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A new extension on the theorem of Bor

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Abstract. In [8], Bor has obtained a main theorem dealing with Riesz summability factors of infinite series and Fourier series. In this paper, we generalized that theorem to $|A, \theta_n|_k$ summability method for taking power increasing sequence. Also some new and known results are obtained dealing with some basic summability methods.

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1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums $(s_n)$. By $u_n^\alpha$ and $t_n^\alpha$ we denote the nth Cesàro means of order $\alpha$, with $\alpha > -1$, of the sequence $(s_n)$ and $(na_n)$, respectively, that is (see [9])

$$
\begin{align*}
  u_n^\alpha &= \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v \\
  t_n^\alpha &= \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v+1}^{\alpha-1} v a_v,
\end{align*}
$$

where

$$
A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\ldots(\alpha+n)}{n!} = O(n^\alpha), \quad A_{-n}^\alpha = 0 \quad \text{for} \quad n > 0.
$$

The series $\sum a_n$ is said to be summable $|C, \alpha|_k$, $k \geq 1$, if (see [11, 13])

$$
\sum_{n=1}^{\infty} n^{k-1} |u_n^\alpha - u_{n-1}^\alpha|^k \leq \sum_{n=1}^{\infty} \frac{1}{n} |t_n^\alpha|^k < \infty.
$$

If we take $\alpha = 1$, then $|C, \alpha|_k$ summability reduces to $|C, 1|_k$ summability.

Let $(p_n)$ be a sequence of positive real numbers such that

$$
P_n = \sum_{\nu=0}^{n} p_\nu \to \infty \quad \text{as} \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1).
$$

The sequence-to-sequence transformation

$$
w_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu s_\nu, \quad P_n \neq 0.
$$

defines the sequence $(w_n)$ of the Riesz mean or simply the $(\overline{N}, p_n)$ mean of the sequence $(s_n)$ generated by the sequence of coefficients $(p_n)$ (see [12]).
Let \( (\theta_n) \) be any sequence of positive constants. The series \( \sum a_n \) is said to be summable \(|\mathbb{N}, p_n; \theta_n|_k, k \geq 1\), if (see [18])

\[
\sum_{n=1}^{\infty} |\theta_n^{k-1}| w_n - w_{n-1}|^k < \infty. \tag{6}
\]

In the special case if we take \( \theta_n = p_n! p_n \), then \(|\mathbb{N}, p_n; \theta_n|_k \) summability reduces to \(|\mathbb{N}, p_n|_k \) summability (see [2]). When \( \theta_n = n \) and \( p_n = 1 \) for all values of \( n \), then we get \(|C, 1|_k \) summability. Furthermore, if we take \( \theta_n = n \), then \(|\mathbb{N}, p_n; \theta_n|_k \) summability reduces to \(|R, p_n|_k \) summability (see [3]).

For any sequence \((\lambda_n)\) we write that

\[
\Delta^2 \lambda_n = \Delta \lambda_n - \Delta \lambda_{n+1} \quad \text{and} \quad \Delta \lambda_n = \lambda_n - \lambda_{n+1}.
\]

A sequence \((\lambda_n)\) is said to be of bounded variation, denoted by \((\lambda_n) \in \mathcal{BV}\), if

\[
\sum_{n=1}^{\infty} |\Delta \lambda_n| < \infty.
\]

Let \( A = (a_{nv}) \) be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then \( A \) defines the sequence-to-sequence transformation, mapping the sequence \( s = (s_n) \) to \( A s = (A_n(s)) \), where

\[
A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, \ldots \tag{7}
\]

Let \( (\theta_n) \) be any sequence of positive real numbers. The series \( \sum a_n \) is said to be summable \(|A, \theta_n|_k, k \geq 1\), if (see [16, 17])

\[
\sum_{n=1}^{\infty} |\theta_n^{k-1}| \Delta A_n(s)|^k < \infty, \tag{8}
\]

where

\[
\Delta A_n(s) = A_n(s) - A_{n-1}(s). \tag{9}
\]

If we take \( a_{nv} = \frac{p_n!}{p_n} \), then \(|A, \theta_n|_k \) summability reduces to \(|\mathbb{N}, p_n; \theta_n|_k \) summability. If we take \( \theta_n = \frac{p_n!}{p_n} \), then \(|A, \theta_n|_k \) summability reduces to \(|A, p_n|_k \) summability (see [19]). And also if we take \( \theta_n = \frac{p_n!}{p_n} \) and \( a_{nv} = \frac{p_n!}{p_n} \), then \(|A, \theta_n|_k \) summability reduces to \(|\mathbb{N}, p_n|_k \) summability. Furthermore, if we take \( \theta_n = n \), \( a_{nv} = \frac{p_n!}{p_n} \) and \( p_n = 1 \) for all values of \( n \), then \(|A, \theta_n|_k \) summability reduces to \(|C, 1|_k \) summability (see [11]). Finally, if we take \( \theta_n = n \) and \( a_{nv} = \frac{p_n!}{p_n} \), then \(|A, \theta_n|_k \) summability reduces to \(|R, p_n|_k \) summability (see [3]).

**Definition 1 (cf. [1]).** A positive sequence \((b_n)\) is said to be an almost increasing sequence if there exists a positive increasing sequence \((c_n)\) and two positive constants \(M\) and \(N\) such that \(Mc_n \leq b_n \leq Nc_n\).

**Definition 2 (cf. [20]).** A positive sequence \(X = (X_n)\) is said to be quasi-\(f\)-power increasing sequence if there exists a constant \(K = K(X, f) \geq 1\) such that \(K f_n X_n \geq f_m X_m\) for all \( n \geq m \geq 1\), where \(f = |f_n(\sigma, \beta)| = \langle n^\sigma \log n \rangle^\beta, \beta \geq 0, 0 < \sigma < 1\).

If we take \( \beta = 0\), then we have a quasi-\(\sigma\)-power increasing sequence (see [15]). Every almost increasing sequence is a quasi-\(\sigma\)-power increasing sequence for any non-negative \(\sigma\), but the converse is not true for \(\sigma > 0\).
2. The Known Results

Recently, many papers have been done for absolute matrix summability factors of infinite series and Fourier series (see [5–7, 14, 22, 23]). From these, in [14], Lee explained history of summability of infinite series and Hüseyin Bor briefly. Now we also used Bor’s new theorem dealing with the Fourier series and we will extend following theorem.

**Theorem 3 (cf. [8]).** Let \( (\theta_n p_n, P_n) \) be a non-increasing sequence. Let \( (p_n) \) be a sequence of positive numbers such that

\[
P_n = O(np_n) \quad \text{as} \quad n \to \infty.
\]

Let \( (X_n) \) be a positive increasing sequence. If the conditions

\[
\lambda_n = o(1) \quad \text{as} \quad n \to \infty,
\]

\[
\sum_{n=1}^{m} nX_n |\Delta^2 \lambda_n| = O(1) \quad \text{as} \quad m \to \infty,
\]

\[
\sum_{n=1}^{m} \theta_n^{k-1} \left( \frac{p_n}{P_n^k} \right)^{k} \frac{|t_n|^k}{X_n^{k-1}} = O(X_m) \quad \text{as} \quad m \to \infty,
\]

are satisfied, then the series \( \sum a_n \lambda_n \) is summable \( \|N, p_n, \theta_n\|_k \), \( k \geq 1 \).

If we take \( \theta_n = P_n/p_n \), then we get a theorem dealing with \( \|N, p_n\|_k \) summability (see [6]).

3. The Main Result

Given a normal matrix \( A = (a_{nv}) \), we associate two lower semimatrices \( \bar{A} = (\bar{a}_{nv}) \) and \( \hat{A} = (\hat{a}_{nv}) \) as follows:

\[
\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, \ldots
\]

and

\[
\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, \ldots
\]

It may be noted that \( \bar{A} \) and \( \hat{A} \) are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

\[
\bar{A}_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \bar{a}_{nv} a_v
\]

and

\[
\bar{A}_n(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v.
\]

By using above notations, we generalize Theorem 3 for \( |A, \theta_n|_k \) summability method by taking \( (X_n) \) as a quasi-\( f \)-power increasing sequence.

**Theorem 4.** Let \( k \geq 1 \) and \( A = (a_{nv}) \) be a positive normal matrix such that

\[
\bar{a}_{n0} = 1, \quad n = 0, 1, \ldots
\]

\[
a_{n-1,v} \geq a_{nv}, \quad \text{for} \quad n \geq v + 1,
\]

\[
1 = O(na_{nn}),
\]

\[
\sum_{v=1}^{n-1} a_{vv} |\bar{a}_{n,v+1}| = O(a_{nn}).
\]
Let \((\theta_n a_{nn})\) be a non-increasing sequence and \((X_n)\) be a quasi-\(f\)-power increasing sequence for some \(\sigma (0 < \sigma < 1)\). If the conditions (11)–(12) of Theorem 3 and \((\theta_n)\) holds for the following condition,

\[
\sum_{n=1}^{m} \theta_n^{k-1} a_{nn}^{k} t_n^{k} X_n^{k-1} = O(X_m) \quad \text{as} \quad m \to \infty,
\]

are satisfied, then the series \(\sum a_n \lambda_n\) is summable \(|A, \theta_n|_k, k \geq 1\).

We need the following lemmas for the proof of Theorem 4.

**Lemma 5 (cf. [21]).** By using conditions (14), (15), (18) and (19), we have

\[
\sum_{v=1}^{n-1} |\Delta_v(\tilde{a}_{nv})| \leq a_{nn},
\]

\[
\sum_{n=v+1}^{m+1} |\Delta_v(\tilde{a}_{nv})| \leq a_{vv},
\]

\[
\sum_{n=v+1}^{m+1} |\tilde{a}_{n,v+1}| = O(1).
\]

**Lemma 6 (cf. [4]).** Under the conditions of Theorem 3 we have the following

\[
nX_n|\Delta_{\lambda_n}| = O(1) \quad \text{as} \quad n \to \infty,
\]

\[
\sum_{n=1}^{\infty} nX_n|\Delta_{\lambda_n}| < \infty,
\]

\[
X_n|\lambda_n| = O(1) \quad \text{as} \quad n \to \infty.
\]

**Proof of Theorem 4.** Let \((I_n)\) denotes the A-transform of the series \(\sum_{n=1}^{\infty} a_n \lambda_n\). Then, by (16) and (17), we have

\[
\tilde{\Delta} I_n = \sum_{v=1}^{n} \tilde{a}_{nv} a_v \lambda_v.
\]

Applying Abel’s transformation to this sum, we have that

\[
\tilde{\Delta} I_n = \sum_{v=1}^{n} \tilde{a}_{nv} a_v \lambda_v \frac{v}{v} = \sum_{v=1}^{n-1} \Delta \left( \frac{\tilde{a}_{nv} \lambda_v}{v} \right) \sum_{r=1}^{v} r \tilde{a}_r + \frac{\tilde{a}_{nn} \lambda_n}{n} \sum_{v=1}^{n} v a_v
\]

\[
= \sum_{v=1}^{n-1} \Delta \left( \frac{\tilde{a}_{nv} \lambda_v}{v} \right) (v+1) t_v + \frac{\tilde{a}_{nn} \lambda_n}{n} t_n
\]

\[
= \sum_{v=1}^{n-1} \Delta_v(\tilde{a}_{nv}) \lambda_v t_v \frac{v+1}{v} + \sum_{v=1}^{n-1} \tilde{a}_{n,v+1} \Delta \lambda_v t_v \frac{v+1}{v} + \sum_{v=1}^{n-1} \tilde{a}_{n,v+1} \lambda_v t_v \frac{n+1}{v} + \tilde{a}_{n} \lambda_n t_n \frac{n+1}{n}
\]

\[
= I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4}.
\]

To complete the proof of Theorem 4, by Minkowski’s inequality, it is sufficient to show that

\[
\sum_{n=1}^{\infty} \theta_n^{k-1} |I_{n,r}|^k < \infty, \quad \text{for} \quad r = 1, 2, 3, 4.
\]
First, by applying Hölder’s inequality with indices \(k\) and \(k'\), where \(k > 1\) and \(\frac{1}{k} + \frac{1}{k'} = 1\), we have that

\[
\sum_{n=2}^{m+1} \theta_{n}^{k-1} |I_{n,1}|^{k} \leq \sum_{n=2}^{m+1} \theta_{n}^{k-1} \left\{ \sum_{v=1}^{n-1} \frac{\nu + 1}{\nu} |\Delta_{v}(\tilde{A}_{nv})||\lambda_v||t_v| \right\}^{k}
= O(1) \sum_{n=2}^{m+1} \theta_{n}^{k-1} \left\{ \sum_{v=1}^{n-1} |\Delta_{v}(\tilde{A}_{nv})||\lambda_v||t_v| \right\}^{k-1}
= O(1) \sum_{n=2}^{m+1} (\theta_{n} a_{nn})^{k-1} \left\{ \sum_{v=1}^{n-1} |\Delta_{v}(\tilde{A}_{nv})||\lambda_v||t_v| \right\}^{k-1}
= O(1) \sum_{v=1}^{m} |\lambda_v|^{k-1} |\lambda_v||t_v|^{k} \sum_{n=v+1}^{m+1} (\theta_{n} a_{nn})^{k-1} |\Delta_{v}(\tilde{A}_{nv})|
= O(1) \sum_{v=1}^{m} \left( \theta_{v} a_{vv} \right)^{k-1} \frac{1}{X_{v}^{k-1}} |\lambda_v||t_v|^{k} \sum_{n=v+1}^{m+1} |\Delta_{v}(\tilde{A}_{nv})|
= O(1) \sum_{v=1}^{m} \left( \theta_{v} a_{vv} \right)^{k-1} \frac{1}{X_{v}^{k-1}} |\lambda_v||t_v|^{k} a_{vv}
= O(1) \sum_{v=1}^{m+1} \Delta |\lambda_v| \sum_{r=1}^{v} \theta_{v}^{k-1} a_{rr}^{1-k} \frac{|t_v|^{k}}{X_{r}^{k-1}} + O(1) |\lambda_m| \sum_{v=1}^{m} \theta_{v}^{k-1} a_{vv}^{1-k} \frac{|t_v|^{k}}{X_{v}^{k-1}}
= O(1) \sum_{v=1}^{m+1} \Delta |\lambda_v| X_v + O(1) |\lambda_m| X_m
= O(1) \quad \text{as} \quad m \to \infty,
\]

by virtue of the hypotheses of Theorem 4, Lemma 5, and Lemma 6. Now using Hölder’s inequality, we have that

\[
\sum_{n=2}^{m+1} \theta_{n}^{k-1} |I_{n,2}|^{k} \leq \sum_{n=2}^{m+1} \theta_{n}^{k-1} \left\{ \sum_{v=1}^{n-1} \frac{\nu + 1}{\nu} |\tilde{a}_{n,v+1}||\Delta\lambda_v||t_v| \right\}^{k}
= O(1) \sum_{n=2}^{m+1} \theta_{n}^{k-1} \left\{ \sum_{v=1}^{n-1} |\tilde{a}_{n,v+1}||\Delta\lambda_v||t_v| \right\}^{k-1}
= O(1) \sum_{n=2}^{m+1} (\theta_{n} a_{nn})^{k-1} \left\{ \sum_{v=1}^{n-1} a_{vv}^{1-k} \tilde{a}_{n,v+1}||\Delta\lambda_v||t_v|^{k} \right\}^{k-1}
= O(1) \sum_{v=1}^{m+1} (\theta_{v} a_{vv})^{k-1} \left\{ \sum_{r=1}^{v} a_{rr}^{1-k} \tilde{a}_{n,v+1}||\Delta\lambda_v||t_v|^{k} \right\}^{k-1}
= O(1) \sum_{v=1}^{m} |\lambda_v|^{k-1} |\lambda_v||t_v|^{k} \sum_{n=v+1}^{m+1} (\theta_{n} a_{nn})^{k-1} \tilde{a}_{n,v+1}||\Delta\lambda_v|
= O(1) \sum_{v=1}^{m} \left( \theta_{v} a_{vv} \right)^{k-1} |t_v|^{k} a_{vv}^{1-k} |\Delta\lambda_v|^{k} \sum_{n=v+1}^{m+1} \tilde{a}_{n,v+1}||\Delta\lambda_v|
= O(1) \sum_{v=1}^{m} \left( \theta_{v} a_{vv} \right)^{k-1} |t_v|^{k} a_{vv}^{1-k} |\Delta\lambda_v|^{k} \sum_{n=v+1}^{m+1} \tilde{a}_{n,v+1}||\Delta\lambda_v|
= O(1) \sum_{v=1}^{m} \left( \theta_{v} a_{vv} \right)^{k-1} |t_v|^{k} a_{vv}^{1-k} \frac{1}{X_{v}^{k-1}} |t_v|^{k} (|\Delta\lambda_v|)
= O(1) \sum_{v=1}^{m} \left( \theta_{v} a_{vv} \right)^{k-1} a_{vv}^{1-k} \frac{1}{X_{v}^{k-1}} |t_v|^{k} (|\Delta\lambda_v|)
\]

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by virtue of the hypotheses of Theorem 4, Lemma 5, and Lemma 6. Again, as in \( I_{n,1} \), we have that

\[
\sum_{n=2}^{m+1} \theta_n^{k-1} |I_{n,3}|^k = \sum_{n=2}^{m+1} \theta_n^{k-1} \left| \sum_{v=1}^{n-1} \tilde{a}_{n,v+1} \lambda_{v+1} \frac{t_v}{v} \right|^k
\leq \sum_{n=2}^{m+1} \theta_n^{k-1} \left\{ \sum_{v=1}^{n-1} \frac{a_{v,n+1}^{k-1} |\lambda_{v+1}|^k |t_v|^k}{v} \right\}
\]

\[
= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left\{ \sum_{v=1}^{n-1} \frac{a_{v,n+1}^{k-1} |\lambda_{v+1}|^k |t_v|^k}{v} \right\}
\]

\[
= O(1) \sum_{n=2}^{m+1} \frac{\theta_n^{k-1} |\lambda_{v+1}|^k |t_v|^k}{\sum_{v=1}^{m+1} \theta_n^{k-1} |\lambda_{v+1}|^k |t_v|^k}
\]

\[
= O(1) \sum_{v=1}^{m+1} \frac{\theta_v^{k-1} |\lambda_{v+1}|^k |t_v|^k}{\sum_{v=1}^{m+1} \theta_v^{k-1} |\lambda_{v+1}|^k |t_v|^k}
\]

\[
= O(1) \sum_{v=1}^{m+1} \frac{\theta_v^{k-1} |\lambda_{v+1}|^k |t_v|^k}{\sum_{v=1}^{m+1} \theta_v^{k-1} |\lambda_{v+1}|^k |t_v|^k}
\]

\[
= O(1) \sum_{v=1}^{m+1} \frac{\theta_v^{k-1} |\lambda_{v+1}|^k |t_v|^k}{\sum_{v=1}^{m+1} \theta_v^{k-1} |\lambda_{v+1}|^k |t_v|^k}
\]

\[
= O(1) \sum_{v=1}^{m+1} \frac{\theta_v^{k-1} |\lambda_{v+1}|^k |t_v|^k}{\sum_{v=1}^{m+1} \theta_v^{k-1} |\lambda_{v+1}|^k |t_v|^k}
\]

\[
= O(1) \sum_{v=1}^{m+1} \frac{\theta_v^{k-1} |\lambda_{v+1}|^k |t_v|^k}{\sum_{v=1}^{m+1} \theta_v^{k-1} |\lambda_{v+1}|^k |t_v|^k}
\]

by virtue of the hypotheses of Theorem 4, Lemma 5, and Lemma 6. Finally, as in \( I_{n,1} \), we have that

\[
\sum_{n=1}^{m} \theta_n^{k-1} |I_{n,4}|^k = O(1) \sum_{n=1}^{m} \theta_n^{k-1} \left| a_n^k \lambda_n |t_n| \right|^k = O(1) \sum_{n=1}^{m} \theta_n^{k-1} \left| a_n^k \lambda_n |t_n| \right|^k
\]

\[
= O(1) \sum_{n=1}^{m} \theta_n^{k-1} \left| a_n^k \lambda_n |t_n| \right|^k = O(1) \sum_{n=1}^{m} \theta_n^{k-1} \left| a_n^k \lambda_n |t_n| \right|^k
\]

\[
= O(1) \sum_{n=1}^{m} \theta_n^{k-1} \left| a_n^k \lambda_n |t_n| \right|^k = O(1) \sum_{n=1}^{m} \theta_n^{k-1} \left| a_n^k \lambda_n |t_n| \right|^k
\]

\[
= O(1) \sum_{n=1}^{m} \theta_n^{k-1} \left| a_n^k \lambda_n |t_n| \right|^k = O(1) \sum_{n=1}^{m} \theta_n^{k-1} \left| a_n^k \lambda_n |t_n| \right|^k
\]

by virtue of hypotheses of Theorem 4, Lemma 5, and Lemma 6. This completes the proof of

Theorem 4.

\[\square\]

4. Application of absolute matrix summability to Fourier series

Let \( f \) be a periodic function with period \( 2\pi \) and integrable \((L)\) over \((-\pi, \pi)\). The trigonometric Fourier series of \( f \) is defined as

\[
f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} c_n(x).
\] (30)
Set
$$
\phi(t) = \frac{1}{2} \{ f(x + t) + f(x - t) \},
$$
$$
\phi_\alpha(t) = \frac{\alpha}{t^\alpha} \int_0^t (t - u)^{\alpha - 1} \phi(u) \, du, \quad (\alpha > 0).
$$

It is well known that if $\phi_1(t) \in BV(0, \pi)$, then $t_n(x) = O(1)$, where $t_n(x)$ is the $(C, 1)$ mean of the sequence $(nC_n(x))$ (see [10]).

The following theorem is known dealing with $|N, p_n, \theta_n|_k$ summability factors of Fourier series.

**Theorem 7** (cf. [8]). Let $\left( \frac{\theta_n p_n}{P_n} \right)$ be a non-increasing sequence. If $\phi_1(t) \in BV(0, \pi)$ and the sequences $(p_n)$, $(\lambda_n)$, and $(X_n)$ satisfy the conditions of Theorem 3, then the series $\sum C_n(x) \lambda_n$ is summable $|N, p_n, \theta_n|_k$, $k \geq 1$.

Now, we generalize Theorem 7 for $|A, \theta_n|_k$ summability method in the following form.

**Theorem 8.** Let $(\theta_n a_{n\nu})$ be a non-increasing sequence, and $A$ be a positive normal matrix as in Theorem 4, and $(X_n)$ be a quasi-$f$-power increasing sequence for some $\sigma$ ($0 < \sigma < 1$). If $\phi_1(t) \in BV(0, \pi)$, and the sequences $(p_n)$, $(\lambda_n)$, and $(X_n)$ satisfy the conditions of Theorem 4, then the series $\sum C_n(x) \lambda_n$ is summable $|A, \theta_n|_k$, $k \geq 1$.

It should be noted that if we take $(X_n)$ as a positive increasing sequence and $a_{n\nu} = \frac{p_v}{P_n}$ in Theorem 8, then we have Theorem 7.

**Applications.**

1. If we write $\sum_{n=0}^N p_v / P_n$, then $(X_n)$ is a positive increasing sequence tending to infinity as $n \rightarrow \infty$. In this case, if we take $(X_n)$ is a positive increasing sequence and $a_{n\nu} = \frac{p_v}{P_n}$ in Theorem 4, then we have Theorem 3.

2. If we take $\theta_n = n, a_{n\nu} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of $n$ in Theorem 4, then we have a new result concerning $|C, \Gamma|_k$ summability method.

3. If we take $\theta_n = n$ and $a_{n\nu} = \frac{p_v}{P_n}$ in Theorem 4, then we get a new result dealing with $|R, p_n|_k$ summability method.

4. If we take $\beta = 0$ and $a_{n\nu} = \frac{p_v}{P_n}$ in Theorem 4, then we have new theorem dealing with quasi-$\sigma$-power increasing sequence.

5. If we take $\beta = 0$ and $a_{n\nu} = \frac{p_v}{P_n}$ in Theorem 8, then we have new theorem dealing with quasi-$\sigma$-power increasing sequence and Fourier series.

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