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Comptes Rendus
Mathématique

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Volume 359, issue 5 (2021), p. 555-562

<https://doi.org/10.5802/crmath.195>

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https://doi.org/10.5802/crmath.195

Numerical analysis / *Analyse numérique*

A new extension on the theorem of Bor

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Abstract. In [8], Bor has obtained a main theorem dealing with Riesz summability factors of infinite series and Fourier series. In this paper, we generalized that theorem to $|A,\theta_n|_k$ summability method for taking power increasing sequence. Also some new and known results are obtained dealing with some basic summability methods.

2020 Mathematics Subject Classification. 26D15, 42A24, 40F05, 40G99.

Manuscript received 10th October 2020, revised 26th November 2020, accepted 1st March 2021.

1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n). By u_n^α and t_n^α we denote the nth Cesàro means of order α, with $\alpha > -1$, of the sequence (s_n) and (na_n), respectively, that is (see [9])

$$u_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v \quad \text{and} \quad t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \tag{1}$$

where

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+n)}{n!} = O(n^\alpha), \qquad A_{-n}^\alpha = 0 \quad \text{for} \quad n > 0. \tag{2}$$

The series $\sum a_n$ is said to be summable $|C,\alpha|_k$, $k \geq 1$, if (see [11, 13])

$$\sum_{n=1}^{\infty} n^{k-1} |u_n^\alpha - u_{n-1}^\alpha|^k = \sum_{n=1}^{\infty} \frac{1}{n} |t_n^\alpha|^k < \infty. \tag{3}$$

If we take $\alpha = 1$, then $|C,\alpha|_k$ summability reduces to $|C,1|_k$ summability.

Let (p_n) be a sequence of positive real numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1). \tag{4}$$

The sequence-to-sequence transformation

$$w_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v, \quad P_n \neq 0. \tag{5}$$

defines the sequence (w_n) of the Riesz mean or simply the $(\overline{N}, p_n)$ mean of the sequence (s_n) generated by the sequence of coefficients (p_n) (see [12]).

Let (θ_n) be any sequence of positive constants. The series $\sum a_n$ is said to be summable $|\overline{N}, p_n; \theta_n|_k$, $k \geq 1$, if (see [18])

$$\sum_{n=1}^{\infty} \theta_n^{k-1} |w_n - w_{n-1}|^k < \infty. \tag{6}$$

In the special case if we take $\theta_n = P_n/p_n$, then $|\overline{N}, p_n; \theta_n|_k$ summability reduces to $|\overline{N}, p_n|_k$ summability (see [2]). When $\theta_n = n$ and $p_n = 1$ for all values of n, then we get $|C, 1|_k$ summability. Furthermore, if we take $\theta_n = n$, then $|\overline{N}, p_n; \theta_n|_k$ summability reduces to $|R, p_n|_k$ summability (see [3]).

For any sequence (λ_n) we write that

$$\Delta^2 \lambda_n = \Delta \lambda_n - \Delta \lambda_{n+1} \quad \text{and} \quad \Delta \lambda_n = \lambda_n - \lambda_{n+1}.$$

A sequence (λ_n) is said to be of bounded variation, denoted by $(\lambda_n) \in \mathscr{BV}$, if

$$\sum_{n=1}^{\infty} |\Delta \lambda_n| < \infty.$$

Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, \ldots \tag{7}$$

Let (θ_n) be any sequence of positive real numbers. The series $\sum a_n$ is said to be summable $|A, \theta_n|_k$, $k \geq 1$, if (see [16, 17])

$$\sum_{n=1}^{\infty} \theta_n^{k-1} |\overline{\Delta} A_n(s)|^k < \infty, \tag{8}$$

where

$$\overline{\Delta} A_n(s) = A_n(s) - A_{n-1}(s). \tag{9}$$

If we take $a_{nv} = \frac{p_v}{P_n}$, then $|A, \theta_n|_k$ summability reduces to $|\overline{N}, p_n; \theta_n|_k$ summability. If we take $\theta_n = \frac{P_n}{p_n}$, then $|A, \theta_n|_k$ summability reduces to $|A, p_n|_k$ summability (see [19]). And also if we take $\theta_n = \frac{P_n}{p_n}$ and $a_{nv} = \frac{p_v}{P_n}$, then $|A, \theta_n|_k$ summability reduces to $|\overline{N}, p_n|_k$ summability. Furthermore, if we take $\theta_n = n$, $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n, then $|A, \theta_n|_k$ summability reduces to $|C, 1|_k$ summability (see [11]). Finally, if we take $\theta_n = n$ and $a_{nv} = \frac{p_v}{P_n}$, then $|A, \theta_n|_k$ summability reduces to $|R, p_n|_k$ summability (see [3]).

Definition 1 (cf. [1]). *A positive sequence (b_n) is said to be an almost increasing sequence if there exists a positive increasing sequence (c_n) and two positive constants M and N such that $Mc_n \leq b_n \leq Nc_n$.*

Definition 2 (cf. [20]). *A positive sequence $X = (X_n)$ is said to be quasi-f-power increasing sequence if there exists a constant $K = K(X, f) \geq 1$ such that $Kf_n X_n \geq f_m X_m$ for all $n \geq m \geq 1$, where $f = \{f_n(\sigma, \beta)\} = \{n^{\sigma}(\log n)^{\beta}, \beta \geq 0, 0 < \sigma < 1\}$.*

If we take $\beta = 0$, then we have a quasi-σ-power increasing sequence (see [15]). Every almost increasing sequence is a quasi-σ-power increasing sequence for any non-negative σ, but the converse is not true for $\sigma > 0$.

2. The Known Results

Recently, many papers have been done for absolute matrix summability factors of infinite series and Fourier series (see [5–7, 14, 22, 23]). From these, in [14], Lee explained history of summability of infinite series and Hüseyin Bor briefly. Now we also used Bor's new theorem dealing with the Fourier series and we will extend following theorem.

Theorem 3 (cf. [8]). *Let $(\theta_n p_n/P_n)$ be a non-increasing sequence. Let (p_n) be a sequence of positive numbers such that*

$$P_n = O(np_n) \quad as \quad n \to \infty. \tag{10}$$

Let (X_n) be a positive increasing sequence. If the conditions

$$\lambda_n = o(1) \qquad as \quad n \to \infty, \tag{11}$$

$$\sum_{n=1}^{m} nX_n|\Delta^2\lambda_n| = O(1) \qquad as \quad m \to \infty, \tag{12}$$

$$\sum_{n=1}^{m} \theta_n^{k-1}\left(\frac{p_n}{P_n}\right)^k \frac{|t_n|^k}{X_n^{k-1}} = O(X_m) \qquad as \quad m \to \infty, \tag{13}$$

are satisfied, then the series $\sum a_n\lambda_n$ is summable $|\overline{N}, p_n, \theta_n|_k$, $k \geq 1$.

If we take $\theta_n = P_n/p_n$, then we get a theorem dealing with $|\overline{N}, p_n|_k$ summability (see [6]).

3. The Main Result

Given a normal matrix $A = (a_{nv})$, we associate two lower semimatrices $\overline{A} = (\overline{a}_{nv})$ and $\widehat{A} = (\widehat{a}_{nv})$ as follows:

$$\overline{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, \ldots \tag{14}$$

and

$$\widehat{a}_{00} = \overline{a}_{00} = a_{00}, \quad \widehat{a}_{nv} = \overline{a}_{nv} - \overline{a}_{n-1,v}, \quad n = 1, 2, \ldots \tag{15}$$

It may be noted that $\overline{A}$ and $\widehat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv}s_v = \sum_{v=0}^{n} \overline{a}_{nv}a_v \tag{16}$$

and

$$\overline{\Delta}A_n(s) = \sum_{v=0}^{n} \widehat{a}_{nv}a_v. \tag{17}$$

By using above notations, we generalize Theorem 3 for $|A, \theta_n|_k$ summability method by taking (X_n) as a quasi-f- power increasing sequence.

Theorem 4. *Let $k \geq 1$ and $A = (a_{nv})$ be a positive normal matrix such that*

$$\overline{a}_{n0} = 1, \qquad n = 0, 1, \ldots, \tag{18}$$

$$a_{n-1,v} \geq a_{nv}, \qquad \text{for } n \geq v + 1, \tag{19}$$

$$1 = O(na_{nn}), \tag{20}$$

$$\sum_{v=1}^{n-1} a_{vv}|\widehat{a}_{n,v+1}| = O(a_{nn}). \tag{21}$$

Let $(\theta_n a_{nn})$ be a non-increasing sequence and (X_n) be a quasi-f-power increasing sequence for some σ $(0<\sigma<1)$. If the conditions (11)–(12) *of Theorem 3 and (θ_n) holds for the following condition,*

$$\sum_{n=1}^{m} \theta_n^{k-1} a_{nn}^k \frac{|t_n|^k}{X_n^{k-1}} = O(X_m) \quad as \quad m\to\infty, \tag{22}$$

are satisfied, then the series $\sum a_n\lambda_n$ is summable $|A,\theta_n|_k$, $k\geq 1$.

We need the following lemmas for the proof of Theorem 4.

Lemma 5 (cf. [21]). *By using conditions* (14), (15), (18) *and* (19), *we have*

$$\sum_{v=1}^{n-1} |\Delta_v(\widehat{a}_{nv})| \leq a_{nn}, \tag{23}$$

$$\sum_{n=v+1}^{m+1} |\Delta_v(\widehat{a}_{nv})| \leq a_{vv}, \tag{24}$$

$$\sum_{n=v+1}^{m+1} |\widehat{a}_{n,v+1}| = O(1). \tag{25}$$

Lemma 6 (cf. [4]). *Under the conditions of Theorem 3 we have the following*

$$nX_n|\Delta\lambda_n| = O(1) \quad as \quad n\to\infty, \tag{26}$$

$$\sum_{n=1}^{\infty} X_n|\Delta\lambda_n| < \infty, \tag{27}$$

$$X_n|\lambda_n| = O(1) \quad as \quad n\to\infty. \tag{28}$$

Proof of Theorem 4. Let (I_n) denotes the A-transform of the series $\sum_{n=1}^{\infty} a_n\lambda_n$. Then, by (16) and (17), we have

$$\bar{\Delta} I_n = \sum_{v=1}^{n} \widehat{a}_{nv} a_v \lambda_v.$$

Applying Abel's transformation to this sum, we have that

$$\begin{aligned}
\bar{\Delta} I_n &= \sum_{v=1}^{n} \widehat{a}_{nv} a_v \lambda_v \frac{v}{v} = \sum_{v=1}^{n-1} \Delta\left(\frac{\widehat{a}_{nv}\lambda_v}{v}\right) \sum_{r=1}^{v} r a_r + \frac{\widehat{a}_{nn}\lambda_n}{n} \sum_{v=1}^{n} v a_v \\
&= \sum_{v=1}^{n-1} \Delta\left(\frac{\widehat{a}_{nv}\lambda_v}{v}\right)(v+1)t_v + \widehat{a}_{nn}\lambda_n \frac{n+1}{n} t_n \\
&= \sum_{v=1}^{n-1} \Delta_v(\widehat{a}_{nv})\lambda_v t_v \frac{v+1}{v} + \sum_{v=1}^{n-1} \widehat{a}_{n,v+1}\Delta\lambda_v t_v \frac{v+1}{v} + \sum_{v=1}^{n-1} \widehat{a}_{n,v+1}\lambda_{v+1}\frac{t_v}{v} + a_{nn}\lambda_n t_n \frac{n+1}{n} \\
&= I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4}.
\end{aligned}$$

To complete the proof of Theorem 4, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \theta_n^{k-1} |I_{n,r}|^k < \infty, \quad \text{for} \quad r=1,2,3,4. \tag{29}$$

First, by applying Hölder's inequality with indices k and k', where $k>1$ and $\frac{1}{k}+\frac{1}{k'}=1$, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\theta_n^{k-1}|I_{n,1}|^k &\leq \sum_{n=2}^{m+1}\theta_n^{k-1}\left\{\sum_{v=1}^{n-1}\left|\frac{v+1}{v}\right||\Delta_v(\widehat{a}_{nv})||\lambda_v||t_v|\right\}^k\\
&= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\left\{\sum_{v=1}^{n-1}|\Delta_v(\widehat{a}_{nv})||\lambda_v|^k|t_v|^k\right\}\times\left\{\sum_{v=1}^{n-1}|\Delta_v(\widehat{a}_{nv})|\right\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}(\theta_n a_{nn})^{k-1}\left\{\sum_{v=1}^{n-1}|\Delta_v(\widehat{a}_{nv})||\lambda_v|^k|t_v|^k\right\}\\
&= O(1)\sum_{v=1}^{m}|\lambda_v|^{k-1}|\lambda_v||t_v|^k\sum_{n=v+1}^{m+1}(\theta_n a_{nn})^{k-1}|\Delta_v(\widehat{a}_{nv})|\\
&= O(1)\sum_{v=1}^{m}(\theta_v a_{vv})^{k-1}\frac{1}{X_v^{k-1}}|\lambda_v||t_v|^k\sum_{n=v+1}^{m+1}|\Delta_v(\widehat{a}_{nv})|\\
&= O(1)\sum_{v=1}^{m}(\theta_v a_{vv})^{k-1}\frac{1}{X_v^{k-1}}|\lambda_v||t_v|^k a_{vv}\\
&= O(1)\sum_{v=1}^{m-1}\Delta|\lambda_v|\sum_{r=1}^{v}\theta_r^{k-1}a_{rr}^k\frac{|t_r|^k}{X_r^{k-1}}+O(1)|\lambda_m|\sum_{v=1}^{m}\theta_v^{k-1}a_{vv}^k\frac{|t_v|^k}{X_v^{k-1}}\\
&= O(1)\sum_{v=1}^{m-1}|\Delta\lambda_v|X_v+O(1)|\lambda_m|X_m\\
&= O(1)\quad\text{as}\quad m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 4, Lemma 5, and Lemma 6. Now using Hölder's inequality, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\theta_n^{k-1}|I_{n,2}|^k &\leq \sum_{n=2}^{m+1}\theta_n^{k-1}\left\{\sum_{v=1}^{n-1}|\frac{v+1}{v}||\widehat{a}_{n,v+1}||\Delta\lambda_v||t_v|\right\}^k\\
&= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\left\{\sum_{v=1}^{n-1}|\widehat{a}_{n,v+1}||\Delta\lambda_v||t_v|\right\}^k\\
&= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\left\{\sum_{v=1}^{n-1}a_{vv}^{1-k}|\widehat{a}_{n,v+1}||\Delta\lambda_v|^k|t_v|^k\right\}\times\left\{\sum_{v=1}^{n-1}a_{vv}|\widehat{a}_{n,v+1}|\right\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}(\theta_n a_{nn})^{k-1}\sum_{v=1}^{n-1}|\Delta\lambda_v|^k a_{vv}^{1-k}|\widehat{a}_{n,v+1}||t_v|^k\\
&= O(1)\sum_{v=1}^{m}|t_v|^k a_{vv}^{1-k}|\Delta\lambda_v|^k\sum_{n=v+1}^{m+1}(\theta_n a_{nn})^{k-1}|\widehat{a}_{n,v+1}|\\
&= O(1)\sum_{v=1}^{m}(\theta_v a_{vv})^{k-1}|t_v|^k a_{vv}^{1-k}|\Delta\lambda_v|^k\sum_{n=v+1}^{m+1}|\widehat{a}_{n,v+1}|\\
&= O(1)\sum_{v=1}^{m}(\theta_v a_{vv})^{k-1}|t_v|^k a_{vv}^{1-k}|\Delta\lambda_v|^k\\
&= O(1)\sum_{v=1}^{m}\theta_v^{k-1}a_{vv}^k|t_v|^k(v|\Delta\lambda_v|)^{k-1}(v|\Delta\lambda_v|)\\
&= O(1)\sum_{v=1}^{m}\theta_v^{k-1}a_{vv}^k\frac{1}{X_v^{k-1}}|t_v|^k(v|\Delta\lambda_v|)
\end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m-1}\Delta(v|\Delta\lambda_v|)\sum_{r=1}^{v}\theta_r^{k-1}a_{rr}^k\frac{1}{X_r^{k-1}}|t_r|^k + O(1)m|\Delta\lambda_m|\sum_{v=1}^{m}\theta_v^{k-1}a_{vv}^k\frac{1}{X_v^{k-1}}|t_v|^k\\
&= O(1)\sum_{v=1}^{m-1}|\Delta(v|\Delta\lambda_v|)|X_v + O(1)m|\Delta\lambda_m|X_m\\
&= O(1)\sum_{v=1}^{m-1}vX_v|\Delta^2\lambda_v| + O(1)\sum_{v=1}^{m-1}X_v|\Delta\lambda_v| + O(1)m|\Delta\lambda_m|X_m\\
&= O(1)\quad \text{as}\quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 4, Lemma 5, and Lemma 6. Again, as in $I_{n,1}$, we have that

$$
\begin{aligned}
\sum_{n=2}^{m+1}\theta_n^{k-1}|I_{n,3}|^k &= \sum_{n=2}^{m+1}\theta_n^{k-1}\left|\sum_{v=1}^{n-1}\widehat{a}_{n,v+1}\lambda_{v+1}\frac{t_v}{v}\right|^k\\
&\le \sum_{n=2}^{m+1}\theta_n^{k-1}\left\{\sum_{v=1}^{n-1}|\widehat{a}_{n,v+1}||\lambda_{v+1}|\frac{|t_v|}{v}\right\}^k\\
&= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\left\{\sum_{v=1}^{n-1}a_{vv}^{1-k}\frac{1}{v^k}|\widehat{a}_{n,v+1}||\lambda_{v+1}|^k|t_v|^k\right\}\times\left\{\sum_{v=1}^{n-1}a_{vv}|\widehat{a}_{n,v+1}|\right\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}(\theta_n a_{nn})^{k-1}\sum_{v=1}^{n-1}a_{vv}^{1-k}\frac{1}{v^k}|\widehat{a}_{n,v+1}||\lambda_{v+1}|^k|t_v|^k\\
&= O(1)\sum_{v=1}^{m}a_{vv}|\lambda_{v+1}|^k|t_v|^k\sum_{n=v+1}^{m+1}(\theta_n a_{nn})^{k-1}|\widehat{a}_{n,v+1}|\\
&= O(1)\sum_{v=1}^{m}(\theta_v a_{vv})^{k-1}a_{vv}|\lambda_{v+1}|^k|t_v|^k\sum_{n=v+1}^{m+1}|\widehat{a}_{n,v+1}|\\
&= O(1)\sum_{v=1}^{m}\theta_v^{k-1}a_{vv}^k|\lambda_{v+1}|^k|t_v|^k\\
&= O(1)\sum_{v=1}^{m}\theta_v^{k-1}a_{vv}^k|t_v|^k|\lambda_{v+1}|^{k-1}|\lambda_{v+1}|\\
&= O(1)\sum_{v=1}^{m}\theta_v^{k-1}a_{vv}^k\frac{1}{X_v^{k-1}}|\lambda_{v+1}||t_v|^k\\
&= O(1)\quad \text{as}\quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 4, Lemma 5, and Lemma 6. Finally, as in $I_{n,1}$, we have that

$$
\begin{aligned}
\sum_{n=1}^{m}\theta_n^{k-1}|I_{n,4}|^k = O(1)\sum_{n=1}^{m}\theta_n^{k-1}a_{nn}^k|\lambda_n|^k|t_n|^k &= O(1)\sum_{n=1}^{m}\theta_n^{k-1}a_{nn}^k|\lambda_n|^{k-1}|\lambda_n||t_n|^k\\
&= O(1)\sum_{n=1}^{m}\theta_n^{k-1}a_{nn}^k\frac{1}{X_n^{k-1}}|\lambda_n||t_n|^k = O(1)\quad \text{as}\quad m\to\infty,
\end{aligned}
$$

by virtue of hypotheses of Theorem 4, Lemma 5, and Lemma 6. This completes the proof of Theorem 4. □

4. An application of absolute matrix summability to Fourier series

Let f be a periodic function with period 2π and integrable (L) over $(-\pi,\pi)$. The trigonometric Fourier series of f is defined as

$$
f(x)\sim\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=\sum_{n=0}^{\infty}C_n(x). \tag{30}
$$

Set

$$\phi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\}, \tag{31}$$

$$\phi_\alpha(t) = \frac{\alpha}{t^\alpha}\int_0^t (t-u)^{\alpha-1}\phi(u)\,\mathrm{d}u, \quad (\alpha > 0). \tag{32}$$

It is well known that if $\phi_1(t) \in \mathscr{BV}(0,\pi)$, then $t_n(x) = O(1)$, where $t_n(x)$ is the $(C,1)$ mean of the sequence $(nC_n(x))$ (see [10]).

The following theorem is known dealing with $|\overline{N}, p_n, \theta_n|_k$ summability factors of Fourier series.

Theorem 7 (cf. [8]). *Let $\left(\frac{\theta_n p_n}{P_n}\right)$ be a non-increasing sequence. If $\phi_1(t) \in \mathscr{BV}(0,\pi)$ and the sequences (p_n), (λ_n), and (X_n) satisfy the conditions of Theorem 3,then the series $\sum C_n(x)\lambda_n$ is summable $|\overline{N}, p_n, \theta_n|_k$, $k \geq 1$.*

Now, we generalize Theorem 7 for $|A, \theta_n|_k$ summability method in the following form.

Theorem 8. *Let $(\theta_n a_{nn})$ be a non-increasing sequence, and A be a positive normal matrix as in Theorem 4, and (X_n) be a quasi-f-power increasing sequence for some σ $(0 < \sigma < 1)$. If $\phi_1(t) \in \mathscr{BV}(0,\pi)$, and the sequences (p_n), (λ_n), and (X_n) satisfy the conditions of Theorem 4, then the series $\sum C_n(x)\lambda_n$ is summable $|A, \theta_n|_k$, $k \geq 1$.*

It should be noted that if we take (X_n) as a positive increasing sequence and $a_{nv} = \frac{p_v}{P_n}$ in Theorem 8, then we have Theorem 7.

Applications.

(1) If we write $\sum_{v=0}^{n} p_v/P_v$, then (X_n) is a positive increasing sequence tending to infinity as $n \to \infty$. In this case, if we take (X_n) is a positive increasing sequence and $a_{nv} = \frac{p_v}{P_n}$ in Theorem 4, then we have Theorem 3.

(2) If we take $\theta_n = n$, $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n in Theorem 4, then we have a new result concerning $|C,1|_k$ summability method.

(3) If we take $\theta_n = n$ and $a_{nv} = \frac{p_v}{P_n}$ in Theorem 4, then we get a new result dealing with $|R, p_n|_k$ summability method.

(4) If we take $\beta = 0$ and $a_{nv} = \frac{p_v}{P_n}$ in Theorem 4, then we have new theorem dealing with quasi-σ-power increasing sequence.

(5) If we take $\beta = 0$ and $a_{nv} = \frac{p_v}{P_n}$ in Theorem 8, then we have new theorem dealing with quasi-σ-power increasing sequence and Fourier series.

Acknowledgments

The author wishes her sincerest thanks to the referee for invaluable suggestions for the improvement of this paper.

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