



INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus

Mathématique


Jun-Yu Zhou and Quan-Hui Yang

On the structure of the h -fold sumsets

Volume 359, issue 4 (2021), p. 493-500

<<https://doi.org/10.5802/crmath.191>>

© Académie des sciences, Paris and the authors, 2021.
Some rights reserved.

 This article is licensed under the
CREATIVE COMMONS ATTRIBUTION 4.0 INTERNATIONAL LICENSE.
<http://creativecommons.org/licenses/by/4.0/>



Les Comptes Rendus. Mathématique sont membres du
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org



Number theory / *Théorie des nombres*

On the structure of the h -fold sumsets

Jun-Yu Zhou^a and Quan-Hui Yang^{*, a}

^a School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China

E-mails: 1045908839@qq.com, yangquanhui01@163.com

Abstract. Let A be a set of nonnegative integers. Let $(hA)^{(t)}$ be the set of all integers in the sumset hA that have at least t representations as a sum of h elements of A . In this paper, we prove that, if $k \geq 2$, and $A = \{a_0, a_1, \dots, a_k\}$ is a finite set of integers such that $0 = a_0 < a_1 < \dots < a_k$ and $\gcd(a_1, a_2, \dots, a_k) = 1$, then there exist integers c_t, d_t and sets $C_t \subseteq [0, c_t - 2]$, $D_t \subseteq [0, d_t - 2]$ such that

$$(hA)^{(t)} = C_t \cup [c_t, ha_k - d_t] \cup (ha_{k-1} - D_t)$$

for all $h \geq \sum_{i=2}^k (ta_i - 1) - 1$. This improves a recent result of Nathanson with the bound $h \geq (k-1)(ta_k - 1)a_k + 1$.

2020 Mathematics Subject Classification. 11B13.

Funding. This author was supported by the National Natural Science Foundation for Youth of China, Grant No. 11501299, the Natural Science Foundation of Jiangsu Province, Grant Nos. BK20150889, 15KJB110014.

Manuscript received 4th January 2021, revised 12th February 2021, accepted 15th February 2021.

1. Introduction

Let A and B be sets of integers. The sumsets and difference sets are defined by

$$A + B = \{a + b : a \in A, b \in B\}, \quad A - B = \{a - b : a \in A, b \in B\}$$

respectively. For any integer t , we define the sets

$$t + A = \{t\} + A, \quad t - A = \{t\} - A.$$

For $h \geq 2$, we denote by hA the h -fold sumset of A , which is the set of all integers n of the form $n = a_1 + a_2 + \dots + a_h$, where a_1, a_2, \dots, a_h are elements of A and not necessarily distinct.

In [3, 4], Nathanson proved the following fundamental beautiful theorem on the structure of h -fold sumsets.

* Corresponding author.

Nathanson’s Theorem A. *Let $A = \{a_0, a_1, \dots, a_k\}$ be a finite set of integers such that*

$$0 = a_0 < a_1 < \dots < a_k \quad \text{and} \quad \gcd(A) = 1.$$

Let $h_1 = (k - 1)(a_k - 1)a_k + 1$. There are nonnegative integers c_1 and d_1 and finite sets C_1 and D_1 with $C_1 \subseteq [0, c_1 - 2]$ and $D_1 \subseteq [0, d_1 - 2]$ such that

$$hA = C_1 \cup [c_1, ha_k - d_1] \cup (ha_k - D_1)$$

for all $h \geq h_1$.

Later, Wu, Chen and Chen [6] improved the lower bound of h_1 to $\sum_{i=2}^k a_i - k$. Recently, Granville and Shakan [1], and Granville and Walker [2] gave some further results on this topic.

Let A be a set of integers. For every positive integer h , the h -fold representation function $r_{A,h}(n)$ counts the number of representations of n as the sum of h elements of A . Thus,

$$r_{A,h}(n) = \#\left\{ (a_{j_1}, \dots, a_{j_h}) \in A^h : n = \sum_{i=1}^h a_{j_i} \text{ and } a_{j_1} \leq \dots \leq a_{j_h} \right\}.$$

For every positive integer t , let $(hA)^{(t)}$ be the set of all integers n that have at least t representations as the sum of h elements of A , that is,

$$(hA)^{(t)} = \{n \in \mathbf{Z} : r_{A,h}(n) \geq t\}$$

Recently, Nathanson [5] found that the sumsets $(hA)^{(t)}$ have the same structure as the sumset hA and proved the following theorem.

Nathanson’s Theorem B. *Let $k \geq 2$, and let $A = \{a_0, a_1, \dots, a_k\}$ be a finite set of integers such that $0 = a_0 < a_1 < \dots < a_k$ and $\gcd(A) = 1$. For every positive integer t , let $h_t = (k - 1)(ta_k - 1)a_k + 1$. There are nonnegative integers c_t and d_t , and finite sets C_t and D_t with $C_t \subseteq [0, c_t - 2]$ and $D_t \subseteq [0, d_t - 2]$ such that*

$$(hA)^{(t)} = C_t \cup [c_t, ha_k - d_t] \cup (ha_k - D_t)$$

for all $h \geq h_t$.

In this paper, motivated by the idea of Wu, Chen and Chen [6], we improved the lower bound of h in Nathanson’s Theorem B.

Theorem 1. *Let $k \geq 2$, and let $A = \{a_0, a_1, \dots, a_k\}$ be a finite set of integers such that*

$$0 = a_0 < a_1 < \dots < a_k \quad \text{and} \quad \gcd(A) = 1$$

For every positive integer t , let

$$h_t = \sum_{i=2}^k (ta_i - 1) - 1$$

There are nonnegative integers c_t and d_t and finite sets C_t and D_t with

$$C_t \subseteq [0, c_t - 2] \quad \text{and} \quad D_t \subseteq [0, d_t - 2]$$

such that

$$(hA)^{(t)} = C_t \cup [c_t, ha_k - d_t] \cup (ha_k - D_t) \tag{1}$$

for all $h \geq h_t$.

Remark 2. Theorem 1 is optimal.

We shall prove Theorem 1 and Remark 2 in Section 3. In Section 2, we give some lemmas.

2. Some Lemmas

Lemma 3 (see [5, Lemma 1]). *Let A be a set of integers. For any positive integer h and t , we have*

$$(hA)^{(t)} + A \subseteq ((h + 1)A)^{(t)}.$$

Lemma 4. *Let $k \geq 2$, and let $A = \{a_0, a_1, \dots, a_k\}$ be a set of integers satisfying $0 = a_0 < a_1 < \dots < a_k$ and $\gcd(A) = 1$. For every positive integer t , let $h_t = \sum_{i=2}^k (ta_i - 1) - 1$ and $c'_t = \sum_{i=1}^{k-1} a_i (ta_{i+1} - 1)$. If $c'_t - a_k < n < c'_t$, then there exist at least t distinct nonnegative k -tuples $(x_{1,s}, x_{2,s}, \dots, x_{k,s})$ ($1 \leq s \leq t$) satisfying*

$$n = x_{1,s}a_1 + x_{2,s}a_2 + \dots + x_{k,s}a_k$$

and $x_{1,s} + x_{2,s} + \dots + x_{k,s} \leq h_t$ for $s = 1, 2, \dots, t$.

Proof. Since $\gcd(a_1, \dots, a_k) = 1$, there exist integers x_1, \dots, x_k such that

$$n = x_1 a_1 + \dots + x_k a_k.$$

For any positive integer s , $[(s - 1)a_2, sa_2 - 1]$ is a complete residue system modulo a_2 . Hence there exists an integer q such that $x_1 = a_2 q + x_{1,s}$ with $(s - 1)a_2 \leq x_{1,s} \leq sa_2 - 1$. This gives

$$n = x_{1,s}a_1 + (a_1 q + x_2)a_2 + \dots + x_k a_k.$$

Let $x'_2 = a_1 q + x_2$. Similarly, there exists an integer q' such that $x'_2 = a_3 q' + x_{2,s}$ with $(s - 1)a_3 \leq x_{2,s} \leq sa_3 - 1$. Now we have

$$n = x_{1,s}a_1 + x_{2,s}x_2 + (a_2 q' + x_3)a_3 + \dots + x_k a_k.$$

By continuing this process, we obtain

$$n = x_{1,s}a_1 + x_{2,s}a_2 + \dots + x_{k,s}a_k$$

with $(s - 1)a_{i+1} \leq x_{i,s} \leq sa_{i+1} - 1$ for $i = 1, \dots, k - 1$ and $x_{k,s}$ is some integer. Hence, for any integer $s \in [1, t]$, we have

$$0 \leq x_{i,s} \leq ta_{i+1} - 1.$$

Since $n > c'_t - a_k$, it follows that

$$\begin{aligned} x_{k,s}a_k &= n - (x_{1,s}a_1 + x_{2,s}a_2 + \dots + x_{k-1,s}a_{k-1}) \\ &\geq n - (ta_2 - 1)a_1 - \dots - (ta_k - 1)a_{k-1} = n - c'_t > -a_k, \end{aligned}$$

and then $x_{k,s} > -1$. Noting that $x_{k,s}$ is an integer, we have $x_{k,s} \geq 0$. By the bound of $x_{i,s}$, the following nonnegative k -tuples

$$(x_{1,s}, x_{2,s}, \dots, x_{k-1,s}, x_{k,s}) \quad (1 \leq s \leq t)$$

are distinct.

Next, we shall prove that $x_{1,s} + x_{2,s} + \dots + x_{k,s} \leq h_t$ for $s = 1, 2, \dots, t$.

For any integer $s \in [1, t]$, let $x_{1,s} + x_{2,s} + \dots + x_{k,s} = u_s$. Since $n < c'_t$, it follows that

$$\begin{aligned} n &= x_{1,s}a_1 + x_{2,s}a_2 + \dots + x_{k,s}a_k \\ &= x_{1,s}a_1 + \dots + x_{k-1,s}a_{k-1} + (u_s - x_{1,s} - x_{2,s} - \dots - x_{k-1,s})a_k \\ &= u_s a_k - x_{1,s}(a_k - a_1) - \dots - x_{k-1,s}(a_k - a_{k-1}) \\ &\geq u_s a_k - (ta_2 - 1)(a_k - a_1) - \dots - (ta_k - 1)(a_k - a_{k-1}) \\ &= u_s a_k - a_k [(ta_2 - 1) + \dots + (ta_k - 1)] + a_1 (ta_2 - 1) + \dots + a_{k-1} (ta_k - 1) \\ &= u_s a_k - (h_t + 1)a_k + c'_t \\ &> u_s a_k - (h_t + 1)a_k + n. \end{aligned}$$

Hence $u_s a_k - (h_t + 1)a_k < 0$, and then $u_s < h_t + 1$. Therefore, $u_s \leq h_t$.

This completes the proof of Lemma 4. □

Lemma 5. *Let c'_t and h_t be defined in Lemma 4. Then*

$$c'_t = \sum_{i=1}^{k-1} a_i (ta_{i+1} - 1) \in ((h_t + 1)A)^{(t)}.$$

Proof. For $i = 1, 2, \dots, k - 1$, let $p_i = ta_{i+1} - 1$. Then

$$c'_t = (ta_2 - 1)a_1 + \dots + (ta_k - 1)a_{k-1} = p_1a_1 + \dots + p_{k-1}a_{k-1}.$$

Noting that

$$p_1 + \dots + p_{k-1} = \sum_{i=2}^k (ta_i - 1) = h_t + 1,$$

we have $c'_t \in (h_t + 1)A$.

Moreover, for any integer $r \in [0, t - 1]$, we have

$$\begin{aligned} c'_t &= \sum_{i=1}^{k-1} (ta_{i+1} - 1)a_i = \sum_{i=1}^{k-1} ((t-r)a_{i+1} - 1)a_i + r \sum_{i=1}^{k-1} a_i a_{i+1} \\ &= ((t-r)a_2 - 1)a_1 + \sum_{i=2}^{k-1} ((t-r)a_{i+1} - 1 + ra_{i-1})a_i + ra_{k-1}a_k \\ &:= p_{1,r}a_1 + p_{2,r}a_2 + \dots + p_{k-1,r}a_{k-1} + p_{k,r}a_k, \end{aligned}$$

where $p_{1,r} = (t-r)a_2 - 1$, $p_{k,r} = ra_{k-1}$ and $p_{i,r} = (t-r)a_{i+1} - 1 + ra_{i-1}$ ($2 \leq i \leq k - 1$). Hence $p_{i,r} \geq 0$ for all $i \in [1, k]$ and

$$\begin{aligned} \sum_{i=1}^k p_{i,r} &= (t-r)a_2 - 1 + (t-r)a_3 - 1 + ra_1 + \dots + (t-r)a_k - 1 + ra_{k-2} + ra_{k-1} \\ &= h_t + 1 - r(a_2 + \dots + a_k) + r(a_1 + \dots + a_{k-1}) \\ &= h_t + 1 - r(a_k - a_1) \leq h_t + 1. \end{aligned}$$

Thus, $r_{A, h_t+1}(c'_t) \geq t$, and so $c'_t \in ((h_t + 1)A)^{(t)}$. □

Lemma 6. *Let n and a_1, a_2 be positive integers with $\gcd(a_1, a_2) = 1$. For any positive integer t , if $n > ta_1a_2 - a_1 - a_2$, then the diophantine equation*

$$a_1x + a_2y = n \tag{2}$$

has at least t nonnegative integer solutions. The lower bound of n is also best possible.

Proof. Suppose that $n > ta_1a_2 - a_1 - a_2$. Let (x_0, y_0) be a solution of the equation (2). Then all the integer solutions of the equation (2) is

$$\begin{cases} x = x_0 + ka_2, \\ y = y_0 - ka_1, \end{cases} \quad k \in \mathbb{Z}. \tag{3}$$

In order to have $x \geq 0$ and $y \geq 0$, we only need $x > -1$ and $y > -1$, that is,

$$\frac{-1 - x_0}{a_2} < k < \frac{y_0 + 1}{a_1}. \tag{4}$$

Since

$$\frac{y_0 + 1}{a_1} - \frac{-1 - x_0}{a_2} = \frac{a_1 + a_2 + a_1x_0 + a_2y_0}{a_1a_2} = \frac{a_1 + a_2 + n}{a_1a_2} > \frac{a_1 + a_2 + ta_1a_2 - a_1 - a_2}{a_1a_2} = t,$$

there exist at least t integers k such that (4) holds.

Therefore, the equation (2) has at least t nonnegative integer solutions.

Now suppose that $l = ta_1a_2 - a_1 - a_2$. Then $(ta_2 - 1, -1)$ is a solution of (2). Take $x_0 = ta_2 - 1$ and $y_0 = -1$. Then (3) becomes

$$\begin{cases} x = ta_2 - 1 - ka_2, \\ y = -1 + ka_1, \end{cases} \quad k \in \mathbb{Z}.$$

Since $x \geq 0$ and $y \geq 0$, it follows that $1 \leq k \leq t - 1$. Hence there exist at most $t - 1$ nonnegative integer solutions.

This completes the proof of Lemma 6. □

3. Proofs

Proof of Theorem 1. Let $c'_t = \sum_{i=1}^{k-1} a_i(ta_{i+1} - 1)$. By Lemma 4, there exist smallest integers c_t and d_t satisfying

$$[c'_t - a_k + 1, c'_t - 1] \subseteq [c_t, h_t a_k - d_t] \subseteq (h_t A)^{(t)}.$$

It follows that $c_t - 1 \notin (h_t A)^{(t)}$ and $h_t a_k - d_t + 1 \notin (h_t A)^{(t)}$. Additionally

$$c_t \leq c'_t - a_k + 1, \tag{5}$$

$$c'_t - 1 \leq h_t a_k - d_t. \tag{6}$$

Define the finite sets C_t and D_t by

$$C_t = (h_t A)^{(t)} \cap [0, c_t - 2]$$

and

$$h_t a_k - D_t = (h_t A)^{(t)} \cap [h_t a_k - (d_t - 2), h_t a_k].$$

Then

$$(h_t A)^{(t)} = C_t \cup [c_t, h_t a_k - d_t] \cup (h_t a_k - D_t). \tag{7}$$

Therefore, (1) holds for $h = h_t$.

Now we prove (1) by induction on h . Suppose that (1) holds for some $h \geq h_t$. Define

$$B^{(t)} = C_t \cup [c_t, (h + 1)a_k - d_t] \cup ((h + 1)a_k - D_t).$$

Firstly we prove that $B^{(t)} \subseteq ((h + 1)A)^{(t)}$.

Take an arbitrary integer $b \in B^{(t)}$.

Case 1: $b \in C_t \cup [c_t, h_t a_k - d_t]$. By (7), we have

$$b \in (h_t A)^{(t)} \subseteq ((h + 1)A)^{(t)}.$$

Case 2: $b \in [c_t + a_k, (h + 1)a_k - d_t] \cup ((h + 1)a_k - D_t)$. It follows that

$$b - a_k \in [c_t, h_t a_k - d_t] \cup (h_t a_k - D_t) \subseteq (h_t A)^{(t)}.$$

Thus, By Lemma 3, $b \in (h_t A)^{(t)} + a_k \subseteq ((h + 1)A)^{(t)}$.

Case 3: $h_t a_k - d_t + 1 \leq b \leq c_t + a_k - 1$. By (5) and (6), we have

$$c_t + a_k - 1 \leq c'_t \leq h_t a_k - d_t + 1. \tag{8}$$

Thus $b = c'_t$. By Lemma 5, we have

$$b = c'_t \in ((h_t + 1)A)^{(t)} \subseteq ((h + 1)A)^{(t)}.$$

Therefore, $B^{(t)} \subseteq ((h + 1)A)^{(t)}$.

Next we shall prove that $((h + 1)A)^{(t)} \subseteq B^{(t)}$. Take an arbitrary integer $a \in ((h + 1)A)^{(t)}$.

Case 1: $a = c'_t$. By (8) and $h \geq h_t$, we have

$$c_t \leq c'_t \leq h_t a_k - d_t + 1 \leq (h + 1) a_k - d_t.$$

Hence $a = c'_t \in B^{(t)}$.

Case 2: $a \neq c'_t$ and $a \notin (hA)^{(t)}$. Since $a \in ((h + 1)A)^{(t)}$, there exist t nonnegative integer k -tuples $(x_{1,s}, x_{2,s}, \dots, x_{k,s})$ ($1 \leq s \leq t$) satisfying

$$a = x_{1,s} a_1 + x_{2,s} a_2 + \dots + x_{k,s} a_k \quad \text{and} \quad x_{1,s} + x_{2,s} + \dots + x_{k,s} \leq h + 1.$$

Furthermore, we can get

$$0 \leq x_{i,s} \leq t a_{i+1} - 1, \quad 1 \leq i \leq k - 1, \quad 1 \leq s \leq t. \tag{9}$$

Otherwise, without loss of generality, assume that $x_{1,1} \geq t a_2$, then for $j = 1, 2, \dots, t$, we have

$$\begin{aligned} a &= x_{1,1} a_1 + x_{2,1} a_2 + \dots + x_{k,1} a_k \\ &= (x_{1,1} - j a_2) a_1 + (x_{2,1} + j a_1) a_2 + \dots + x_{k,1} a_k. \end{aligned}$$

Noting that for $j = 1, 2, \dots, t$,

$$(x_{1,1} - j a_2) + (x_{2,1} + j a_1) + x_{3,1} + \dots + x_{k,1} = h + 1 - j(a_2 - a_1) < h + 1,$$

we have $a \in (hA)^{(t)}$, a contradiction. Hence the inequality (9) holds.

By $a \notin (hA)^{(t)}$, there exists $s \in [1, t]$ such that $a = x_{1,s} a_1 + x_{2,s} a_2 + \dots + x_{k,s} a_k$ and

$$x_{1,s} + x_{2,s} + \dots + x_{k,s} = h + 1.$$

By (9), we have

$$\begin{aligned} a &= x_{1,s} a_1 + x_{2,s} a_2 + \dots + x_{k,s} a_k \\ &= x_{1,s} a_1 + \dots + x_{k-1,s} a_{k-1} + (h + 1 - x_{1,s} - x_{2,s} - \dots - x_{k-1,s}) a_k \\ &= (h + 1) a_k - x_{1,s} (a_k - a_1) - \dots - x_{k-1,s} (a_k - a_{k-1}) \\ &\geq (h + 1) a_k - (t a_2 - 1) (a_k - a_1) - \dots - (t a_k - 1) (a_k - a_{k-1}) \\ &= (h + 1) a_k - a_k [(t a_2 - 1) + \dots + (t a_k - 1)] + a_1 (t a_2 - 1) + \dots + a_{k-1} (t a_k - 1) \\ &= (h + 1) a_k - (h_t + 1) a_k + c'_t \\ &\geq c'_t. \end{aligned}$$

Since $a \neq c'_t$, it follows that $a \geq c'_t + 1$. By (5), we have $a \geq c'_t + 1 \geq c_t + a_k$.

If $x_{k,s} = 0$ for some s with $1 \leq s \leq t$, by (9), then

$$a \leq (t a_2 - 1) a_1 + \dots + (t a_k - 1) a_{k-1} = c'_t,$$

a contradiction.

Hence $x_{k,s} \geq 1$ for all integers $s = 1, 2, \dots, t$.

Therefore, $a - a_k \in (hA)^{(t)}$ and $a - a_k \geq c_t$. By the induction hypothesis,

$$a \in a_k + [c_t, h a_k - d_t] \cup (h a_k - D_t) = [c_t + a_k, (h + 1) a_k - d_t] \cup ((h + 1) a_k - D_t) \subseteq B^{(t)}.$$

Case 3: $a \neq c'_t$ and $a \in (hA)^{(t)}$. By the induction hypothesis, we have

$$(hA)^{(t)} = C_t \cup [c_t, h a_k - d_t] \cup (h a_k - D_t).$$

Since $C_t \cup [c_t, (h + 1) a_k - d_t] \subseteq B^{(t)}$, we can suppose that $a > (h + 1) a_k - d_t$. By $a \in (hA)^{(t)}$, there exist at least t distinct nonnegative k -tuples $(x_{1,s}, x_{2,s}, \dots, x_{k,s})$ ($1 \leq s \leq t$) such that

$$a = x_{1,s} a_1 + x_{2,s} a_2 + \dots + x_{k,s} a_k$$

and

$$x_{1,s} + x_{2,s} + \dots + x_{k,s} \leq h.$$

As in the proof of Lemma 4, assume that $0 \leq x_{i,s} \leq ta_{i+1} - 1$ for $i = 1, 2, \dots, k-1$. If $x_{k,s} \leq 0$, then by (6) we have

$$\begin{aligned} a &\leq x_{1,s}a_1 + x_{2,s}a_2 + \dots + x_{k-1,s}a_{k-1} \\ &\leq a_1(ta_2 - 1) + \dots + a_{k-1}(ta_k - 1) \\ &= c'_t \leq h_t a_k - d_t + 1 \\ &\leq (h_t + 1)a_k - d_t \leq (h + 1)a_k - d_t, \end{aligned}$$

which contradicts with $a > (h + 1)a_k - d_t$. Therefore $x_{k,s} \geq 1$ and $a - a_k \in (hA)^{(t)}$. Since $a > (h + 1)a_k - d_t$, it follows that $a - a_k \in ha_k - D_t$. Hence

$$a \in (h + 1)a_k - D_t \subseteq B^{(t)},$$

and so $((h + 1)A)^{(t)} \subseteq B^{(t)}$.

This completes the proof of Theorem 1. □

Proof of Remark 2. Let $n \geq 3$ be an integer and $A = \{0, n, n + 1\}$. By Theorem 1, there exist integers c_t, d_t and sets $C_t \subseteq [0, c_t - 2], D_t \subseteq [0, d_t - 2]$ such that

$$(hA)^{(t)} = C_t \cup [c_t, ha_k - d_t] \cup (ha_k - D_t)$$

for all $h \geq h_t = t(n + 1) - 2$.

For any integer $m \geq c_t$, choose an integer $h' \geq t(n + 1) - 2$ such that $h'a_k - d_t \geq m$, then we have $m \in (h'A)^{(t)}$.

Hence, there exist t nonnegative integer tuples (u_i, v_i) ($1 \leq i \leq t$) such that $m = u_i n + v_i(n + 1)$.

On the other hand, there does not exist t nonnegative integer tuples (u_i, v_i) ($1 \leq i \leq t$) such that $c_t - 1 = u_i n + v_i(n + 1)$. Otherwise, if exist, choose $h > \max_{1 \leq i \leq t} \{u_i + v_i\}$, then we have $c_t - 1 \in (hA)^{(t)}$, a contradiction. Hence, by Lemma 6, it follows that $c_t - 1 = ta_1 a_2 - a_1 - a_2 = tn(n + 1) - n - (n + 1)$, and then $c_t = tn(n + 1) - 2n$.

Let $p \in ((h_t - 1)A)^{(t)}$. Then there exist t nonnegative integer tuples (u_i, v_i) ($1 \leq i \leq t$) such that $p = u_i n + v_i(n + 1)$ and $u_1 > u_2 > \dots > u_t$ are the maximal t numbers in all the representations. Hence

$$\begin{aligned} p = u_1 n + v_1(n + 1) &= [u_1 - (n + 1)]n + (v_1 + n)(n + 1) \\ &= [u_1 - 2(n + 1)]n + (v_1 + 2n)(n + 1) \\ &= \dots \\ &= [u_1 - (t - 1)(n + 1)]n + [v_1 + (t - 1)n](n + 1). \end{aligned}$$

It follows that $u_t = u_1 - (t - 1)(n + 1), v_t = v_1 + (t - 1)n$. Noting that

$$u_t + v_t < u_{t-1} + v_{t-1} < \dots < u_1 + v_1 \leq h_t - 1,$$

we have

$$\begin{aligned} u_t + v_t &= u_1 - (t - 1)(n + 1) + v_1 + (t - 1)n \\ &= u_1 + v_1 - (t - 1) \leq h_t - 1 - (t - 1) = tn - 2. \end{aligned}$$

Hence, for every $p \in ((h_t - 1)A)^{(t)}$,

$$\begin{aligned} p = u_t n + v_t(n + 1) &\leq (u_t + v_t)(n + 1) \leq (tn - 2)(n + 1) \\ &= tn(n + 1) - 2(n + 1) < tn(n + 1) - 2n = c_t. \end{aligned}$$

By (1), it follows that

$$((h_t - 1)A)^{(t)} \subseteq [0, tn(n + 1) - 2(n + 1)].$$

Therefore, (1) cannot hold for $h = h_t - 1$, and so Theorem 1 is optimal. □

Acknowledgement

We would like to thank the referee for their useful comments.

References

- [1] A. Granville, G. Shakan, “The Frobenius postage stamp problem, and beyond”, <https://arxiv.org/abs/2003.04076>, 2020.
- [2] A. Granville, A. Walker, “A tight structure theorem for sumsets”, <https://arxiv.org/abs/2006.01041>, 2020.
- [3] M. B. Nathanson, “Sums of finite sets of integers”, *Am. Math. Mon.* **79** (1972), p. 1010-1012.
- [4] ———, *Additive Number Theory: Inverse Problems and the Geometry of Sumsets*, Graduate Texts in Mathematics, vol. 165, Springer, 1996.
- [5] ———, “Sums of finite sets of integers, II”, <https://arxiv.org/abs/2005.10809v3>, 2020.
- [6] J. Wu, F. Chen, Y. Chen, “On the structure of the sumsets”, *Discrete Math.* **311** (2011), no. 6, p. 408-412.