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 $A_n$

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Harmonic analysis, Representation theory / *Analyse harmonique, Théorie des représentations*

# Sharp Estimates of Radial Dunkl and Heat Kernels in the Complex Case $A_n$

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**Abstract.** In this article, we consider the radial Dunkl geometric case  $k = 1$  corresponding to flat Riemannian symmetric spaces in the complex case and we prove exact estimates for the positive valued Dunkl kernel and for the radial heat kernel.

**Résumé.** Dans cet article, nous considérons le cas géométrique radial de Dunkl  $k = 1$  correspondant aux espaces symétriques riemanniens plats dans le cas complexe et nous prouvons des estimations exactes pour le noyau de Dunkl à valeur positive et pour le noyau de chaleur radial.

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## 1. Introduction and notation

Finding good estimates of Dunkl heat kernels is a challenging and important subject, developed recently in [1]. Dunkl analysis has been developed intensely in recent years (refer to [7]). Optimal Dunkl heat kernel estimates have important consequences in harmonic analysis (e.g. Hardy spaces, maximal inequalities, see for instance [1]), potential theory (e.g. Green function estimates, Martin boundary description, see [2]) and spectral theory (see [3]). Recall that the Dunkl heat kernel is the probability transition function of the Dunkl stochastic process (refer to [5] and [7]). Consequently, Dunkl heat kernel estimates have also stochastic applications for the Dunkl process. Establishing estimates of the heat kernels is equivalent to estimating the Dunkl kernel as demonstrated by equation (3) below.

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In this paper we prove exact estimates in the  $W$ -radial Dunkl geometric case of multiplicity  $k = 1$ , corresponding to Cartan motion groups and flat Riemannian symmetric spaces with the ambient complex group  $G$ , the Weyl group  $W$  and the root system  $A_n$ .

We study for the first time the non-centered heat kernel, denoted  $p_t^W(X, Y)$ , on Riemannian symmetric spaces and we provide its sharp estimates. Exact estimates were obtained in [2] in the centered case  $Y = 0$  for all Riemannian symmetric spaces.

We provide exact estimates for the spherical functions  $\psi_\lambda(X)$  in the two variables  $X, \lambda$  when  $\lambda$  is real and, consequently, for the heat kernel  $p_t^W(X, Y)$  in the three variables  $t, X, Y$ .

We recall here some basic terminology and facts about symmetric spaces associated to Cartan motion groups.

Let  $G$  be a semisimple Lie group and let  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  be the Cartan decomposition of  $G$ . We recall the definition of the Cartan motion group and the flat symmetric space associated with the semisimple Lie group  $G$  with maximal compact subgroup  $K$ . The Cartan motion group is the semi-direct product  $G_0 = K \ltimes \mathfrak{p}$  where the multiplication is defined by  $(k_1, X_1) \cdot (k_2, X_2) = (k_1 k_2, \text{Ad}(k_1)(X_2) + X_1)$ . The associated flat symmetric space is then  $M = \mathfrak{p} \simeq G_0/K$  (the action of  $G_0$  on  $\mathfrak{p}$  is given by  $(k, X) \cdot Y = \text{Ad}(k)(Y) + X$ ).

The spherical functions for the symmetric space  $M$  are then given by

$$\psi_\lambda(X) = \int_K e^{\lambda(\text{Ad}(k)(X))} dk$$

where  $\lambda$  is a complex linear functional on  $\mathfrak{a} \subset \mathfrak{p}$ , a Cartan subalgebra of the Lie algebra of  $G$ . To extend  $\lambda$  to  $X \in \text{Ad}(K)\mathfrak{a} = \mathfrak{p}$ , one uses  $\lambda(X) = \lambda(\pi_\mathfrak{a}(X))$  where  $\pi_\mathfrak{a}$  is the orthogonal projection with respect to the Killing form (denoted throughout this paper by  $\langle \cdot, \cdot \rangle$ ). Note that in [9–11],  $\lambda$  is replaced by  $i\lambda$ .

Throughout this paper, we usually assume that  $G$  is a semisimple complex Lie group. The complex root systems are respectively  $A_n$  for  $n \geq 1$  (where  $\mathfrak{p}$  consists of the  $n \times n$  hermitian matrices with trace 0),  $B_n$  for  $n \geq 2$  (where  $\mathfrak{p} = i\mathfrak{so}(2n+1)$ ),  $C_n$  for  $n \geq 3$  (where  $\mathfrak{p} = i\mathfrak{sp}(n)$ ) and  $D_n$  for  $n \geq 4$  (where  $\mathfrak{p} = i\mathfrak{so}(2n)$ ) for the classical cases and the exceptional root systems  $E_6, E_7, E_8, F_4$  and  $G_2$ .

The radial heat kernel is considered with respect to the invariant measure  $\mu(dY) = \pi^2(Y) dY$  on  $M$ , where  $\pi(Y) = \prod_{\alpha > 0} \alpha(Y)$ .

Note also that in the curved case  $M_0 = G/K$ , the spherical functions for the symmetric space  $M_0$  are then given by

$$\phi_\lambda(e^X) = \int_K e^{(\lambda - \rho)H(e^X k)} dk$$

where  $\rho$  is the half-sum of the roots counted with their multiplicities and  $H(g)$  is the abelian component in the Iwasawa decomposition of  $g$ :  $g = k e^{H(g)} n$ .

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## 2. Estimates of spherical functions and of the heat kernel

We will be developing a sharp estimate for the spherical function  $\psi_\lambda(X)$ . We introduce the following useful convention. We will write

$$f(t, X, \lambda) \asymp g(t, X, \lambda)$$

in a given domain of  $f$  and  $g$  if there exist constants  $C_1 > 0$  and  $C_2 > 0$  independent of  $t, X$  and  $\lambda$  such that  $C_1 f(t, X, \lambda) \leq g(t, X, \lambda) \leq C_2 f(t, X, \lambda)$  in the domain of consideration.

We conjecture the following global estimate for the spherical function in the complex case.

**Conjecture 1.** *On flat Riemannian symmetric spaces with complex group  $G$  we have,*

$$\psi_\lambda(X) \asymp \frac{e^{\langle \lambda, X \rangle}}{\prod_{\alpha > 0} (1 + \alpha(\lambda)\alpha(X))}, \quad \lambda \in \bar{\mathfrak{a}}^+, X \in \bar{\mathfrak{a}}^+.$$

**Remark 2.** Recall that, denoting  $\delta(X) = \prod_{\alpha > 0} \sinh^2 \alpha(X)$ , we have

$$\phi_\lambda(e^X) = \frac{\pi(X)}{\delta^{1/2}(X)} \psi_\lambda(X). \tag{1}$$

Since

$$\delta^{1/2}(X) \asymp \frac{e^{\rho(X)} \pi(X)}{\prod_{\alpha > 0} (1 + \alpha(X))}$$

in the complex case, Conjecture 1 therefore becomes

$$\phi_\lambda(e^X) \asymp e^{(\lambda-\rho)(X)} \prod_{\alpha > 0} \frac{1 + \alpha(X)}{1 + \alpha(\lambda)\alpha(X)} \tag{2}$$

in the curved complex case.

Let us compare the estimate (2) we conjecture for  $\phi_\lambda$  with the one obtained in [12, Theorem 3.4], cf. also [17, Remark 3.1]. The estimates in [12] apply in all the generality of hypergeometric functions of Heckman and Opdam. The authors show that there exist constants  $C_1(\lambda) > 0$ ,  $C_2(\lambda) > 0$  such that

$$C_1(\lambda) e^{(\lambda-\rho)(X)} \prod_{\substack{\alpha > 0, \\ \alpha(\lambda)=0}} (1 + \alpha(X)) \leq \phi_\lambda(e^X) \leq C_2(\lambda) e^{(\lambda-\rho)(X)} \prod_{\substack{\alpha > 0, \\ \alpha(\lambda)=0}} (1 + \alpha(X)).$$

Given (1), corresponding estimates clearly also hold in the flat case for  $\psi_\lambda(X)$ . The interest of our result, in the case  $A_n$ , lies in the fact that our estimate is universal in both  $\lambda$  and  $X$ .

The results of [12, 17] and our estimates in the  $A_n$  case strongly suggest that the Conjecture 1 is true for any complex root system.

Note that asymptotics of  $\psi_\lambda(tX)$  when  $\lambda$  and  $X$  are singular and  $t \rightarrow \infty$  were proven in [8] for all classical complex root systems and the systems  $F_4$  and  $G_2$ .

Consider the relationship between the Dunkl kernel  $E_k(X, Y)$  and the Dunkl heat kernel  $p_t(X, Y)$ , as given in [13, Lemma 4.5]

$$p_t(X, Y) = \frac{1}{2^{\gamma+d/2} c_k} t^{-\frac{d}{2}-\gamma} e^{\frac{-|X|^2-|Y|^2}{4t}} E_k\left(X, \frac{Y}{2t}\right), \tag{3}$$

where  $\gamma$  is the number of positive roots and the constant  $c_k$  is the Macdonald–Mehta–Selberg integral. The formula (3) remains true for the  $W$ -invariant kernels  $p_t^W$  and  $E^W$ . In the geometric cases  $k = \frac{1}{2}, 1$  and  $2$ , by [4], the  $W$ -invariant formula (3) translates in a similar relationship between the spherical function  $\psi_\lambda$  and the heat kernel  $p_t^W(X, Y)$ :

$$p_t^W(X, Y) = \frac{1}{2^{\gamma+d/2} c_k} t^{-\frac{d}{2}-\gamma} e^{\frac{-|X|^2-|Y|^2}{4t}} \psi_X\left(\frac{Y}{2t}\right). \tag{4}$$

A simple direct proof of (4) for  $k = 1$  is given in [8, Remark 2.9].

Equation (4) and Conjecture 1 bring us to an equivalent conjecture for the heat kernel  $p_t^W(X, Y)$ .

**Conjecture 3.** *We have*

$$p_t^W(X, Y) \asymp t^{-\frac{d}{2}} \frac{e^{\frac{-|X-Y|^2}{4t}}}{\prod_{\alpha > 0} (t + \alpha(X)\alpha(Y))}.$$

Consider also the relationship between the heat kernel  $p_t^W(X, Y)$  and the heat kernel  $\tilde{p}_t^W(X, Y)$  in the curved case. We have

$$\tilde{p}_t^W(X, Y) = e^{-|\rho|^2 t} \frac{\pi(X)\pi(Y)}{\delta^{1/2}(X)\delta^{1/2}(Y)} p_t^W(X, Y). \tag{5}$$

This relation follows directly from the fact that the respective radial Laplacians and radial measures are  $\pi^{-1}L_\alpha \circ \pi$  and  $\pi(X) dX$  in the flat case and  $\delta^{-1/2}(L_\alpha - |\rho|^2) \circ \delta^{1/2}$  and  $\delta(X) dX$  in the curved case ( $L_\alpha$  stands for the Euclidean Laplacian on  $\mathfrak{a}$ ).

In the curved complex case, Conjecture 3 becomes

$$\tilde{p}_t^W(X, Y) \asymp e^{-|\rho|^2 t} t^{-\frac{d}{2}} e^{-\rho(X+Y)} \prod_{\alpha>0} \frac{(1 + \alpha(X))(1 + \alpha(Y))}{(t + \alpha(X)\alpha(Y))} e^{-\frac{|X-Y|^2}{4t}}.$$

**Remark 4.** In [6], sharp estimates of  $W$ -invariant Poisson and Newton kernels in the complex Dunkl case were obtained, by exploiting the method of construction of these  $W$ -invariant kernels by alternating sums. When a root system  $\Sigma$  acts in  $\mathbf{R}^d$ , the sharp estimates of [6] have the common form

$$\mathcal{K}^W(X, Y) \asymp \frac{\mathcal{K}^{\mathbf{R}^d}(X, Y)}{\prod_{\alpha>0} (|X - Y|^2 + \alpha(X)\alpha(Y))}, \quad X, Y \in \bar{\mathfrak{a}}^+, \tag{6}$$

where  $\mathcal{K}^W(X, Y)$  is the  $W$ -invariant kernel in Dunkl setting and  $\mathcal{K}^{\mathbf{R}^d}(X, Y)$  is the classical kernel on  $\mathbf{R}^d$ . Let us observe a common pattern in the appearance of the classical kernels  $\mathcal{K}^{\mathbf{R}^d}$  and of products of roots  $\alpha(X)\alpha(Y)$  in formulas (6) and of the Fourier kernel  $e^{\langle \lambda, X \rangle}$  and the classical Gaussian heat kernel and of products  $\alpha(\lambda)\alpha(X)$  in the estimates given in Conjecture 1 and Conjecture 3.

### 2.1. Proof of Conjecture 1 in some cases

We start with a practical result.

**Proposition 5.** Let  $\alpha_1, \dots, \alpha_r$  be the simple roots and let  $A_{\alpha_i}$  be such that  $\langle X, A_{\alpha_i} \rangle = \alpha_i(X)$  for  $X \in \mathfrak{a}$ . Suppose  $X \in \mathfrak{a}^+$  and  $w \in W \setminus \{id\}$ . Then we have

$$Y - wY = \sum_{i=1}^r 2 \frac{a_i^w(Y)}{|\alpha_i|^2} A_{\alpha_i} \tag{7}$$

where  $a_i^w$  is a linear combination of simple roots with non-negative integer coefficients for each  $i$ .

**Proof.** Refer to [6, Proposition 3.5]. □

**Remark 6.** Note that  $a_i^w(Y)/|\alpha_i|^2$  is bounded by  $C \max_k |\alpha_k(Y)|$  where  $C$  is a constant depending only on  $w \in W$  and, ultimately, on  $W$ .

**Corollary 7.** Let  $Y \in \bar{\mathfrak{a}}^+$  and  $w \in W$ . Consider the decomposition (7) of  $Y - wY$ . If  $a_k^w(Y) \neq 0$  then  $\alpha_k$  appears in  $a_k^w$ , i.e.  $a_k^w = \sum_{i=1}^r n_i \alpha_i$  with  $n_k > 0$ .

**Proof.** Refer to [6, Corollary 3.10]. □

As a prelude to the next Proposition, we give here a description of the Abel transform  $\mathcal{A}(f)$  of a function  $f$  on  $\mathfrak{a}$  in the case of a flat symmetric space (in the case of the curved symmetric space refer for example to [15]). The dual of the Abel transform  $\mathcal{A}^*(f)$  can be defined as follows

$$\mathcal{A}^*(f)(X) = \int_K f(\pi_\alpha(\text{Ad}(k)(X))) dk$$

where, as before,  $\pi_\alpha$  is the orthogonal projection with respect to the Killing form. We can then define (in analogy with the curved case)

$$\int_{\mathfrak{a}} f(X) \mathcal{A}(g)(X) dX = \int_{\mathfrak{a}} \mathcal{A}^*(f)(X) g(X) w(X) dX$$

where  $w(X) = \prod_{\alpha>0} |\alpha(X)|^{m_\alpha}$  and  $m_\alpha$  is the multiplicity of  $\alpha$ . From [9, Chap IV, Theorem 10.2], we can deduce that the support of the dual Abel transform is  $C(X)$ , the convex hull of  $W \cdot X$ . The set  $C(X)$  has non-empty interior as long as not all roots are 0 on  $X$ . From there, we conclude that we can write

$$\mathcal{A}^*(f)(X) = \int_{C(X)} f(Y) K(X, Y) dY$$

and that  $K(X, Y) dY$  is a probability measure supported by  $C(X)$ .

**Proposition 8.** *Let  $\delta > 0$ . Suppose  $\alpha_i(\lambda) \alpha_j(X) \leq \delta$  for all  $i, j$ . Then  $\psi_\lambda(X) \asymp e^{\lambda(X)}$  (the constants involved only depend on  $\delta$ ).*

**Proof.** Notice that

$$e^{w_{min} \lambda(X)} \leq \psi_\lambda(X) = \int_{C(X)} e^{\lambda(Y)} K(X, Y) dY \leq e^{\lambda(X)} \tag{8}$$

where  $w_{min}$  is the element of the Weyl group giving the minimum value of  $w \lambda(X)$ . Now, using Proposition 5 and Remark 6 with  $Y = \lambda$ , we see that for any  $w \in W$

$$\begin{aligned} e^{\lambda(X)} &\geq e^{w \lambda(X)} = e^{\langle w\lambda - \lambda, X \rangle} e^{\langle \lambda, X \rangle} = \prod_{i=1}^r e^{-2 \frac{a_i^w(\lambda)}{|\alpha_i|^2} \alpha_i(X)} e^{\langle \lambda, X \rangle} \\ &\geq \prod_{i=1}^r e^{-2C(\max_k \alpha_k(\lambda)) \alpha_i(X)} e^{\langle \lambda, X \rangle} \geq \prod_{i=1}^r e^{-2C\delta} e^{\langle \lambda, X \rangle}. \end{aligned} \quad \square$$

**Remark 9.** This case and this method apply for any radial Dunkl case; it suffices to replace  $K(X, Y) dY$  by the so-called Rösler measure  $\mu_X(dY)$  in the integral in (8), see [14].

**Proposition 10.** *A spherical function  $\psi_\lambda(X)$  on  $M$  is given by the formula*

$$\psi_\lambda(X) = \frac{\pi(\rho)}{2\gamma\pi(\lambda)\pi(X)} \sum_{w \in W} \epsilon(w) e^{\langle w\lambda, X \rangle}, \tag{9}$$

where  $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha = \sum_{\alpha \in \Sigma^+} \alpha$  and  $\gamma = |\Sigma^+|$  is the number of positive roots (refer to [9, Chap. IV, Proposition 4.8 and Chap. II, Theorem 5.35]).

**Proposition 11.** *Suppose  $\alpha(\lambda) \alpha(X) \geq (\log|W|)/2$  for all  $\alpha > 0$ . Then*

$$\psi_\lambda(X) \asymp \frac{e^{\lambda(X)}}{\pi(\lambda)\pi(X)}.$$

We are assuming here that  $|\alpha_i| \geq 1$  for each  $i$ .

**Proof.** Suppose  $w \in W$  is not the identity. In that case,  $a_i^w(\lambda)$  is not equal to 0 for some  $i$ . By Proposition 5 with  $y = \lambda$  and Corollary 7,  $\lambda(X) - w \lambda(X) \geq 2 a_i^w(\lambda) \alpha_i(X) / |\alpha_i|^2 \geq 2 \alpha_i(\lambda) \alpha_i(X) \geq \log|W|$ . Each term  $e^{\langle w\lambda, X \rangle}$  in the alternating sum (9) corresponding to  $w \neq \text{id}$  is bounded by  $e^{-\log|W|} e^{\lambda(X)} = e^{\lambda(X)} / |W|$ . Hence, since only half the terms in the sum are negative,

$$|W| e^{\lambda(X)} \geq \sum_{w \in W} \epsilon(w) e^{\langle w\lambda, X \rangle} \geq e^{\lambda(X)} - \frac{|W|}{2} e^{\lambda(X)} / |W| = \frac{1}{2} e^{\lambda(X)}. \quad \square$$

### 3. The conjecture in the case of the root system $A_n$

We will prove the conjecture in the case of the root system of type  $A$ .

**Theorem 12.** *In the case of the root system of type  $A_n$  in the complex case, we have*

$$\psi_\lambda(e^X) = \frac{e^{(\lambda, X)}}{\prod_{i < j} (1 + (\lambda_i - \lambda_j)(x_i - x_j))}, \quad \lambda, X \in \bar{\mathfrak{a}}^+ \tag{10}$$

where  $X = [x_1, \dots, x_{n+1}]$  with  $x_i \geq x_{i+1}$  and  $\lambda = [\lambda_1, \dots, \lambda_{n+1}]$  with  $\lambda_i \geq \lambda_{i+1}$ .

**Corollary 13.**

$$\begin{aligned} \phi_\lambda(e^X) &= e^{(\lambda - \rho)(X)} \prod_{i < j} \frac{1 + x_i - x_j}{1 + (x_i - x_j)(\lambda_i - \lambda_j)}, \\ p_t^W(X, Y) &= t^{-\frac{d}{2}} \frac{e^{-\frac{|X-Y|^2}{4t}}}{\prod_{i < j} (t + (x_i - x_j)(y_i - y_j))}, \\ \tilde{p}_t^W(X, Y) &= e^{-|\rho|^2 t} t^{-\frac{d}{2}} e^{-\rho(X+Y)} \prod_{i < j} \frac{(1 + x_i - x_j)(1 + y_i - y_j)}{(t + (x_i - x_j)(y_i - y_j))} e^{-\frac{|X-Y|^2}{4t}}. \end{aligned}$$

We recall (refer to [16, Theorem 4.1]) the following iterative formula for the spherical functions of type  $A$  in the complex case. Here the Cartan subalgebra  $\mathfrak{a}$  for the root system  $A_{n-1}$  is isomorphic to  $\mathbf{R}^n$ . For  $\lambda, X \in \bar{\mathfrak{a}}^+ \subset \mathbf{R}^n$ , we have

$$\begin{aligned} \psi_\lambda(e^X) &= e^{\lambda(X)} \text{ if } n = 1 \\ \text{and } \psi_\lambda(e^X) &= (n-1)! e^{\lambda_n \sum_{k=1}^n x_k} \left( \prod_{i < j} (x_i - x_j) \right)^{-1} \int_{x_n}^{x_{n-1}} \dots \int_{x_2}^{x_1} \psi_{\lambda_0}(e^Y) \\ &\quad \prod_{i < j < n} (y_i - y_j) dy_1 \dots dy_{n-1} \end{aligned} \tag{11}$$

where  $\lambda_0(U) = \sum_{k=1}^{n-1} (\lambda_k - \lambda_n) u_k$ .

**Remark 14.** Formula (11) represents the action of the root system  $A_{n-1}$  on  $\mathbf{R}^n$ . If we assume

$$\sum_{k=1}^n x_k = 0 = \sum_{k=1}^n \lambda_k,$$

we have then the action of the root system  $A_{n-1}$  on  $\mathbf{R}^{n-1}$ . We can also consider the action of  $A_{n-1}$  on any  $\mathbf{R}^m$  with  $m \geq n-1$  by considering formula (9) and deciding on which entries  $x_k$ , the Weyl group  $W = S_n$  acts. These considerations do not affect the conclusion of Theorem 12.

#### 3.1. Approximate factorization for $A_n$

Before proving the conjecture in the case  $A_n$ , we will prove an interesting “factorization”.

**Proposition 15.** *For  $n \geq 1$ , consider the root system  $A_n$  on  $\mathbf{R}^{n+1}$ . Let  $\lambda, X \in \bar{\mathfrak{a}}^+ \subset \mathbf{R}^{n+1}$ ,  $X' = [x_1, \dots, x_n]$  so that  $X = [X', x_{n+1}]$ . Define*

$$\begin{aligned} I^{(n)} = I^{(n)}(\lambda; X) &= \int_{x_{n+1}}^{x_n} \int_{x_n}^{x_{n-1}} \dots \int_{x_3}^{x_2} \int_{x_2}^{x_1} e^{-\lambda_0(X'-Y)} \\ &\quad \prod_{i < j < n} \frac{(y_i - y_j)(\lambda_i - \lambda_j)}{1 + (y_i - y_j)(\lambda_i - \lambda_j)} dy_1 dy_2 \dots dy_n. \end{aligned}$$

Then the following approximate factorization holds

$$I^{(n)} \simeq \prod_{k=1}^n I_k^{(n)} \tag{12}$$

where

$$I_1^{(n)} = \int_{x_2}^{x_1} e^{-(\lambda_1 - \lambda_{n+1})(x_1 - y_1)} dy_1$$

and

$$I_k^{(n)} = \int_{x_{k+1}}^{x_k} e^{-(\lambda_k - \lambda_{n+1})(x_k - y_k)} \prod_{j=1}^{k-1} \frac{(x_j - y_k)(\lambda_j - \lambda_k)}{1 + (x_j - y_k)(\lambda_j - \lambda_k)} dy_k \text{ for } 1 < k \leq n.$$

**Proof.** Since  $u/(1 + u)$  is an increasing function, we clearly have

$$I^{(n)} \leq \int_{x_{n+1}}^{x_n} \int_{x_n}^{x_{n-1}} \dots \int_{x_3}^{x_2} \int_{x_2}^{x_1} e^{-\lambda_0(X' - Y)} \prod_{i < j < n} \frac{(x_i - y_j)(\lambda_i - \lambda_j)}{1 + (x_i - y_j)(\lambda_i - \lambda_j)} dy_1 dy_2 \dots dy_n.$$

On the other hand,

$$\begin{aligned} I^{(n)} &\geq \int_{(x_n + x_{n+1})/2}^{x_n} \int_{(x_{n-1} + x_n)/2}^{x_{n-1}} \dots \int_{(x_2 + x_3)/2}^{x_2} \int_{(x_1 + x_2)/2}^{x_1} e^{-\lambda_0(X' - Y)} \\ &\quad \prod_{i < j < n} \frac{(y_i - y_j)(\lambda_i - \lambda_j)}{1 + (y_i - y_j)(\lambda_i - \lambda_j)} dy_1 dy_2 \dots dy_n \\ &\geq \int_{(x_n + x_{n+1})/2}^{x_n} \int_{(x_{n-1} + x_n)/2}^{x_{n-1}} \dots \int_{(x_2 + x_3)/2}^{x_2} \int_{(x_1 + x_2)/2}^{x_1} e^{-\lambda_0(X' - Y)} \\ &\quad \prod_{i < j < n} \frac{((x_i + x_{i+1})/2 - y_j)(\lambda_i - \lambda_j)}{1 + ((x_i + x_{i+1})/2 - y_j)(\lambda_i - \lambda_j)} dy_1 dy_2 \dots dy_n \\ &\simeq \int_{(x_n + x_{n+1})/2}^{x_n} \int_{(x_{n-1} + x_n)/2}^{x_{n-1}} \dots \int_{(x_2 + x_3)/2}^{x_2} \int_{(x_1 + x_2)/2}^{x_1} e^{-\lambda_0(X' - Y)} \\ &\quad \prod_{i < j < n} \frac{(x_i - y_j)(\lambda_i - \lambda_j)}{1 + (x_i - y_j)(\lambda_i - \lambda_j)} dy_1 dy_2 \dots dy_n = \\ &= \prod_{k=1}^n \int_{(x_k + x_{k+1})/2}^{x_k} e^{-(\lambda_k - \lambda_{n+1})(x_k - y_k)} \prod_{j=1}^{k-1} \frac{(x_j - y_k)(\lambda_j - \lambda_k)}{1 + (x_j - y_k)(\lambda_j - \lambda_k)} dy_k = \prod_{k=1}^n A_k^{(n)} \end{aligned}$$

since

$$\frac{((x_i + x_{i+1})/2 - y_j)(\lambda_i - \lambda_j)}{1 + ((x_i + x_{i+1})/2 - y_j)(\lambda_i - \lambda_j)} \leq \frac{(x_i - y_j)(\lambda_i - \lambda_j)}{1 + (x_i - y_j)(\lambda_i - \lambda_j)}$$

while

$$\frac{((x_i + x_{i+1})/2 - y_j)(\lambda_i - \lambda_j)}{1 + ((x_i + x_{i+1})/2 - y_j)(\lambda_i - \lambda_j)} \geq \frac{(x_i - y_j)/2 (\lambda_i - \lambda_j)}{1 + (x_i - y_j)/2 (\lambda_i - \lambda_j)} \geq \frac{1}{2} \frac{(x_i - y_j)(\lambda_i - \lambda_j)}{1 + (x_i - y_j)(\lambda_i - \lambda_j)}.$$

Now, let

$$B_k^{(n)} = \int_{x_{k+1}}^{(x_k + x_{k+1})/2} e^{-(\lambda_k - \lambda_{n+1})(x_k - y_k)} \prod_{j=1}^{k-1} \frac{(x_j - y_k)(\lambda_j - \lambda_k)}{1 + (x_j - y_k)(\lambda_j - \lambda_k)} dy_k$$

and note that  $I_k^{(n)} = A_k^{(n)} + B_k^{(n)}$ .



Now, using the change of variable  $2w = x_k - y_k$ , we have

$$\begin{aligned} B_k^{(n)} &= 2 \int_{(x_k - x_{k+1})/4}^{(x_k - x_{k+1})/2} e^{-2(\lambda_k - \lambda_{n+1})w} \prod_{j=1}^{k-1} \frac{(x_j - x_k + 2w)(\lambda_j - \lambda_k)}{1 + (x_j - x_k + 2w)(\lambda_j - \lambda_k)} dw \\ &\leq 4 \int_{(x_k - x_{k+1})/4}^{(x_k - x_{k+1})/2} e^{-2(\lambda_k - \lambda_{n+1})w} \prod_{j=1}^{k-1} \frac{(x_j - x_k + w)(\lambda_j - \lambda_k)}{1 + (x_j - x_k + w)(\lambda_j - \lambda_k)} dw \\ &\leq 4 \int_0^{(x_k - x_{k+1})/2} e^{-(\lambda_k - \lambda_{n+1})w} \prod_{j=1}^{k-1} \frac{(x_j - x_k + w)(\lambda_j - \lambda_k)}{1 + (x_j - x_k + w)(\lambda_j - \lambda_k)} dw = 4A_k^{(n)}, \end{aligned}$$

where the last equality comes from the change of variable  $w = x_k - y_k$  in the expression for  $A_k^{(n)}$ . Therefore  $I_k^{(n)} = A_k^{(n)} + B_k^{(n)} \leq 5A_k^{(n)}$ . The result follows.  $\square$

The next Proposition 16 gives an inductive way of estimating  $I^{(n+1)}$ , knowing  $I^{(n)}$  and  $I^{(n-1)}$ .

**Proposition 16.** Consider the root system  $A_{n+1}$  on  $\mathbf{R}^{n+2}$ . Let  $\lambda, X \in \bar{\mathfrak{a}}^+ \subset \mathbf{R}^{n+2}$ . Assume  $\alpha_1(X) \geq \alpha_{n+1}(X)$ . Then

$$\begin{aligned} I^{(n+1)}(\lambda; X) &= I^{(n)}(\lambda_1, \dots, \lambda_n, \lambda_{n+2}; x_1, \dots, x_{n+1}) \frac{(x_1 - x_{n+1})(\lambda_1 - \lambda_{n+1})}{1 + (x_1 - x_{n+1})(\lambda_1 - \lambda_{n+1})} \\ &\quad \frac{I^{(n)}(\lambda_2, \dots, \lambda_{n+1}, \lambda_{n+2}; x_2, \dots, x_{n+2})}{I^{(n-1)}(\lambda_2, \dots, \lambda_n, \lambda_{n+2}; x_2, \dots, x_{n+1})}. \end{aligned}$$

**Proof.** We start with an outline of the proof.

- (i)  $I^{(n+1)}$  is estimated by a product of  $n+1$  factors  $I_k^{(n+1)}(\lambda; X)$ .
- (ii) The product of the first  $n$  factors  $I_1^{(n+1)}(\lambda; X), \dots, I_n^{(n+1)}(\lambda; X)$  gives an estimate of the term  $I^{(n)}(\lambda_1, \dots, \lambda_n, \lambda_{n+2}; X')$  by Proposition 15.
- (iii) In the last factor  $I_{n+1}^{(n+1)}(\lambda; X)$ , we “draw off” one term from under the integral, using the additional hypothesis  $\alpha_1(X) \geq \alpha_{n+1}(X)$ . The remaining integral corresponds to  $I_n^{(n)}(\lambda_2, \dots, \lambda_{n+2}; x_2, \dots, x_{n+2})$ .
- (iv) The last factor  $I_n^{(n)}$  of  $I^{(n)}$  is estimated by  $I^{(n)}/I^{(n-1)}$ , up to a change of variables (we re-use the idea of (ii)).

Since  $x_{n+2} \leq y_{n+1} \leq x_{n+1}$  and  $x_{n+1} - x_{n+2} \leq x_1 - x_2$ , we get  $x_1 - x_{n+1} \leq x_1 - y_{n+1} \leq x_1 - x_{n+2} \leq 2(x_1 - x_{n+1})$  and we have

$$\begin{aligned} I_{n+1}^{(n+1)} &= \int_{x_{n+2}}^{x_{n+1}} e^{-(\lambda_{n+1} - \lambda_{n+2})(x_{n+1} - y_{n+1})} \frac{(x_1 - y_{n+1})(\lambda_1 - \lambda_{n+1})}{1 + (x_1 - y_{n+1})(\lambda_1 - \lambda_{n+1})} \\ &\quad \prod_{j=2}^n \frac{(x_j - y_{n+1})(\lambda_j - \lambda_{n+1})}{1 + (x_j - y_{n+1})(\lambda_j - \lambda_{n+1})} (\lambda_j - \lambda_{n+1}) dy_{n+1} \\ &\asymp \frac{(x_1 - x_{n+1})(\lambda_1 - \lambda_{n+1})}{1 + (x_1 - x_{n+1})(\lambda_1 - \lambda_{n+1})} \int_{x_{n+2}}^{x_{n+1}} e^{-(\lambda_{n+1} - \lambda_{n+2})(x_{n+1} - y_{n+1})} \\ &\quad \prod_{j=2}^n \frac{(x_j - y_{n+1})(\lambda_j - \lambda_{n+1})}{1 + (x_j - y_{n+1})(\lambda_j - \lambda_{n+1})} dy_{n+1}. \end{aligned}$$

Hence, noting that  $I_1^{(n+1)}(\lambda; X) \cdots I_n^{(n+1)}(\lambda; X) = I^{(n)}(\lambda_1, \dots, \lambda_n, \lambda_{n+2}; X')$ , we have

$$\begin{aligned} I^{(n+1)}(\lambda; X) &= I^{(n)}(\lambda_1, \dots, \lambda_n, \lambda_{n+2}; X') \frac{(x_1 - x_{n+1})(\lambda_1 - \lambda_{n+1})}{1 + (x_1 - x_{n+1})(\lambda_1 - \lambda_{n+1})} \\ &\quad \int_{x_{n+2}}^{x_{n+1}} e^{-(\lambda_{n+1} - \lambda_{n+2})(x_{n+1} - y_{n+1})} \prod_{j=2}^n \frac{(x_j - y_{n+1})(\lambda_j - \lambda_{n+1})}{1 + (x_j - y_{n+1})(\lambda_j - \lambda_{n+1})} dy_{n+1}. \end{aligned}$$

Finally,

$$\begin{aligned} & \int_{x_{n+2}}^{x_{n+1}} e^{-(\lambda_{n+1}-\lambda_{n+2})(x_{n+1}-y_{n+1})} \prod_{j=2}^n \frac{(x_j - y_{n+1})(\lambda_j - \lambda_{n+1})}{1 + (x_j - y_{n+1})(\lambda_j - \lambda_{n+1})} dy_{n+1} \\ &= \frac{\prod_{k=1}^n \int_{x_{k+2}}^{x_{k+1}} e^{-(\lambda_{k+1}-\lambda_{n+2})(x_{k+1}-y_{k+1})} \prod_{j=1}^{k-1} \frac{(x_{j+1} - y_{k+1})(\lambda_{j+1} - \lambda_{k+1})}{1 + (x_{j+1} - y_{k+1})(\lambda_{j+1} - \lambda_{k+1})} dy_{k+1}}{\prod_{k=1}^{n-1} \int_{x_{k+2}}^{x_{k+1}} e^{-(\lambda_{k+1}-\lambda_{n+2})(x_{k+1}-y_{k+1})} \prod_{j=1}^{k-1} \frac{(x_{j+1} - y_{k+1})(\lambda_{j+1} - \lambda_{k+1})}{1 + (x_{j+1} - y_{k+1})(\lambda_{j+1} - \lambda_{k+1})} dy_{k+1}} \\ &= \frac{I^{(n)}(\lambda_2, \dots, \lambda_{n+1}, \lambda_{n+2}; x_2, \dots, x_{n+2})}{I^{(n-1)}(\lambda_2, \dots, \lambda_n, \lambda_{n+2}; x_2, \dots, x_{n+1})}. \quad \square \end{aligned}$$

**Remark 17.** When  $n = 1$ , the result of Proposition 16 remains valid if we set  $I^{(0)} = 1$ .

We now prove our main result.

**Proof of Theorem 12.** We use induction on the rank. In the case of  $A_1$ , we have

$$\begin{aligned} \psi_\lambda(e^X) &= e^{\lambda_2(x_1+x_2)} (x_1 - x_2)^{-1} \int_{x_2}^{x_1} e^{(\lambda_1-\lambda_2)y} dy \\ &= e^{\lambda_2(x_1+x_2)} (x_1 - x_2)^{-1} \frac{e^{(\lambda_1-\lambda_2)x_1} - e^{(\lambda_1-\lambda_2)x_2}}{\lambda_1 - \lambda_2} \\ &= e^{\lambda_1 x_1 + \lambda_2 x_2} \frac{1 - e^{-(\lambda_1-\lambda_2)(x_1-x_2)}}{(\lambda_1 - \lambda_2)(x_1 - x_2)} \simeq e^{\lambda_1 x_1 + \lambda_2 x_2} \frac{1}{1 + (\lambda_1 - \lambda_2)(x_1 - x_2)} \end{aligned}$$

since  $1 - e^{-u} \simeq u/(1 + u)$  for  $u \geq 0$ .

Assume that the result is true for  $A_r$ ,  $1 \leq r \leq n$ ,  $n \geq 1$ . Using (11) and the induction hypothesis, we have for  $r = 1, \dots, n + 1$  and  $\lambda, X$  in positive Weyl chamber in  $\mathbf{R}^{r+1}$

$$\begin{aligned} & \pi(X) \pi(\lambda') e^{-\lambda(X)} \psi_\lambda([x_1, \dots, x_r, x_{r+1}]) \\ &= r! \pi(\lambda') e^{-\lambda(X)} e^{\lambda_{r+1} \sum_{k=1}^{r+1} x_k} \int_{x_{r+1}}^{x_r} \dots \int_{x_2}^{x_1} \psi_{\lambda_0}(e^Y) \prod_{i < j < r+1} (y_i - y_j) dy_1 \dots dy_r \\ &\simeq \int_{x_{r+1}}^{x_r} \int_{x_r}^{x_{r-1}} \dots \int_{x_3}^{x_2} \int_{x_2}^{x_1} e^{-\lambda_0(X'-Y)} \prod_{i < j < r+1} \frac{(y_i - y_j)(\lambda_i - \lambda_j)}{1 + (y_i - y_j)(\lambda_i - \lambda_j)} dy_1 dy_2 \dots dy_r \end{aligned}$$

where  $X' = \text{diag}[x_1, \dots, x_r]$  and  $\lambda' = [\lambda_1, \dots, \lambda_r]$ . Using the notation introduced in Proposition 15, we have

$$\pi(X) \pi(\lambda') e^{-\lambda(X)} \psi_{[\lambda_1, \dots, \lambda_{r+1}]}([x_1, \dots, x_{r+1}]) = r! I^{(r)}(\lambda_1, \dots, \lambda_{r+1}, x_1, \dots, x_{r+1}).$$

Still using the induction hypothesis, we have

$$\begin{aligned} \pi(X) \pi(\lambda') e^{-\lambda(X)} \psi_{[\lambda_1, \dots, \lambda_{r+1}]}([x_1, \dots, x_{r+1}]) &= r! I^{(r)}(\lambda_1, \dots, \lambda_{r+1}; x_1, \dots, x_{r+1}) \\ &\simeq \frac{\pi(X) \pi(\lambda')}{\prod_{i < j \leq r+1} (1 + (\lambda_i - \lambda_j)(x_i - x_j))} \end{aligned} \tag{13}$$

for  $r = 1, \dots, n$ .

It remains to show that (13) holds for  $r = n + 1$ , i.e. that

$$I^{(n+1)}(\lambda_1, \dots, \lambda_{n+2}; x_1, \dots, x_{n+2}) \simeq \frac{\pi(X) \pi(\lambda')}{\prod_{i < j \leq n+2} (1 + (\lambda_i - \lambda_j)(x_i - x_j))}.$$

It is sufficient to prove the last formula under the hypothesis that  $\alpha_1(X) \geq \alpha_{n+1}(X)$  since the case  $\alpha_1(X) \leq \alpha_{n+1}(X)$  is symmetric. Now, according to Proposition 16 and (13),

$$\begin{aligned}
 I^{(n+1)}(\lambda; X) &= \frac{(x_1 - x_{n+1})(\lambda_1 - \lambda_{n+1})}{1 + (x_1 - x_{n+1})(\lambda_1 - \lambda_{n+1})} I^{(n)}(\lambda_1, \dots, \lambda_n, \lambda_{n+2}; x_1, \dots, x_{n+1}) \\
 &\quad I^{(n)}(\lambda_2, \dots, \lambda_{n+1}, \lambda_{n+2}; x_2, \dots, x_{n+2}) \left( I^{(n-1)}(\lambda_2, \dots, \lambda_n, \lambda_{n+2}; x_2, \dots, x_{n+1}) \right)^{-1} \\
 &= \frac{(x_1 - x_{n+1})(\lambda_1 - \lambda_{n+1})}{1 + (x_1 - x_{n+1})(\lambda_1 - \lambda_{n+1})} \\
 &\quad \frac{\prod_{i < j \leq n+1} (x_i - x_j) \prod_{i < j < n+1} (\lambda_i - \lambda_j)}{\prod_{i < j \leq n} (1 + (x_i - x_j)(\lambda_i - \lambda_j)) \prod_{i=1}^n (1 + (x_i - x_{n+1})(\lambda_i - \lambda_{n+2}))} \\
 &\quad \frac{\prod_{1 < i < j \leq n+2} (x_i - x_j) \prod_{1 < i < j \leq n+1} (\lambda_i - \lambda_j)}{\prod_{1 < i < j \leq n+2} (1 + (x_i - x_j)(\lambda_i - \lambda_j))} \\
 &\quad \frac{\prod_{1 < i < j \leq n} (1 + (x_i - x_j)(\lambda_i - \lambda_j)) \prod_{i=2}^n (1 + (x_i - x_{n+1})(\lambda_i - \lambda_{n+2}))}{\prod_{1 < i < j \leq n+1} (x_i - x_j) \prod_{1 < i < j < n+1} (\lambda_i - \lambda_j)} \\
 &= \frac{x_1 - x_{n+1}}{x_1 - x_{n+2}} \frac{1 + (x_1 - x_{n+1})(\lambda_1 - \lambda_2)}{1 + (x_1 - x_{n+1})(\lambda_1 - \lambda_2)} \frac{\prod_{i < j \leq n+2} (x_i - x_j) \prod_{i < j \leq n+1} (\lambda_i - \lambda_j)}{\prod_{i < j \leq n+2} (1 + (\lambda_i - \lambda_j)(x_i - x_j))}.
 \end{aligned}$$

The result follows since  $x_1 - x_{n+1} \asymp x_1 - x_{n+2}$  given that  $x_1 - x_2 \geq x_{n+1} - x_{n+2}$ .  $\square$

#### 4. Comparison with the estimates of Anker et al. in [1]. Conjecture for Dunkl setting

In [1, Theorem 4.1 p. 2372 and Theorem 4.4, p. 2377] the following estimates were proven for the heat kernel  $p_t(X, Y)$  in the Dunkl setting on  $\mathbf{R}^n$ . There exist positive constants  $c_1, c_2, C_1$  and  $C_2$  such that for all  $X, Y \in \mathfrak{a}^+$

$$\frac{C_1 e^{-c_1 |X-Y|^2/t}}{\min\{w(B(X, \sqrt{t})), w(B(Y, \sqrt{t}))\}} \leq p_t(X, Y) \leq \frac{C_2 e^{-c_2 |X-Y|^2/t}}{\max\{w(B(X, \sqrt{t})), w(B(Y, \sqrt{t}))\}} \quad (14)$$

where  $w$  is the  $W$ -invariant reference measure (in our paper  $w = \pi(X)^2 dX$ ) and the  $w$ -volume of a ball satisfies the estimate ([1, p. 2365])

$$w(B(X, r)) \asymp r^n \prod_{\alpha > 0} (r + \alpha(X))^{2k(\alpha)}.$$

The same estimates follow for  $p_t^W(X, Y)$ . Our sharp estimates in Corollary 13 for  $k(\alpha) = 1$  in the  $W$ -radial case  $A_n$  suggest that  $c_1 = c_2 = 1/4$  in (14) and that products of terms  $(t + \alpha(X)\alpha(Y))^{k(\alpha)}$  are natural in place of separate terms  $w(B(X, \sqrt{t}))$  and  $w(B(Y, \sqrt{t}))$ . On the other hand, estimates (14) and in Corollary 13 suggest that the following conjecture is true in the Dunkl setting.

**Conjecture 18.** *The Weyl-invariant heat kernel for a root system  $\Sigma$  in  $\mathbf{R}^d$  satisfies the following estimates*

$$p_t^W(X, Y) \asymp t^{-\frac{d}{2}} \frac{e^{-\frac{|X-Y|^2}{4t}}}{\prod_{\alpha > 0} (t + \alpha(X)\alpha(Y))^{k(\alpha)}}. \quad (15)$$

Formula (3) then implies that the  $W$ -invariant Dunkl kernel satisfies the estimate

$$E_k^W(X, Y) \asymp \frac{e^{\lambda(X)}}{\prod_{\alpha>0} (1 + \alpha(X) \alpha(\lambda))^{k(\alpha)}}.$$

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