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Partial differential equations / *Equations aux dérivées partielles*

# On the hydrostatic approximation of compressible anisotropic Navier–Stokes equations

*Sur l'approximation hydrostatique des équations de Navier–Stokes anisotropes compressibles*

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**Abstract.** In this work, we obtain the hydrostatic approximation by taking the small aspect ratio limit to the Navier–Stokes equations. The aspect ratio (the ratio of the depth to horizontal width) is a geometrical constraint in general large scale motions meaning that the vertical scale is significantly smaller than horizontal. We use the versatile relative entropy inequality to prove rigorously the limit from the compressible Navier–Stokes equations to the compressible Primitive Equations. This is the first work to use relative entropy inequality for proving hydrostatic approximation and derive the compressible Primitive Equations.

**Résumé.** Dans ce travail, nous obtenons l'approximation hydrostatique en prenant la limite du petit rapport d'aspect des équations de Navier–Stokes. Le rapport d'aspect (le rapport de la profondeur à la largeur horizontale) est une contrainte géométrique dans les mouvements géophysiques signifiant que l'échelle verticale est nettement plus petite que l'horizontale. Nous utilisons l'inégalité d'entropie relative pour prouver rigoureusement la limite des équations de Navier–Stokes compressibles aux équations primitives compressibles. C'est le premier travail qui utilise l'inégalité d'entropie relative pour prouver l'approximation hydrostatique et dériver les équations primitives compressibles.

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### 1. Introduction

In this paper, we consider the following compressible anisotropic Navier–Stokes equations

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\rho) = \mu_x \Delta_x \mathbf{u} + \mu_z \partial_{zz} \mathbf{u}, \end{cases} \tag{1}$$

in the thin domain  $(0, T) \times \Omega_\epsilon$ . We consider  $\Omega_\epsilon = \{(x, z) | x \in \mathbb{T}^2, -\epsilon < z < \epsilon\}$ ,  $x$  denotes the horizontal direction and  $z$  denotes the vertical direction, while,  $\mu_x$  and  $\mu_z$  are given constant horizontal viscous coefficient and vertical viscous coefficient. The velocity  $\mathbf{u} = (\mathbf{v}, w)$ , where  $\mathbf{v}(t, x, z) \in \mathbb{R}^2$  and  $w(t, x, z) \in \mathbb{R}$  represent the horizontal velocity and vertical velocity respectively. The pressure  $p(\rho)$  satisfies the barotropic pressure law where the pressure and the density are related by the formula:  $p(\rho) = \rho^\gamma$  ( $\gamma > 1$ ). Through out this paper, we use  $\operatorname{div} \mathbf{u} = \operatorname{div}_x \mathbf{v} + \partial_z w$  and  $\nabla = (\nabla_x, \partial_z)$  to denote the three-dimensional spatial divergence and gradient respectively, and  $\Delta_x$  stands for horizontal Laplacian. Similar to the assumptions by [1, 8], we suppose  $\mu_x = 1$  and  $\mu_z = \epsilon^2$ . As was stressed by Azérad and Guillén [1], it is necessary to consider the above anisotropic viscosities scaling, which is the fundamental assumption for the derivation of Primitive Equations (PE). Under this assumption, the system is rewritten as the following

$$\begin{cases} \partial_t \rho + \operatorname{div}_x(\rho \mathbf{v}) + \partial_z(\rho w) = 0, \\ \rho \partial_t \mathbf{v} + \rho(\mathbf{u} \cdot \nabla) \mathbf{v} - \Delta_x \mathbf{v} - \epsilon^2 \partial_{zz} \mathbf{v} + \nabla_x p(\rho) = 0, \\ \rho \partial_t w + \rho \mathbf{u} \cdot \nabla w - \Delta_x w - \epsilon^2 \partial_{zz} w + \partial_z p(\rho) = 0. \end{cases} \tag{2}$$

Inspired by [1, 8] and following geophysical assumption such as Pedlosky [12], we introduce the following new unknowns,

$$\mathbf{u}_\epsilon = (\mathbf{v}_\epsilon, w_\epsilon), \quad \mathbf{v}_\epsilon(x, z, t) = \mathbf{v}(x, \epsilon z, t), \quad w_\epsilon = \frac{1}{\epsilon} w(x, \epsilon z, t), \quad \rho_\epsilon = \rho(x, t),$$

for any  $(x, z) \in \Omega := \mathbb{T}^2 \times (-1, 1)$ . It was observed that atmosphere and ocean are the thin layers, where the fluid layer depth is small compared to radius of sphere. Pedlosky [12] pointed out that “the pressure difference between any two points of the same vertical line depends only on the weight of the fluid between these points . . .”. Here we neglect the gravity and suppose the pressure satisfies the barotropic pressure law. Therefore we assume the density to be independent of the vertical variable. Then the system (2) becomes the following compressible scaled Navier–Stokes equations (CNS):

$$\begin{cases} \partial_t \rho_\epsilon + \operatorname{div}_x(\rho_\epsilon \mathbf{v}_\epsilon) + \partial_z(\rho_\epsilon w_\epsilon) = 0, \\ \rho_\epsilon \partial_t \mathbf{v}_\epsilon + \rho_\epsilon(\mathbf{u}_\epsilon \cdot \nabla) \mathbf{v}_\epsilon - \Delta_x \mathbf{v}_\epsilon - \partial_{zz} \mathbf{v}_\epsilon + \nabla_x p(\rho_\epsilon) = 0, \\ \epsilon^2(\rho_\epsilon \partial_t w_\epsilon + \rho_\epsilon \mathbf{u}_\epsilon \cdot \nabla w_\epsilon - \Delta_x w_\epsilon - \partial_{zz} w_\epsilon) + \partial_z p(\rho_\epsilon) = 0. \end{cases} \tag{3}$$

We supplement the CNS with the following boundary and initial conditions:

$$\begin{aligned} \rho_\epsilon, \mathbf{u}_\epsilon \text{ are periodic in } x, y, z, \\ (\rho_\epsilon, \mathbf{u}_\epsilon)|_{t=0} = (\rho_0, \mathbf{u}_0). \end{aligned} \tag{4}$$

The goal of this work is to investigate the limit process  $\epsilon \rightarrow 0$  in the system of (3) converges in a certain sense to the following compressible Primitive Equations(CPE):

$$\begin{cases} \partial_t \rho + \operatorname{div}_x(\rho \mathbf{v}) + \partial_z(\rho w) = 0, \\ \partial_t(\rho \mathbf{v}) + \operatorname{div}_x(\rho \mathbf{v} \otimes \mathbf{v}) + \partial_z(\rho \mathbf{v} w) + \nabla_x p(\rho) = \Delta_x \mathbf{v} + \partial_{zz} \mathbf{v}, \\ \partial_z p(\rho) = 0. \end{cases} \tag{5}$$

The PE model is a fundamental model to understand the atmosphere and ocean. There is a large literatures dedicated to PE model see [3, 9] and references therein. The hydrostatic approximation is one of the important feature of PE model. Rigorous justification of the limit

passage from anisotropic Navier–Stokes equations to its hydrostatic approximation via the small aspect limit seems to be of obvious practical importance. There are numerous studies of the incompressible convergence. For example, Azérad and Guillén [1] proved the weak solutions of anisotropic Navier–Stokes equation converge to weak solutions of PE. Then Li and Titi [8] used the method of weak-strong uniqueness to prove the aspect ratio limit of incompressible anisotropic Navier–Stokes equations, that is from weak solutions of anisotropic Navier–Stokes equations to strong solutions of incompressible PE model. Then Giga, Hieber and Kashiwabara et al. [6] extended the results into maximal regularity spaces. However, for the compressible fluids flows, to the best of authors’ knowledge, there are no results concerning the convergence from CNS to CPE. Our result generalizes the results by Azérad and Guillén [1] and Li and Titi [8] in the incompressible case to compressible case. However the compressible case is very different because one cannot derive information on the vertical velocity with the incompressible constraint.

Our goal is to rigorously derive the limit in the framework of weak solutions of CNS. Recently, Maltese and Novotný [11] proved the convergence from 3D compressible Navier–Stokes equations to 2D compressible Navier–Stokes equations in thin domain. See also result by Ducomet et al. [4]. Heuristically, inspired by their works, we develop the corresponding idea of relative entropy inequality for the compressible Navier–Stokes equations, and utilize the special structure of CPE to prove the convergence. There are huge differences of mathematical structure between the Navier–Stokes equations and the CPE model. Due to the hydrostatic approximation, there is no information for the vertical velocity in the momentum equation of CPE model, and the vertical velocity is determined by the horizontal velocity via the continuity equation, so it is very difficult to analyze the CPE model. Therefore, the classical method used in Navier–Stokes system can not be applied straightforwardly to CPE. This is the first work to use the relative entropy inequality for proving an hydrostatic approximation at compressible case.

## 2. Main result

Before showing our main result, we give the definition of weak solutions for CNS and the strong solution for CPE. Recently, Bresch and Jabin [2] consider different compactness method from Lions or Feireisl which can be applied to the anisotropical stress tensor. They obtain the global existence of weak solutions for non-monotone pressure. Let us recall their definitions here.

### 2.1. Dissipative weak solutions of CNS

**Definition 1.** We say that  $[\rho_\epsilon, \mathbf{u}_\epsilon]$  with  $\mathbf{u}_\epsilon = (\mathbf{v}_\epsilon, w_\epsilon)$  is a finite energy weak solution to the system of (3), supplemented with initial data (4) if  $\rho_\epsilon = \rho_\epsilon(x, t)$  and

$$\begin{aligned} \mathbf{u}_\epsilon &\in L^2(0, T; H^1(\Omega)), \quad \rho|\mathbf{u}_\epsilon|^2 \in L^\infty(0, T; L^1(\Omega)), \\ \rho_\epsilon &\in L^\infty(0, T; L^Y(\Omega)) \cap C([0, T], L^1(\Omega)), \end{aligned} \tag{6}$$

- the continuity equation

$$\left[ \int_\Omega \rho_\epsilon \psi \, dx dz \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \rho_\epsilon \partial_t \psi + \rho_\epsilon \mathbf{v}_\epsilon \cdot \nabla_x \psi + \rho_\epsilon w_\epsilon \partial_z \psi \, dx dz dt, \tag{7}$$

holds for all  $\psi \in C_c^\infty([0, T] \times \Omega)$ ;

- the momentum equation

$$\begin{aligned} & \left[ \int_{\Omega} \rho_{\epsilon} \mathbf{v}_{\epsilon} \varphi_{\mathbf{H}} dx dz \right]_{t=0}^{t=\tau} - \int_0^{\tau} \int_{\Omega} \rho_{\epsilon} \mathbf{v}_{\epsilon} \partial_t \varphi_{\mathbf{H}} dx dz dt - \int_0^{\tau} \int_{\Omega} \rho_{\epsilon} \mathbf{u}_{\epsilon} \mathbf{v}_{\epsilon} \cdot \nabla \varphi_{\mathbf{H}} dx dz dt \\ & \quad + \int_0^{\tau} \int_{\Omega} \nabla \mathbf{v}_{\epsilon} \cdot \nabla \varphi_{\mathbf{H}} dx dz dt - \int_0^{\tau} \int_{\Omega} p(\rho_{\epsilon}) \operatorname{div}_x \varphi_{\mathbf{H}} dx dz dt = 0, \quad (8) \end{aligned}$$

and

$$\begin{aligned} & \epsilon^2 \left[ \int_{\Omega} \rho_{\epsilon} w_{\epsilon} \varphi_3 dx dz \right]_{t=0}^{t=\tau} - \epsilon^2 \int_0^{\tau} \int_{\Omega} \rho_{\epsilon} w_{\epsilon} \partial_t \varphi_3 dx dz dt - \epsilon^2 \int_0^{\tau} \int_{\Omega} \rho_{\epsilon} \mathbf{u}_{\epsilon} w_{\epsilon} \cdot \nabla \varphi_3 dx dz dt \\ & \quad + \epsilon^2 \int_0^{\tau} \int_{\Omega} \nabla w_{\epsilon} \cdot \nabla \varphi_3 dx dz dt - \int_0^{\tau} \int_{\Omega} p(\rho_{\epsilon}) \partial_z \varphi_3 dx dz dt = 0, \quad (9) \end{aligned}$$

holds for all  $\varphi_{\mathbf{H}}, \varphi_3 \in C_c^{\infty}([0, T] \times \Omega)$ . Combining (8)–(9), we obtain

$$\begin{aligned} & \left[ \int_{\Omega} \rho_{\epsilon} \mathbf{v}_{\epsilon} \varphi_{\mathbf{H}} dx dz + \epsilon^2 \int_{\Omega} \rho_{\epsilon} w_{\epsilon} \varphi_3 dx dz \right]_{t=0}^{t=\tau} - \int_0^{\tau} \int_{\Omega} \rho_{\epsilon} \mathbf{v}_{\epsilon} \partial_t \varphi_{\mathbf{H}} dx dz dt \\ & \quad - \epsilon^2 \int_0^{\tau} \int_{\Omega} \rho_{\epsilon} w_{\epsilon} \partial_t \varphi_3 dx dz dt - \int_0^{\tau} \int_{\Omega} \rho_{\epsilon} \mathbf{v}_{\epsilon} \otimes \mathbf{v}_{\epsilon} : \nabla_x \varphi_{\mathbf{H}} dx dz dt \\ & \quad - \int_0^{\tau} \int_{\Omega} \rho_{\epsilon} \mathbf{v}_{\epsilon} w_{\epsilon} \cdot \partial_z \varphi_{\mathbf{H}} dx dz dt - \epsilon^2 \int_0^{\tau} \int_{\Omega} \rho_{\epsilon} \mathbf{v}_{\epsilon} w_{\epsilon} \cdot \nabla_x \varphi_3 dx dz dt \\ & \quad - \epsilon^2 \int_0^{\tau} \int_{\Omega} \rho_{\epsilon} w_{\epsilon}^2 \partial_z \varphi_3 dx dz dt + \int_0^{\tau} \int_{\Omega} \nabla \mathbf{v}_{\epsilon} : \nabla \varphi_{\mathbf{H}} dx dz dt \\ & \quad + \epsilon^2 \int_0^{\tau} \int_{\Omega} \nabla w_{\epsilon} \cdot \nabla \varphi_3 dx dz dt - \int_0^{\tau} \int_{\Omega} p(\rho_{\epsilon}) \operatorname{div} \varphi dx dz dt = 0, \quad (10) \end{aligned}$$

where  $\varphi = (\varphi_{\mathbf{H}}, \varphi_3) \in C_c^{\infty}([0, T] \times \Omega)$  and  $\operatorname{div} \varphi = \operatorname{div}_H \varphi_{\mathbf{H}} + \partial_z \varphi_3$ ,

- the energy inequality

$$\left[ \int_{\Omega} \frac{1}{2} \rho_{\epsilon} |\mathbf{v}_{\epsilon}|^2 + \frac{\epsilon^2}{2} \rho_{\epsilon} |w_{\epsilon}|^2 + P(\rho_{\epsilon}) dx dz \right]_{t=0}^{t=\tau} + \int_0^{\tau} \int_{\Omega} (|\nabla \mathbf{v}_{\epsilon}|^2 + \epsilon^2 |\nabla w_{\epsilon}|^2) dx dz dt \leq 0, \quad (11)$$

holds for a.a.  $\tau \in (0, T)$ , where  $P(\rho) = \rho \int_1^{\rho} \frac{p(z)}{z^2} dz$ .

## 2.2. Strong solution of CPE

We say that  $(r, \mathbf{u})$ ,  $\mathbf{u} = (\mathbf{v}, W)$  is a strong solution to the CPE system (5) in  $(0, T) \times \Omega$ , if

$$\begin{aligned} & r^{\frac{1}{2}} \in L^{\infty}(0, T; H^2(\Omega)), \partial_t r^{\frac{1}{2}} \in L^{\infty}(0, T; H^1(\Omega)), r > 0 \text{ for all } (t, x), \\ & \mathbf{v} \in L^{\infty}(0, T; H^3(\Omega)) \cap L^2(0, T; H^4(\Omega)), \partial_t \mathbf{v} \in L^2(0, T; H^2(\Omega)), \end{aligned}$$

with initial data  $r_0^{\frac{1}{2}} \in H^2(\Omega)$ ,  $r_0 > 0$  and  $\mathbf{v}_0 \in H^3(\Omega)$ . Liu and Titi [10] have proved the local existence of strong solution to CPE system (5).

### 2.3. Relative entropy inequality

Motivated by [5], for any finite energy weak solution  $(\rho, \mathbf{u})$ , where  $\mathbf{u} = (\mathbf{v}, w)$ , to the CNS system, we introduce the relative energy functional

$$\begin{aligned} \mathcal{E}(\rho, \mathbf{u}|r, \mathbf{u}) &= \int_{\Omega} \left[ \frac{1}{2} \rho |\mathbf{v} - \mathbf{v}|^2 + \frac{\epsilon^2}{2} \rho |w - W|^2 + P(\rho) - P'(r)(\rho - r) - P(r) \right] dx dz \\ &= \int_{\Omega} \left( \frac{1}{2} \rho |\mathbf{v}|^2 + \frac{\epsilon^2}{2} \rho |w|^2 + P(\rho) \right) dx dz - \int_{\Omega} (\rho \mathbf{v} \cdot \mathbf{v} + \epsilon^2 \rho w W) dx dz \\ &\quad + \int_{\Omega} \left[ \rho \frac{|\mathbf{v}|^2}{2} + \frac{\epsilon^2}{2} \rho |W|^2 - P'(r) \right] dx dz + \int_{\Omega} p(r) dx dz \\ &= \sum_{i=1}^4 I_i, \end{aligned} \tag{12}$$

where  $r > 0$ ,  $\mathbf{u} = (\mathbf{v}, W)$  are smooth “test” functions,  $r, \mathbf{u}$  compactly supported in  $\Omega$ .

**Lemma 2.** *Let  $(\rho, \mathbf{v}, w)$  be a dissipative weak solution introduced in Definition 1. Then  $(\rho, \mathbf{v}, w)$  satisfy the relative entropy inequality*

$$\begin{aligned} \mathcal{E}(\rho, \mathbf{u}|r, \mathbf{u})|_{t=0}^{t=\tau} &+ \int_0^\tau \int_{\Omega} (\nabla \mathbf{v} \cdot (\nabla \mathbf{v} - \nabla \mathbf{v}) + \epsilon^2 |\nabla w|^2) dx dz dt \\ &\leq \int_0^\tau \int_{\Omega} \rho (\partial_t \mathbf{v} + \mathbf{v} \nabla_x \mathbf{v} + w \partial_z \mathbf{v}) (\mathbf{v} - \mathbf{v}) dx dz dt \\ &\quad + \epsilon^2 \int_0^\tau \int_{\Omega} \rho (\partial_t W + \mathbf{v} \nabla_x W + w \partial_z W) (W - w) dx dz dt + \epsilon^2 \int_0^\tau \int_{\Omega} \nabla w \cdot \nabla W dx dz dt \\ &\quad - \int_0^\tau \int_{\Omega} P''(r) ((\rho - r) \partial_t r + \rho \mathbf{v} \nabla_x r) dx dz dt - \int_0^\tau \int_{\Omega} p(r) \operatorname{div}_x \mathbf{v} dx dz dt. \end{aligned} \tag{13}$$

### 2.4. Main result

Now, we are ready to state our main result.

**Theorem 3.** *Let  $\gamma > 4$ ,  $T_{max} > 0$  be the life time of strong solution to CPE system (3) corresponding to initial data  $[r_0, \mathbf{v}_0]$ . Let  $(\rho_\epsilon, \mathbf{u}_\epsilon)$ ,  $\mathbf{u}_\epsilon = (\mathbf{v}_\epsilon, w_\epsilon)$  be a sequence of dissipative weak solutions to the CNS system (2) from the initial data  $(\rho_{0,\epsilon}, \mathbf{u}_{0,\epsilon})$ . Suppose that*

$$\mathcal{E}(\rho_{0,\epsilon}, \mathbf{u}_{0,\epsilon}|r_0, \mathbf{u}_0) \rightarrow 0,$$

where  $\mathbf{u}_0 = (\mathbf{v}_0, W_0)$ , then

$$\operatorname{ess\,sup}_{t \in (0, T_{max})} \mathcal{E}(\rho_\epsilon, \mathbf{u}_\epsilon|r, \mathbf{u}) \rightarrow 0,$$

where  $\mathbf{u} = (\mathbf{v}, W)$  and the couple  $(r, \mathbf{u})$  satisfy the CPE system (3) on the time interval  $[0, T_{max})$ .

For the proof see [7].

**Remark 4.** There is a constraint for the pressure index at Bresch and Jabin [2] that is  $\gamma > \frac{3}{2} \left( \frac{4}{3} + \frac{\sqrt{10}}{3} \right) \simeq 3.5$ . Thus, our condition satisfies their assumption.

**Remark 5.** It is important to point out that our convergence holds on a fixed time interval due to the local existence of CPE. Some results [6, 8] concerning the incompressible PE model were shown for the global convergence based on the global existence in time under assumptions on the smallness of initial data.

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