}$ is of the form (8).

Finally we show that $\operatorname{supp}\mathcal{G}\subseteq E\overline{co}\mathcal{G}$.

Let $\hat{g}\in\mathcal{G}$ be any support point of $\mathcal{G}$. Since $\mathcal{G}\subset\overline{co}\mathcal{G}$, it follows that $\hat{g}\in\overline{co}\mathcal{G}$. Thus there exists a continuous linear functional $\widehat{\mathcal{L}}$ on $\mathcal{H}$, with $\operatorname{Re}\widehat{\mathcal{L}}$ non-constant on $\mathcal{G}$ such that

$$\operatorname{Re}\widehat{\mathcal{L}}(\hat{g})=\max\left\{\operatorname{Re}\widehat{\mathcal{L}}(g): g\in\mathcal{G}\right\}.$$

Since $\mathcal{G}$ is compact, Lemma 2 gives

$$\operatorname{Re}\widehat{\mathcal{L}}(\hat{g})=\max\left\{\operatorname{Re}\widehat{\mathcal{L}}(g): g\in\overline{co}\mathcal{G}\right\}.$$

It is easy to see that $\hat{g}$ is a support point of $\overline{co}\mathcal{G}$, and is therefore the form (8), i.e.,

$$\sum_{k=1}^{m}\lambda_k\frac{z-\left(\dfrac{x_k}{2}\right)z^2}{(1-x_kz)^2}.$$

If $\lambda_k\neq 0$ for at least two values of k, then $\hat{g}$ has at least two poles at $\bar{x}_k$ on the unit circle each of order 2, and such a function is not univalent in $\mathbb{D}$. Therefore,

$$\hat{g}(z)=\frac{z-\left(\dfrac{x}{2}\right)z^2}{(1-xz)^2}\quad\text{where } |x|=1.$$

Thus $\hat{g}\in E\overline{co}\mathcal{G}$, which shows that

$$\operatorname{supp}\mathcal{G}\subseteq E\overline{co}\mathcal{G}. \tag{9}$$

Hence by (7) and (9) we obtain

$$\operatorname{supp}\mathcal{G}=E\overline{co}\mathcal{G}=\left\{\frac{z-\frac{x}{2}z^2}{(1-xz)^2}: |x|=1\right\},$$

which completes the proof of Theorem 6. □

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