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Control theory / *Théorie du contrôle*

Controllability to trajectories of a Ladyzhenskaya model for a viscous incompressible fluid

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Abstract. We consider the controllability of a viscous incompressible fluid modeled by the Navier–Stokes system with a nonlinear viscosity. To prove the controllability to trajectories, we linearize around a trajectory and the corresponding linear system includes a nonlocal spatial term. Our main result is a Carleman estimate for the adjoint of this linear system. This estimate yields in a standard way the null controllability of the linear system and the local controllability to trajectories. Our method to obtain the Carleman estimate is completely general and can be adapted to other parabolic systems when a Carleman estimate is available.

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1. Introduction

The aim of this article is to consider the controllability to trajectories of a model for the motion of a viscous incompressible fluid. This model was considered and studied by Ladyzhenskaya in [11]. The null-controllability of this system is obtained in [7], and the controllability to stationary trajectories is proved in [14] in the one-dimensional case (that is with the Burgers viscous equation instead of the Navier–Stokes system).

Here our aim is to complete the previous results and to show the local null controllability to trajectories. Besides the interest of the corresponding result, our aim consists in showing how we can derive Carleman estimates for a parabolic system with a nonlocal spatial term. Such terms can appear naturally in fluid mechanics to model the turbulence, but with more complicated models and we can also see such terms in biology, see for instance [15, Section 11.5]. Previous results have been obtained for parabolic systems with nonlocal spatial terms, see, for instance [1, 8, 9, 13], etc. Let us note that in these references, the nonlocal spatial term is an

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integral term with a general kernel $K(x, y)$ and here we only treat the case of a kernel of the form $K = a \otimes b$, see (6) below. Nevertheless with this type of kernels, we are able to show a Carleman estimate whereas the previous references are considering a compactness-uniqueness argument that does not permit to deduce directly a controllability result on the nonlinear systems. In [8], the authors consider a nonlinear heat equation where the nonlinearity contains a nonlocal term similar to the one here. Their method consists in showing the approximate controllability of the linearized system by using a compactness-uniqueness argument and then deduce the approximate controllability of the nonlinear system by using a Kakutani fixed point argument. Then the local exact controllability to trajectories is obtained with a passage to the limit. With our approach, the Carleman estimate for the adjoint system implies in a standard way the local exact controllability to trajectories with a Banach fixed point argument and without any passage to the limit.

Let us present the model of Ladyzhenskaya for the motion of a viscous incompressible fluid. Assume Ω is a smooth domain of $\mathbb{R}^3$ and $T > 0$. We consider the following system

$$\begin{cases} \partial_t v - \left(\nu_0 + \nu_1 \int_\Omega \operatorname{curl}(v)^2 \, dx\right) \Delta v + \nabla p + (v \cdot \nabla) v = 1_\omega u & \text{in } (0,T)\times\Omega, \\ \operatorname{div} v = 0 & \text{in } (0,T)\times\Omega, \\ v = 0 & \text{on } (0,T)\times\partial\Omega, \\ v(0,\cdot) = v^0 & \text{in } \Omega. \end{cases} \tag{1}$$

In the above system, v and p are respectively the velocity and the pressure of the fluid. The viscosity of the fluid is not constant and depends on the velocity of the fluid. Such a model where a nonlocal spatial dependence appears has some common features with models for the turbulence (see, for instance [2,12], etc.) The constants ν_0 and ν_1 are assumed to be positive. The control u of this system is supported in a subdomain $\omega \Subset \Omega$ and we want to use it to obtain the controllability to the following given trajectory:

$$\begin{cases} \partial_t \overline{v} - \left(\nu_0 + \nu_1 \int_\Omega \operatorname{curl}(\overline{v})^2 \, dx\right) \Delta \overline{v} + \nabla \overline{p} + (\overline{v} \cdot \nabla) \overline{v} = 0 & \text{in } (0,T)\times\Omega, \\ \operatorname{div} \overline{v} = 0 & \text{in } (0,T)\times\Omega, \\ \overline{v} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \overline{v}(0,\cdot) = \overline{v}^0 & \text{in } \Omega. \end{cases} \tag{2}$$

This means that we search a control u such that $v(T,\cdot) = \overline{v}(T,\cdot)$. In order to do this, we set

$$z = v - \overline{v}, \quad q = p - \overline{p},$$

$$\mu := \nu_0 + \nu_1 \int_\Omega \operatorname{curl}(\overline{v})^2 \, dx, \quad a = 2\nu_1 \Delta\overline{v}, \quad b = \Delta\overline{v}, \quad z^0 = v^0 - \overline{v}^0 \tag{3}$$

so that

$$\begin{cases} \partial_t z - \mu\Delta z + \left(\int_\Omega b \cdot z \, dx\right) a + z \cdot \nabla\overline{v} + \overline{v} \cdot \nabla z + \nabla q = F(z) + 1_\omega u & \text{in } (0,T)\times\Omega, \\ \operatorname{div} z = 0 & \text{in } (0,T)\times\Omega, \\ z = 0 & \text{on } (0,T)\times\partial\Omega, \\ z(0,\cdot) = z^0 & \text{in } \Omega, \end{cases} \tag{4}$$

where

$$F(z) = \nu_1 \left(\int_\Omega |\operatorname{curl} z|^2 \, dx\right) \Delta z + \nu_1 \left(\int_\Omega |\operatorname{curl} z|^2 \, dx\right) \Delta\overline{v} + 2\nu_1 \left(\int_\Omega (\operatorname{curl}\overline{v}) \cdot \operatorname{curl} z \, dx\right) \Delta z - z \cdot \nabla z. \tag{5}$$

We are then reduced to show the null-controllability of the nonlinear system (4). A standard method to prove the local null-controllability of (4) consists in showing the null-controllability of the linearized system of (4), that is

$$\begin{cases} \partial_t z - \mu\Delta z + \left(\int_\Omega b\cdot z\,dy\right) a + z\cdot\nabla\overline{v} + \overline{v}\cdot\nabla z + \nabla q = f + 1_\omega u & \text{in } (0,T)\times\Omega,\\ \operatorname{div} z = 0 & \text{in } (0,T)\times\Omega,\\ z = 0 & \text{on } (0,T)\times\partial\Omega,\\ z(0,\cdot) = z^0 & \text{in } \Omega, \end{cases} \tag{6}$$

where f is a given source term. To show the null-controllability of the above system, we need to prove an observability inequality for the adjoint system of (6) given by

$$\begin{cases} -\partial_t \varphi - \mu\Delta \varphi + \left(\int_\Omega a\cdot \varphi\,dx\right) b + \left(\nabla\overline{v}\right)^\top \varphi - \overline{v}\cdot\nabla \varphi + \nabla \pi = g & \text{in } (0,T)\times\Omega,\\ \operatorname{div} \varphi = 0 & \text{in } (0,T)\times\Omega,\\ \varphi = 0 & \text{on } (0,T)\times\partial\Omega,\\ \varphi(T,\cdot) = \varphi^T & \text{in } \Omega. \end{cases} \tag{7}$$

A classical way to obtain this observability inequality relies on the Carleman estimates (see, for instance [5, 6, 10], etc.). The nice feature of this method is that the lower order terms can be neglected during the proof. However, here the nonlocal spatial term

$$\left(\int_\Omega a\cdot\varphi\,dx\right) b$$

can not be absorbed in a direct way and one has to work differently to handle this term.

First, we assume that

$$\overline{v} \in W^{1,\infty}\left(0,T;H^1(\Omega)\right)\cap H^1\left(0,T;W^{1,\infty}(\Omega)\right)\cap H^1\left(0,T;H^2(\Omega)\right)\cap L^2\left(0,T;H^4(\Omega)\right). \tag{8}$$

In particular, we have

$$\mu\in W^{1,\infty}(0,T),\quad a\in H^1\left(0,T;L^2(\Omega)\right),\quad b\in H^1\left(0,T;L^2(\Omega)\right)\cap L^2\left(0,T;H^2(\Omega)\right) \tag{9}$$

and

$$\mu(t)\geqslant \nu_0>0.$$

The domain of the control ω is a nonempty open set of Ω and we assume that

$$\operatorname{curl} b\not\equiv 0 \quad \text{in } (0,T)\times\omega \tag{10}$$

and more precisely that there exists a non empty open set $(T_1,T_2)\times\omega_0\subset(0,T)\times\omega$ such that

$$|\operatorname{curl} b|\geqslant c_*>0 \quad \text{in } (T_1,T_2)\times\omega_0. \tag{11}$$

If $\operatorname{curl} b\in C^0([0,T]\times\Omega)$, then (10) implies (11) but in the general case, condition (11) is stronger. In the controllability of (4), one can always consider the case where in (11), $T_1=0$ and $T_2=T$ by considering a control $u=0$ outside (T_1,T_2). Therefore we assume in what follows that

$$\omega_0\Subset\omega, \tag{12}$$

and

$$|\operatorname{curl} b|\geqslant c_*>0 \quad \text{in } (0,T)\times\omega_0. \tag{13}$$

Our first result is a Carleman estimate for (7). In order to state this result we first introduce some standard weights. We choose ω_1 a nonempty open set such that

$$\omega_1\Subset\omega_0\Subset\omega \tag{14}$$

There exists

$$\eta\in C^2(\overline{\Omega}),\quad \eta>0\quad\text{in }\Omega,\quad \eta=0\quad\text{on }\partial\Omega,\quad \left|\nabla\eta\right|>0\quad\text{in }\overline{\Omega\setminus\omega_1},\quad \max_{\Omega}\eta=1. \tag{15}$$

Assume $\lambda \geqslant 2\ln 2$, $m \geqslant 4$ and let us set

$$\xi(t,x) = \frac{e^{\lambda(2m+\eta(x))}}{[t(T-t)]^m}, \qquad \alpha(t,x) = \frac{e^{\lambda(2m+2)} - e^{\lambda(2m+\eta(x))}}{[t(T-t)]^m}. \tag{16}$$

We also set

$$\xi_\sharp(t) = \frac{e^{2m\lambda}}{[t(T-t)]^m}, \qquad \xi_\flat(t) = \frac{e^{\lambda(2m+1)}}{[t(T-t)]^m}, \tag{17}$$

$$\alpha_\sharp(t) = \frac{e^{\lambda(2m+2)} - e^{2m\lambda}}{[t(T-t)]^m}, \quad \alpha_\flat(t) = \frac{e^{\lambda(2m+2)} - e^{\lambda(2m+1)}}{[t(T-t)]^m}. \tag{18}$$

We have the following relations for $C>0$ independent of T and λ:

$$\xi_\sharp \leqslant \xi \leqslant \xi_\flat \quad \text{and} \quad e^{-s\alpha_\sharp} \leqslant e^{-s\alpha} \leqslant e^{-s\alpha_\flat} \quad \text{in } (0,T)\times\Omega, \tag{19}$$

$$\alpha_\sharp < \frac{4}{3}\alpha_\flat, \quad \left|\xi_\sharp'\right| \leqslant CT\xi_\sharp^{1+1/m}, \quad \left|\alpha_\sharp'\right| \leqslant CT\xi_\sharp^{1+1/m} \quad \text{in } (0,T)\times\Omega, \tag{20}$$

$$\left|\xi_\sharp''\right| \leqslant CT^2\xi_\sharp^{1+2/m}, \quad \left|\alpha_\sharp''\right| \leqslant CT^2\xi_\sharp^{1+2/m} \quad \text{in } (0,T)\times\Omega. \tag{21}$$

There exists $C>0$ such that if $s \geqslant CT^{2m}$, then

$$s\xi_\sharp \geqslant 1 \quad \text{in } (0,T)\times\Omega. \tag{22}$$

Finally, let us define

$$\rho_1 := \begin{cases} e^{-\frac{3}{2}s\alpha_\sharp} & \text{in } \left[\frac{T}{2}, T\right] \\ \left[e^{-\frac{3}{2}s\alpha_\sharp}\right]\left(\frac{T}{2}\right) & \text{in } \left[0, \frac{T}{2}\right] \end{cases}, \quad \rho_2 := \begin{cases} s^{6/m}\lambda^{-9}\xi_\sharp^{6/m}e^{-4s\alpha_\flat+\frac{3}{2}s\alpha_\sharp} & \text{in } \left[\frac{T}{2}, T\right] \\ \left[s^{6/m}\lambda^{-9}\xi_\sharp^{6/m}e^{-4s\alpha_\flat+\frac{3}{2}s\alpha_\sharp}\right]\left(\frac{T}{2}\right) & \text{in } \left[0, \frac{T}{2}\right] \end{cases}, \tag{23}$$

$$\rho_3 := \begin{cases} s^2\lambda^3\xi_\sharp^2 e^{-\frac{5}{2}s\alpha_\sharp} & \text{in } \left[\frac{T}{2}, T\right] \\ \left[s^2\lambda^3\xi_\sharp^2 e^{-\frac{5}{2}s\alpha_\sharp}\right]\left(\frac{T}{2}\right) & \text{in } \left[0, \frac{T}{2}\right] \end{cases}. \tag{24}$$

Note that $\rho_2(T) = 0$ due to (20). We are now in a position to state the Carleman estimate for (7).

Theorem 1. *Assume* (8), (12) *and* (13). *There exists a constant* $C>0$ *such that for any* φ *solution of* (7)

$$\left\|\rho_3\varphi\right\|_{L^2(0,T;L^2(\Omega))} + \left\|\varphi(0,\cdot)\right\|_{L^2(\Omega)} \leqslant C\left[\left\|\rho_2\varphi\right\|_{L^2(0,T;L^2(\omega))} + \left\|\rho_1 g\right\|_{L^2(0,T;L^2(\Omega))}\right]. \tag{25}$$

Remark 2. Note that if

$$\operatorname{curl} b \equiv 0 \quad \text{in } (0,T)\times\Omega,$$

then Theorem 1 holds true without conditions (12) and (13) and with a weaker condition than (8). In fact, the result also holds true if b is in the kernel of any differential operator corresponding to a composition with the curl operator: for instance, if $\Delta b = 0$ or if $\nabla\Delta b = 0$ in $(0,T)\times\Omega$. In that case, one can easily adapt the proof of Theorem 1 by using the operator Δ or $\nabla\Delta$ instead of the curl operator.

If $b \not\equiv 0$ in $(0,T)\times\Omega$ but

$$b \equiv 0 \quad \text{in } (0,T)\times\omega, \tag{26}$$

so that (10) does not hold, one can show that the unique continuation property is not satisfied so that one can not expect a Carleman estimate in that case. More precisely, there exist a, b (without the relation with $\overline{v}$ given by (3)), and (φ,π) a solution of (7) with $g=0$ such that $\varphi \equiv 0$ in $(0,T)\times\omega$ but $\varphi \not\equiv 0$ in $(0,T)\times\Omega$. The construction is quite standard: we consider $\varphi \in C^2(\overline{\Omega})$, independent in time to simplify, not identically null, with $\operatorname{div}\varphi = 0$ and $\varphi = 0$ on $\partial\Omega$. We also take $\pi \equiv 0$. Then, there exists $a \in L^2(\Omega)$ such that

$$\int_\Omega a \cdot \varphi \, dx \neq 0$$

and we define b by

$$b := \frac{\mu\Delta\varphi - \left(\nabla\overline{v}\right)^\top \varphi + \overline{v} \cdot \nabla\varphi}{\displaystyle\int_\Omega a \cdot \varphi \, dx}.$$

One can check that (a, b, φ, π) satisfies the above hypotheses.

Remark 3. Another important remark about the proof given here to obtain Theorem 1 is that it is quite general and can be adapted to many other parabolic systems. One can for instance consider the controllability of the system considered in [8] (nonlinear parabolic system with nonlinear diffusion) or a system of heat equations with a nonlocal spatial term of the same type as here and in the case where one can show a Carleman estimate without this nonlocal term.

Let us define

$$\rho_0 := \begin{cases} e^{-\frac{11}{8} s\alpha_\sharp} & \text{in } \left[\dfrac{T}{2}, T\right] \\ \left[e^{-\frac{11}{8} s\alpha_\sharp}\right]\left(\dfrac{T}{2}\right) & \text{in } \left[0, \dfrac{T}{2}\right] \end{cases}. \tag{27}$$

As a corollary of Theorem 1, we deduce the following controllability results:

Corollary 4. *Assume* (8), (12) *and* (13). *Suppose*

$$z^0 \in H_0^1(\Omega), \quad \operatorname{div} z^0 = 0, \quad \frac{f}{\rho_3} \in L^2((0,T) \times \Omega).$$

Then there exists $u \in L^2(0,T; L^2(\omega))$ *such that the solution* z *of* (6) *satisfies*

$$\left\| \frac{z}{\rho_0} \right\|_{L^2(0,T;\, H^2(\Omega)) \cap C^0([0,T]; H^1(\Omega)) \cap H^1(0,T; L^2(\Omega))} \leqslant C \left(\left\| \frac{f}{\rho_3} \right\|_{L^2((0,T)\times\Omega)} + \left\| z^0 \right\|_{H^1(\Omega)} \right).$$

In particular, $z(T, \cdot) = 0$.

Moreover, there exists a constant c_0 *such that for any* $\|z^0\|_{H^1(\Omega)} \leqslant c_0$, *there exists* $u \in L^2(0,T; L^2(\omega))$ *such that the solution* z *of* (4) *satisfies*

$$\frac{z}{\rho_0} \in L^2\left(0,T; H^2(\Omega)\right) \cap C^0\left([0,T]; H^1(\Omega)\right) \cap H^1\left(0,T; L^2(\Omega)\right).$$

In particular, $z(T, \cdot) = 0$.

The outline of the article is as follows. In Section 2 we recall some preliminary results: well-posedness of systems of type (6) or (7), and standard Carleman estimates for the gradient, the Laplace and the heat operators. Let us emphasize that the Carleman estimate for the gradient is a key point in the proof of Theorem 1. Section 3 is devoted to the proof of Theorem 1 and we use this result to show the controllability results (Corollary 4) in Section 4.

2. Preliminaries

2.1. *A well-posedness result*

Let us consider the system

$$\begin{cases} \partial_t \phi - \kappa^{(1)} \Delta \phi + \nabla r + \left(\nabla \phi\right) \kappa^{(2)} + \kappa^{(3)} \phi + \left(\int_\Omega \kappa^{(4)} \cdot \phi \, dx\right) \kappa^{(5)} = h & \text{in } (0,T) \times \Omega, \\ \operatorname{div} \phi = 0 & \text{in } (0,T) \times \Omega, \\ \phi = 0 & \text{on } (0,T) \times \partial\Omega, \\ \phi(0,\cdot) = \phi^0 & \text{in } \Omega, \end{cases} \tag{28}$$

where

$$\kappa^{(1)} : (0,T) \to \mathbb{R}_+^*, \quad \kappa^{(i)} : (0,T) \times \Omega \to \mathbb{R}^3 \quad (i = 2,4,5), \quad \kappa^{(3)} : (0,T) \times \Omega \to \mathbb{R}^9,$$

$$\kappa^{(1)} \in W^{1,\infty}(0,T), \quad \kappa^{(1)} \geqslant \nu_0 > 0, \tag{29}$$

$$\kappa^{(2)} \in H^1\left(0,T; L^\infty(\Omega)\right) \cap L^\infty\left(0,T; H^2(\Omega)\right), \tag{30}$$

$$\kappa^{(3)} \in H^1\left(0,T; L^\infty(\Omega)\right) \cap L^2(0,T; H^2(\Omega)), \tag{31}$$

$$\kappa^{(4)} \in H^1\left(0,T; L^2(\Omega)\right), \tag{32}$$

$$\kappa^{(5)} \in H^1(0,T; L^2(\Omega)) \cap L^2\left(0,T; H^2(\Omega)\right). \tag{33}$$

We set

$$X_1 := L^2\left(0,T; H^2(\Omega)\right) \cap C^0\left([0,T]; H^1(\Omega)\right) \cap H^1\left(0,T; L^2(\Omega)\right),$$
$$X_2 := L^2\left(0,T; H^4(\Omega)\right) \cap C^0\left([0,T]; H^3(\Omega)\right) \cap H^1\left(0,T; H^2(\Omega)\right) \cap C^1\left([0,T]; H^1(\Omega)\right) \cap H^2\left(0,T; L^2(\Omega)\right).$$

Then we have the following result that can be obtained by standard methods:

Lemma 5. *With the above assumptions, assume*

$$\phi^0 \in H_0^1(\Omega), \quad \operatorname{div} \phi^0 = 0, \quad h \in L^2((0,T) \times \Omega).$$

Then there exists a unique solution to (28)

$$(\phi, r) \in X_1 \times L^2(0,T; H^1(\Omega)/\mathbb{R})$$

and there exists a constant $C > 0$ *such that*

$$\|\phi\|_{X_1} + \|\nabla r\|_{L^2((0,T)\times\Omega)} \leqslant C\left(\left\|\phi^0\right\|_{H_0^1(\Omega)} + \|h\|_{L^2((0,T)\times\Omega)}\right).$$

Assume

$$\phi^0 \in H^3(\Omega) \cap H_0^1(\Omega), \quad \operatorname{div} \phi^0 = 0, \quad h \in X_1$$

and there exists $r^0 \in H^1(\Omega)$ *such that*

$$\phi^1 := \kappa^{(1)}(0) \Delta \phi^0 - \nabla r^0 - \left(\nabla \phi^0\right) \kappa^{(2)}(0,\cdot) - \kappa^{(3)}(0,\cdot) \phi^0 - \left(\int_\Omega \kappa^{(4)}(0,\cdot) \cdot \phi^0 \, dx\right) \kappa^{(5)}(0,\cdot) + h(0,\cdot)$$

satisfies $\phi^1 = 0$ *on* $\partial\Omega$, $\operatorname{div} \phi^1 = 0$. *Then there exists a unique solution to* (28)

$$\left(\phi, \nabla r\right) \in X_2 \times X_1$$

and there exists a constant $C > 0$ *such that*

$$\|\phi\|_{X_2} + \|\nabla r\|_{X_1} \leqslant C\left(\left\|\phi^0\right\|_{H^3(\Omega)} + \|h\|_{X_1}\right).$$

2.2. *First Carleman estimates*

We recall here some Carleman estimates that were obtained in previous articles. The weights used below are given by (15)-(18).

First, we recall a Carleman estimate for the gradient operator (see, for instance, [3, Lemma 3]):

Lemma 6. *There exists $C>0$ depending on the geometry and on η such that for any $T>0$, $\lambda \geqslant C$, $s \geqslant CT^{2m}$ and $u \in L^2(0,T;H^1(\Omega))$,*

$$\iint_{(0,T)\times\Omega} e^{-2s\alpha}|u|^2\,dx\,dt \leqslant C\left(\frac{1}{s^2\lambda^2}\iint_{(0,T)\times\Omega} \xi^{-2}e^{-2s\alpha}|\nabla u|^2\,dx\,dt + \iint_{(0,T)\times\omega_1} e^{-2s\alpha}|u|^2\,dx\,dt\right).$$

In particular, if $u \in L^2(0,T)$, then the above inequality writes

$$\iint_{(0,T)\times\Omega} e^{-2s\alpha}|u|^2\,dx\,dt \leqslant C\iint_{(0,T)\times\omega_1} e^{-2s\alpha}|u|^2\,dx\,dt. \tag{34}$$

Then, we recall a Carleman estimate for the Laplace operator (see, for instance, [3, Lemma 4]):

Lemma 7. *There exists $C>0$ depending on the geometry and on η such that for any $T>0$, $\lambda \geqslant C$, $s \geqslant CT^{2m}$ and $u \in L^2(0,T;H^2(\Omega)\cap H^1_0(\Omega))$,*

$$\begin{aligned}\iint_{(0,T)\times\Omega} s^4\lambda^6\xi^4 e^{-2s\alpha}|u|^2\,dx\,dt + \iint_{(0,T)\times\Omega} s^2\lambda^4\xi^2 e^{-2s\alpha}|\nabla u|^2\,dx\,dt \\ \leqslant C\left(\iint_{(0,T)\times\Omega} s\lambda^2\xi e^{-2s\alpha}|\Delta u|^2\,dx\,dt + \iint_{(0,T)\times\omega_1} s^4\lambda^6\xi^4 e^{-2s\alpha}|u|^2\,dx\,dt\right).\end{aligned} \tag{35}$$

Finally, we need a Carleman estimates for the heat equation with Neumann boundary conditions:

$$\begin{cases}\partial_t u + \mu\Delta u = f^{(1)} + \operatorname{div} f^{(2)} & \text{in } (0,T)\times\Omega,\\ -\mu\dfrac{\partial u}{\partial n} + f^{(2)}\cdot n = f^{(3)} & \text{on } (0,T)\times\partial\Omega.\end{cases} \tag{36}$$

The following lemma is obtained in [4] (see also [3, Lemma 5]):

Lemma 8. *There exists $C>0$ depending on the geometry and on η such that for any*

$$T>0,\ \lambda\geqslant C,\ s\geqslant C\left(T^{2m}+T^m\right),$$

$$f^{(1)}, f^{(2)} \in L^2\left(0,T;L^2(\Omega)\right),\ f^{(3)} \in L^2\left(0,T;L^2(\partial\Omega)\right),$$

and $u \in L^2(0,T;H^1(\Omega))\cap C^0([0,T];L^2(\Omega))$ weak solution of (36),

$$\begin{aligned}\iint_{(0,T)\times\Omega} s^3\lambda^4\xi^3 e^{-2s\alpha}|u|^2\,dx\,dt + \iint_{(0,T)\times\Omega} s\lambda^2\xi e^{-2s\alpha}|\nabla u|^2\,dx\,dt \\ \leqslant C\left(\iint_{(0,T)\times\Omega} e^{-2s\alpha}\left|f^{(1)}\right|^2\,dx\,dt + \iint_{(0,T)\times\Omega} s^2\lambda^2\xi^2 e^{-2s\alpha}\left|f^{(2)}\right|^2\,dx\,dt\right. \\ \left. + \iint_{(0,T)\times\partial\Omega} s\lambda\xi_\sharp e^{-2s\alpha_\sharp}\left|f^{(3)}\right|^2\,d\gamma\,dt + \iint_{(0,T)\times\omega_1} s^3\lambda^4\xi^3 e^{-2s\alpha}|u|^2\,dx\,dt\right).\end{aligned} \tag{37}$$

3. Proof of Theorem 1

We consider the function $\rho:[0,T]\to\mathbb{R}_+$ defined by

$$\rho := e^{-\frac{3}{2}s\alpha_\sharp}. \tag{38}$$

Note that $\rho(0)=\rho(T)=0$. Then, we consider the following decomposition of the solution of (7):

$$\rho\varphi = \widehat{\varphi} + \widetilde{\varphi} \tag{39}$$

where

$$\begin{cases} -\partial_t \widehat{\varphi} - \mu\Delta\widehat{\varphi} + \left(\int_\Omega a\cdot\widehat{\varphi}\,dx\right) b + \left(\nabla\overline{v}\right)^\top \widehat{\varphi} - \overline{v}\cdot\nabla\widehat{\varphi} + \nabla\widehat{\pi} = \rho g & \text{in } (0,T)\times\Omega,\\ \operatorname{div}\widehat{\varphi} = 0 & \text{in } (0,T)\times\Omega,\\ \widehat{\varphi} = 0 & \text{on } (0,T)\times\partial\Omega,\\ \widehat{\varphi}(T,\cdot) = 0 & \text{in } \Omega, \end{cases} \tag{40}$$

and

$$\begin{cases} -\partial_t \widetilde{\varphi} - \mu\Delta\widetilde{\varphi} + \left(\int_\Omega a\cdot\widetilde{\varphi}\,dx\right) b + \left(\nabla\overline{v}\right)^\top \widetilde{\varphi} - \overline{v}\cdot\nabla\widetilde{\varphi} + \nabla\widetilde{\pi} = -\rho'\varphi & \text{in } (0,T)\times\Omega,\\ \operatorname{div}\widetilde{\varphi} = 0 & \text{in } (0,T)\times\Omega,\\ \widetilde{\varphi} = 0 & \text{on } (0,T)\times\partial\Omega,\\ \widetilde{\varphi}(T,\cdot) = 0 & \text{in } \Omega. \end{cases} \tag{41}$$

3.1. *A priori estimates with weights*

In this section we show the follow result:

Proposition 9. *The solution of* (40) *satisfies*

$$\left\|\widehat{\varphi}\right\|_{X_1} \leqslant C\left\|\rho g\right\|_{L^2(0,T;L^2(\Omega))}. \tag{42}$$

Let us consider

$$\gamma_1 := s^{-2/m}\lambda^3\xi_\sharp^{-2/m}e^{-s\alpha_\sharp}. \tag{43}$$

Then the solution of (41) *satisfies*

$$\left\|\gamma_1\widetilde{\varphi}\right\|_{L^2(0,T;H^4(\Omega))} + \left\|\gamma_1\partial_t\Delta\widetilde{\varphi}\right\|_{L^2(0,T;L^2(\Omega))} + \left\|\gamma_1\partial_{tt}\widetilde{\varphi}\right\|_{L^2(0,T;L^2(\Omega))} \leqslant C\left(\left\|Ts^{2-1/m}\lambda^3\xi^2e^{-s\alpha}\widetilde{\varphi}\right\|_{L^2(0,T;L^2(\Omega))} + \|\rho g\|_{L^2(0,T;L^2(\Omega))}\right). \tag{44}$$

Proof. Relation (42) is a direct consequence of Lemma 5, (3) and (8). Let us set

$$\gamma_0 := s^{1-1/m}\lambda^3\xi_\sharp^{1-1/m}e^{-s\alpha_\sharp}. \tag{45}$$

Then from (20), we deduce that for $s \geqslant CT^{2m}$,

$$\left|\gamma_0'\right| \leqslant CTs^{2-1/m}\lambda^3\xi_\sharp^2e^{-s\alpha_\sharp}, \quad \left|\gamma_0\rho'\right| \leqslant CTs^{2-1/m}\lambda^3\xi_\sharp^2e^{-s\alpha_\sharp}\rho, \tag{46}$$

and thus

$$\left\|\gamma_0'\widetilde{\varphi}\right\|_{L^2(0,T;L^2(\Omega))} \leqslant C\left\|Ts^{2-1/m}\lambda^3\xi^2e^{-s\alpha}\widetilde{\varphi}\right\|_{L^2(0,T;L^2(\Omega))}, \tag{47}$$

$$\|\gamma_0\rho'\varphi\|_{L^2(0,T;L^2(\Omega))} \leqslant C\left(\left\|Ts^{2-1/m}\lambda^3\xi^2e^{-s\alpha}\widetilde{\varphi}\right\|_{L^2(0,T;L^2(\Omega))} + \left\|\widehat{\varphi}\right\|_{L^2(0,T;L^2(\Omega))}\right). \tag{48}$$

Using (41), we deduce that $\gamma_0\widetilde{\varphi}$ solves (28) (with a change of variables $t \mapsto T-t$) with the right-hand side $-\gamma_0'\widetilde{\varphi} - \gamma_0\rho'\varphi$ and with a null final condition. We can apply Lemma 5 and combine it with (47), (48) and (42) to obtain

$$\|\gamma_0\widetilde{\varphi}\|_{X_1} \leqslant C\left(\left\|Ts^{2-1/m}\lambda^3\xi^2e^{-s\alpha}\widetilde{\varphi}\right\|_{L^2(0,T;L^2(\Omega))} + \left\|\rho g\right\|_{L^2(0,T;L^2(\Omega))}\right). \tag{49}$$

Using (20)-(21), we deduce that for $s \geqslant C(T^m + T^{2m})$,

$$\left|\gamma_1'\right| \leqslant C\gamma_0, \quad \left|\gamma_1\rho'\right| \leqslant C\gamma_0\rho, \tag{50}$$

$$|\gamma_1''| \leqslant CT^2s^{2-2/m}\lambda^3\xi^2e^{-s\alpha}, \quad \left|\gamma_1'\rho'\right| + \left|\gamma_1\rho''\right| \leqslant CT^2s^{2-2/m}\lambda^3\xi^2e^{-s\alpha}\rho. \tag{51}$$

From (41), we remark that $\gamma_1\widetilde{\varphi}$ solves (28) (with a change of variables $t \mapsto T-t$) with the right-hand side $-\gamma_1'\widetilde{\varphi} - \gamma_1\rho'\varphi$ and with a null final condition. Applying Lemma 5, we obtain that

$$\left\|\gamma_1\widetilde{\varphi}\right\|_{X_2} \leqslant C\left(\left\|\gamma_1'\widetilde{\varphi}\right\|_{X_1} + \left\|\gamma_1\rho'\varphi\right\|_{X_1}\right). \tag{52}$$

From (46), (50), (51), the above estimate yields

$$\|\gamma_1\widetilde{\varphi}\|_{X_2} \leqslant C\Big(\|\gamma_0\widetilde{\varphi}\|_{X_1} + \big\|\big(T^2 s^{2-2/m} + Ts^{2-1/m}\big)\lambda\xi^2 e^{-s\alpha}\widetilde{\varphi}\big\|_{L^2(0,T;L^2(\Omega))} + \|\widehat{\varphi}\|_{X_1}\Big).$$

The above estimate combined with (49) and (42) implies

$$\|\gamma_1\widetilde{\varphi}\|_{X_2} \leqslant C\Big(\big\|Ts^{2-1/m}\lambda^3\xi^2 e^{-s\alpha}\widetilde{\varphi}\big\|_{L^2(0,T;L^2(\Omega))} + \|\rho g\|_{L^2(0,T;L^2(\Omega))}\Big), \tag{53}$$

for $s \geqslant CT^m$. Combining this with (46), (50), (51), and (49), we deduce (44). □

Remark 10. Let us notice that for $s \geqslant C(T^m + T^{2m})$,

$$e^{-2s\alpha_\flat} \leqslant C\gamma_1^{-2} e^{-4s\alpha_\flat}. \tag{54}$$

Using that $\alpha_\sharp < 2\alpha_\flat$, we deduce that

$$\gamma_1^{-2} e^{-4s\alpha_\flat} = \big(\gamma_1^{-2} e^{-4s\alpha_\flat}\big)' = 0 \quad \text{at } t \in \{0, T\}. \tag{55}$$

3.2. *Carleman estimates for the system* (41)

Taking the curl of the first equation of (41), we obtain

$$-\partial_t \operatorname{curl}\widetilde{\varphi} - \mu\Delta\operatorname{curl}\widetilde{\varphi} = -\operatorname{curl}\left[\left(\int_\Omega a\cdot\widetilde{\varphi}\,dx\right)b + \left((\nabla\overline{v})^\top\widetilde{\varphi} - \overline{v}\cdot\nabla\widetilde{\varphi}\right) + \rho'\varphi\right]. \tag{56}$$

We first apply Lemma 8 (for $f_2 = 0$) and we use that $\overline{v} \in L^\infty(0,T;W^{2,\infty}(\Omega))$:

$$\begin{aligned}
&\iint_{(0,T)\times\Omega} s^3\lambda^4\xi^3 e^{-2s\alpha}\left|\operatorname{curl}\widetilde{\varphi}\right|^2\,dx\,dt + \iint_{(0,T)\times\Omega} s\lambda^2\xi e^{-2s\alpha}\left|\nabla\operatorname{curl}\widetilde{\varphi}\right|^2\,dx\,dt\\
&\leqslant C\left[\iint_{(0,T)\times\Omega} e^{-2s\alpha}\left(|\widetilde{\varphi}|^2 + |\nabla\widetilde{\varphi}|^2 + |\nabla\operatorname{curl}\widetilde{\varphi}|^2 + |\rho'\operatorname{curl}\varphi|^2\right)\,dx\,dt\right.\\
&\quad + \iint_{(0,T)\times\Omega} e^{-2s\alpha}\left(\int_\Omega a\cdot\widetilde{\varphi}\,dx\right)^2 |\operatorname{curl} b|^2\,dx\,dt + \iint_{(0,T)\times\partial\Omega} s\lambda\xi_\sharp e^{-2s\alpha_\sharp}\left|\frac{\partial}{\partial n}\operatorname{curl}\widetilde{\varphi}\right|^2\,d\gamma\,dt\\
&\quad \left. + \iint_{(0,T)\times\omega_1} s^3\lambda^4\xi^3 e^{-2s\alpha}\left|\operatorname{curl}\widetilde{\varphi}\right|^2\,dx\,dt\right].
\end{aligned} \tag{57}$$

Then, we apply Lemma 7:

$$\begin{aligned}
&\iint_{(0,T)\times\Omega} s^4\lambda^6\xi^4 e^{-2s\alpha}|\widetilde{\varphi}|^2\,dx\,dt + \iint_{(0,T)\times\Omega} s^2\lambda^4\xi^2 e^{-2s\alpha}|\nabla\widetilde{\varphi}|^2\,dx\,dt\\
&\qquad \leqslant C\left(\iint_{(0,T)\times\Omega} s\lambda^2\xi e^{-2s\alpha}|\Delta\widetilde{\varphi}|^2\,dx\,dt + \iint_{(0,T)\times\omega_1} s^4\lambda^6\xi^4 e^{-2s\alpha}|\widetilde{\varphi}|^2\,dx\,dt\right).
\end{aligned} \tag{58}$$

Using (38), (39) and (20), we deduce that

$$\begin{aligned}
&\iint_{(0,T)\times\Omega} e^{-2s\alpha}\left|\rho'\operatorname{curl}\varphi\right|^2\,dx\,dt\\
&\qquad \leqslant C\left(\iint_{(0,T)\times\Omega}\left|\operatorname{curl}\widehat{\varphi}\right|^2\,dx\,dt + \iint_{(0,T)\times\Omega} s^2\xi_\sharp^{2+2/m} e^{-2s\alpha}\left|\operatorname{curl}\widetilde{\varphi}\right|^2\,dx\,dt\right)
\end{aligned}$$

and thus, with (42),

$$\begin{aligned}
&\iint_{(0,T)\times\Omega} e^{-2s\alpha}\left|\rho'\operatorname{curl}\varphi\right|^2\,dx\,dt\\
&\qquad \leqslant C\left(\iint_{(0,T)\times\Omega}|\rho g|^2\,dx\,dt + \iint_{(0,T)\times\Omega} s^2\xi_\sharp^{2+2/m} e^{-2s\alpha}\left|\operatorname{curl}\widetilde{\varphi}\right|^2\,dx\,dt\right).
\end{aligned} \tag{59}$$

Combining (57), (58) and (59), and using that $\operatorname{curl}\operatorname{curl}\widetilde{\varphi} = -\Delta\widetilde{\varphi}$, we deduce that

$$I_0\left(s,\lambda,\widetilde{\varphi}\right) := \iint_{(0,T)\times\Omega} s^4\lambda^6\xi^4 e^{-2s\alpha}\left|\widetilde{\varphi}\right|^2\, dx\,dt + \iint_{(0,T)\times\Omega} s^2\lambda^4\xi^2 e^{-2s\alpha}\left|\nabla\widetilde{\varphi}\right|^2\, dx\,dt \\ + \iint_{(0,T)\times\Omega} s^3\lambda^4\xi^3 e^{-2s\alpha}\left|\operatorname{curl}\widetilde{\varphi}\right|^2\, dx\,dt + \iint_{(0,T)\times\Omega} s\lambda^2\xi e^{-2s\alpha}\left|\nabla\operatorname{curl}\widetilde{\varphi}\right|^2\, dx\,dt \tag{60}$$

satisfies for $\lambda \geqslant C$, $s \geqslant CT^{2m}$,

$$I_0(s,\lambda,\widetilde{\varphi}) \leqslant C\left[\iint_{(0,T)\times\Omega}\left|\rho g\right|^2\, dx\,dt + \iint_{(0,T)\times\Omega} e^{-2s\alpha}\left(\int_\Omega a\cdot\widetilde{\varphi}\,dx\right)^2 |\operatorname{curl} b|^2\, dx\,dt \right. \\ + \iint_{(0,T)\times\partial\Omega} s\lambda\xi_\sharp e^{-2s\alpha_\sharp}\left|\frac{\partial}{\partial n}\operatorname{curl}\widetilde{\varphi}\right|^2\, d\gamma\,dt \\ \left. + \iint_{(0,T)\times\omega_1} s^3\lambda^4\xi^3 e^{-2s\alpha}\left|\operatorname{curl}\widetilde{\varphi}\right|^2\, dx\,dt + \iint_{(0,T)\times\omega_1} s^4\lambda^6\xi^4 e^{-2s\alpha}\left|\widetilde{\varphi}\right|^2\, dx\,dt\right]. \tag{61}$$

Here, we have used that $m \geqslant 4$.

In order to deal with the nonlocal term in (61), we apply Lemma 6 and in particular (34): for all $\lambda \geqslant C$, $s \geqslant CT^{2m}$,

$$\iint_{(0,T)\times\Omega} e^{-2s\alpha}\left(\int_\Omega a\cdot\widetilde{\varphi}\,dx\right)^2 |\operatorname{curl} b|^2\, dx\ dt \leqslant C\iint_{(0,T)\times\omega_1} e^{-2s\alpha}\left(\int_\Omega a\cdot\widetilde{\varphi}\,dx\right)^2\, dx\ dt. \tag{62}$$

On the other hand, from (56) and (13),

$$\left|-\partial_t\operatorname{curl}\widetilde{\varphi} - \mu\Delta\operatorname{curl}\widetilde{\varphi} + \rho'\operatorname{curl}\varphi + \operatorname{curl}\left[\left((\nabla\overline{v})^\top\widetilde{\varphi} - \overline{v}\cdot\nabla\widetilde{\varphi}\right)\right]\right|^2 \\ = \left(\int_\Omega a\cdot\widetilde{\varphi}\,dx\right)^2 |\operatorname{curl} b|^2 \geqslant \left(c^*\right)^2\left(\int_\Omega a\cdot\widetilde{\varphi}\,dx\right)^2 \quad \text{in } (0,T)\times\omega_0 \tag{63}$$

and consequently, using (14)

$$\iint_{(0,T)\times\omega_1} e^{-2s\alpha}\left(\int_\Omega a\cdot\widetilde{\varphi}\,dx\right)^2\, dx\ dt \\ \leqslant C\iint_{(0,T)\times\omega_1} e^{-2s\alpha}\Bigg(\left|\partial_t\operatorname{curl}\widetilde{\varphi}\right|^2 + \left|\Delta\operatorname{curl}\widetilde{\varphi}\right|^2 + \left|\rho'\operatorname{curl}\varphi\right|^2 \\ + \left|\operatorname{curl}\left[\left(\nabla\overline{v}\right)^\top\widetilde{\varphi}\right]\right|^2 + \left|\operatorname{curl}\left[\overline{v}\cdot\nabla\widetilde{\varphi}\right]\right|^2\Bigg)\,dx\ dt. \tag{64}$$

Combining the above relation with (62), we deduce

$$\iint_{(0,T)\times\Omega} e^{-2s\alpha}\left(\int_\Omega a\cdot\widetilde{\varphi}\,dx\right)^2 |\operatorname{curl} b|^2\, dx\ dt \\ \leqslant C\iint_{(0,T)\times\omega_1} e^{-2s\alpha}\Bigg(\left|\partial_t\operatorname{curl}\widetilde{\varphi}\right|^2 + \left|\Delta\operatorname{curl}\widetilde{\varphi}\right|^2 + \left|\rho'\operatorname{curl}\varphi\right|^2 \\ + \left|\operatorname{curl}\left[\left(\nabla\overline{v}\right)^\top\widetilde{\varphi}\right]\right|^2 + \left|\operatorname{curl}\left[\overline{v}\cdot\nabla\widetilde{\varphi}\right]\right|^2\Bigg)\,dx\ dt. \tag{65}$$

The last three terms in the right-hand side of (65) can be estimated as previously, and we can focus on the first two terms in the right-hand side of (65). We consider a nonempty open set ω_2 such that

$$\omega_1 \Subset \omega_2 \Subset \omega_0 \tag{66}$$

and a function θ such that

$$\theta \in C_c^\infty(\omega_2;\mathbb{R}_+), \quad \theta \equiv 1 \quad \text{in } \omega_1. \tag{67}$$

Using (19) and (54) and integrating by parts, we deduce

$$\begin{aligned}\iint_{(0,T)\times\omega_1} e^{-2s\alpha}\left|\partial_t \operatorname{curl}\widetilde{\varphi}\right|^2\,dx\,dt &\leqslant C\iint_{(0,T)\times\omega_2} e^{-2s\alpha_\flat}\theta\left|\partial_t\nabla\widetilde{\varphi}\right|^2\,dx\,dt\\ &= C\iint_{(0,T)\times\omega_2} e^{-2s\alpha_\flat}\left(\frac{\Delta\theta}{2}\left|\partial_t\widetilde{\varphi}\right|^2-\theta\partial_t\widetilde{\varphi}\cdot\partial_t\Delta\widetilde{\varphi}\right)\,dx\,dt.\\ &\leqslant C\iint_{(0,T)\times\omega_2}\gamma_1^{-2}e^{-4s\alpha_\flat}|\partial_t\widetilde{\varphi}|^2\,dx\,dt+C\iint_{(0,T)\times\omega_2}\gamma_1^2|\partial_t\Delta\widetilde{\varphi}|^2\,dx\,dt. \qquad (68)\end{aligned}$$

We can estimate the first term of the right-hand side of (68) by integrating by parts in time and by using (55):

$$\begin{aligned}\iint_{(0,T)\times\omega_2}\gamma_1^{-2}e^{-4s\alpha_\flat}\left|\partial_t\widetilde{\varphi}\right|^2\,dx\,dt &= \iint_{(0,T)\times\omega_2}\left(-\gamma_1^{-2}e^{-4s\alpha_\flat}\partial_{tt}\widetilde{\varphi}\cdot\widetilde{\varphi}+\frac{1}{2}\left(\gamma_1^{-2}e^{-4s\alpha_\flat}\right)''\left|\widetilde{\varphi}\right|^2\right)dx\,dt\\ &\leqslant \iint_{(0,T)\times\omega_2}\left(\gamma_1^{-6}e^{-8s\alpha_\flat}\left|\widetilde{\varphi}\right|^2+\frac{1}{2}\left(\gamma_1^{-2}e^{-4s\alpha_\flat}\right)''\left|\widetilde{\varphi}\right|^2+\gamma_1^2\left|\partial_{tt}\widetilde{\varphi}\right|^2\right)dx\,dt. \qquad (69)\end{aligned}$$

Combining the above estimate with (68), we deduce

$$\begin{aligned}\iint_{(0,T)\times\omega_1} e^{-2s\alpha}\left|\partial_t\operatorname{curl}\widetilde{\varphi}\right|^2\,dx\,dt &\leqslant C\iint_{(0,T)\times\omega_2}\left(\gamma_1^{-6}e^{-8s\alpha_\flat}+\frac{1}{2}\left(\gamma_1^{-2}e^{-4s\alpha_\flat}\right)''\right)\left|\widetilde{\varphi}\right|^2\,dx\,dt\\ &\quad + C\iint_{(0,T)\times\omega_2}\gamma_1^2\left(\left|\partial_t\Delta\widetilde{\varphi}\right|^2+\left|\partial_{tt}\widetilde{\varphi}\right|^2\right)dx\,dt. \qquad (70)\end{aligned}$$

From (18) and (54) we have

$$\frac{1}{2}\left(\gamma_1^{-2}e^{-4s\alpha_\flat}\right)''\leqslant C\gamma_1^{-6}e^{-8s\alpha_\flat}$$

and thus combining (70) and (44), we obtain

$$\begin{aligned}\iint_{(0,T)\times\omega_1} e^{-2s\alpha}\left|\partial_t\operatorname{curl}\widetilde{\varphi}\right|^2\,dx\,dt &\leqslant C\iint_{(0,T)\times\omega_2}\gamma_1^{-6}e^{-8s\alpha_\flat}\left|\widetilde{\varphi}\right|^2\,dx\,dt\\ &\quad + C\left(\left\|Ts^{2-1/m}\lambda\xi^2e^{-s\alpha}\widetilde{\varphi}\right\|^2_{L^2(0,T;L^2(\Omega))}+\left\|\rho g\right\|^2_{L^2(0,T;L^2(\Omega))}\right). \qquad (71)\end{aligned}$$

With a similar calculation, we can also show that

$$\iint_{(0,T)\times\omega_1} e^{-2s\alpha}\left|\Delta\operatorname{curl}\widetilde{\varphi}\right|^2\,dx\,dt\leqslant C\iint_{(0,T)\times\omega_2}\gamma_1^{-6}e^{-8s\alpha_\flat}\left|\widetilde{\varphi}\right|^2\,dx\,dt+C\left\|\gamma_1\widetilde{\varphi}\right\|^2_{L^2(0,T;H^4(\Omega))},$$

and thus, with (44), we deduce

$$\begin{aligned}\iint_{(0,T)\times\omega_1} e^{-2s\alpha}\left|\Delta\operatorname{curl}\widetilde{\varphi}\right|^2\,dx\,dt &\leqslant C\iint_{(0,T)\times\omega_2}\gamma_1^{-6}e^{-8s\alpha_\flat}\left|\widetilde{\varphi}\right|^2\,dx\,dt\\ &\quad + C\left(\left\|Ts^{2-1/m}\lambda\xi^2e^{-s\alpha}\widetilde{\varphi}\right\|^2_{L^2(0,T;L^2(\Omega))}+\left\|\rho g\right\|^2_{L^2(0,T;L^2(\Omega))}\right). \qquad (72)\end{aligned}$$

We can also estimate the following local term in (61):

$$\begin{aligned}\iint_{(0,T)\times\omega_1} s^3\lambda^4\xi^3e^{-2s\alpha}\left|\operatorname{curl}\widetilde{\varphi}\right|^2\,dx\,dt &\leqslant \iint_{(0,T)\times\omega_2}\theta s^3\lambda^4\xi^3e^{-2s\alpha}\left|\operatorname{curl}\widetilde{\varphi}\right|^2\,dx\,dt\\ &= \iint_{(0,T)\times\omega_2}\operatorname{curl}\left(\theta s^3\lambda^4\xi^3e^{-2s\alpha}\operatorname{curl}\widetilde{\varphi}\right)\cdot\widetilde{\varphi}\,dx\,dt\\ &\leqslant C\iint_{(0,T)\times\omega_2}s^4\lambda^5\xi^4e^{-2s\alpha}\left|\operatorname{curl}\widetilde{\varphi}\right|\left|\widetilde{\varphi}\right|\,dx\,dt+C\iint_{(0,T)\times\omega_2}s^3\lambda^4\xi^3e^{-2s\alpha}\left|\Delta\widetilde{\varphi}\right|\left|\widetilde{\varphi}\right|\,dx\,dt.\end{aligned}$$

Thus for any $\varepsilon>0$, there exists C_ε such that

$$\begin{aligned}\iint_{(0,T)\times\omega_1} s^3\lambda^4\xi^3e^{-2s\alpha}\left|\operatorname{curl}\widetilde{\varphi}\right|^2\,dx\,dt &\leqslant C_\varepsilon\iint_{(0,T)\times\omega_2}s^5\lambda^6\xi^5e^{-2s\alpha}\left|\widetilde{\varphi}\right|^2\,dx\,dt\\ &\quad +\varepsilon\left(\iint_{(0,T)\times\omega_2}s^3\lambda^4\xi^3e^{-2s\alpha}\left|\operatorname{curl}\widetilde{\varphi}\right|^2\,dx\,dt+\iint_{(0,T)\times\omega_2}s\lambda^2\xi e^{-2s\alpha}\left|\Delta\widetilde{\varphi}\right|^2\,dx\,dt\right). \qquad (73)\end{aligned}$$

Finally, for the boundary term in (61), we use a trace property and an interpolation inequality:

$$\iint_{(0,T)\times\partial\Omega} s\lambda\xi_\sharp e^{-2s\alpha_\sharp}\left|\frac{\partial}{\partial n}\operatorname{curl}\widetilde{\varphi}\right|^2 d\gamma\,dt \leqslant C\int_0^T s\lambda\xi_\sharp e^{-2s\alpha_\sharp}\left\|\nabla\operatorname{curl}\widetilde{\varphi}\right\|_{L^2(\Omega)}\left\|\nabla\operatorname{curl}\widetilde{\varphi}\right\|_{H^1(\Omega)} dt$$
$$\leqslant C\int_0^T s\lambda\xi_\sharp e^{-2s\alpha_\sharp}\left\|\operatorname{curl}\widetilde{\varphi}\right\|_{L^2(\Omega)}\left\|\widetilde{\varphi}\right\|_{H^4(\Omega)} dt.$$

Thus, using that $m\geqslant 4$, we obtain that for any $\varepsilon>0$ there exists $C_\varepsilon>0$ such that

$$\begin{aligned}\iint_{(0,T)\times\partial\Omega} & s\lambda\xi_\sharp e^{-2s\alpha_\sharp}\left|\frac{\partial}{\partial n}\operatorname{curl}\widetilde{\varphi}\right|^2 d\gamma\,dt\\
&\leqslant \varepsilon\iint_{(0,T)\times\Omega} s^3\lambda^4\xi_\sharp^3 e^{-2s\alpha_\sharp}\left|\operatorname{curl}\widetilde{\varphi}\right|^2 dx\,dt + C\lambda^{-8}\left\|\gamma_1\widetilde{\varphi}\right\|^2_{L^2(0,T;H^4(\Omega))}\\
&\leqslant \varepsilon\iint_{(0,T)\times\Omega} s^3\lambda^4\xi_\sharp^3 e^{-2s\alpha_\sharp}\left|\operatorname{curl}\widetilde{\varphi}\right|^2 dx\,dt\\
&\quad + C\lambda^{-8}\left(\left\|Ts^{2-1/m}\lambda\xi^2 e^{-s\alpha}\widetilde{\varphi}\right\|^2_{L^2(0,T;L^2(\Omega))} + \left\|\rho g\right\|^2_{L^2(0,T;L^2(\Omega))}\right).\end{aligned}$$

Gathering the above estimate with (61), (65), (71), (72), (73) yields

$$\begin{aligned}I_0(s,\lambda,\widetilde{\varphi}) \leqslant C\Bigg[&\iint_{(0,T)\times\omega_2}\gamma_1^{-6}e^{-8s\alpha_\flat}\left|\widetilde{\varphi}\right|^2 dx\,dt + \iint_{(0,T)\times\omega_2} s^5\lambda^6\xi^5 e^{-2s\alpha}\left|\widetilde{\varphi}\right|^2 dx\,dt\\
&+\left\|Ts^{2-1/m}\lambda^3\xi^2 e^{-s\alpha}\widetilde{\varphi}\right\|^2_{L^2(0,T;L^2(\Omega))} + \left\|\rho g\right\|^2_{L^2(0,T;L^2(\Omega))}\\
&+\varepsilon\left(\iint_{(0,T)\times\omega_2} s^3\lambda^4\xi^3 e^{-2s\alpha}\left|\operatorname{curl}\widetilde{\varphi}\right|^2 dx\,dt + \iint_{(0,T)\times\omega_2} s\lambda^2\xi e^{-2s\alpha}\left|\Delta\widetilde{\varphi}\right|^2 dx\,dt\right)\Bigg].\end{aligned}$$

From (60), for $s\geqslant C(T^m+T^{2m})$ and $\varepsilon>0$ small enough, the above relation implies

$$\begin{aligned}&I_0(s,\lambda,\widetilde{\varphi})\\
&\leqslant C\left[\iint_{(0,T)\times\omega_2}\gamma_1^{-6}e^{-8s\alpha_\flat}\left|\widetilde{\varphi}\right|^2 dx\,dt + \iint_{(0,T)\times\omega_2} s^5\lambda^6\xi^5 e^{-2s\alpha}\left|\widetilde{\varphi}\right|^2 dx\,dt + \left\|\rho g\right\|^2_{L^2(0,T;L^2(\Omega))}\right].\end{aligned} \tag{74}$$

From (18) and (54), we have

$$s^5\lambda^6\xi^5 e^{-2s\alpha}\leqslant C\gamma_1^{-6}e^{-8s\alpha_\flat}$$

and thus combining this with (38) and (39), yields

$$\begin{aligned}\iint_{(0,T)\times\Omega} & s^4\lambda^6\xi_\sharp^4 e^{-5s\alpha_\sharp}\left|\varphi\right|^2 dx\,dt\\
&\leqslant C\left[\iint_{(0,T)\times\omega_2}\gamma_1^{-6}e^{-8s\alpha_\flat-3s\alpha_\sharp}\left|\varphi\right|^2 dx\,dt + \left\|\rho g\right\|^2_{L^2(0,T;L^2(\Omega))}\right].\end{aligned} \tag{75}$$

4. Proof of Corollary 4

The proof of Corollary 4 is completely standard and we only present the main ideas to prove it from Theorem 1.

We define the space

$$\mathcal{X}_0 := \left\{(\varphi,\pi)\in C^\infty\left([0,T]\times\overline{\Omega}\right) : \operatorname{div}\varphi = 0,\quad \varphi = 0 \text{ on } (0,T)\times\partial\Omega,\quad \int_\Omega \pi\,dx = 0\right\}$$

the operators

$$L^*\varphi := -\partial_t\varphi - \mu\Delta\varphi + \left(\int_\Omega a\cdot\varphi\,dx\right)b + \left(\nabla\overline{v}\right)^\top\varphi - \overline{v}\cdot\nabla\varphi,$$

and
$$\left\langle (\varphi,\pi),(\check{\varphi},\check{\pi})\right\rangle_{\mathcal{X}} := \iint_{(0,T)\times\Omega} \rho_1^2 \left(L^*\varphi + \nabla\pi\right)\cdot\left(L^*\check{\varphi}+\nabla\check{\pi}\right)\,dx\,dt + \iint_{(0,T)\times\omega} \rho_2^2 \varphi\cdot\check{\varphi}\,dx\,dt.$$
From (25), we deduce that
$$\left\|(\varphi,\pi)\right\|_{\mathcal{X}} := \left\langle (\varphi,\pi),(\varphi,\pi)\right\rangle_{\mathcal{X}}^{1/2} \geqslant C\left(\iint_{(0,T)\times\Omega} \rho_3^2 |\varphi|^2\,dx\,dt\right)^{1/2} + C\left\|\varphi(0,\cdot)\right\|_{L^2(\Omega)} \tag{76}$$
and thus $\|\cdot\|_{\mathcal{X}}$ is a norm and we can define the completion $\mathcal{X}$ of $\mathcal{X}_0$ for this norm.

We also define
$$\ell\left((\check{\varphi},\check{\pi})\right) := \iint_{(0,T)\times\Omega} f\cdot\check{\varphi}\,dx\,dt + \int_\Omega z^0\cdot\check{\varphi}(0,\cdot)\,dx.$$
From (76), we deduce that ℓ is a linear continuous form of $\mathcal{X}$ and
$$\|\ell\|_{\mathcal{X}'} \leqslant C\left(\left\|\frac{f}{\rho_3}\right\|_{L^2((0,T)\times\Omega)} + \left\|z^0\right\|_{L^2(\Omega)}\right).$$
Thus from the Riesz theorem, there exists a unique $(\varphi,\pi)\in\mathcal{X}$ such that
$$\forall\,(\check{\varphi},\check{\pi})\in\mathcal{X},\quad \left\langle (\varphi,\pi),(\check{\varphi},\check{\pi})\right\rangle_{\mathcal{X}} = \ell\left((\check{\varphi},\check{\pi})\right). \tag{77}$$
We set
$$z := \rho_1^2\left(L^*\varphi+\nabla\pi\right),\quad u := -\rho_2^2\varphi, \tag{78}$$
and from (77), we deduce that
$$\left\|\frac{z}{\rho_1}\right\|_{L^2((0,T)\times\Omega)} + \left\|\frac{u}{\rho_2}\right\|_{L^2((0,T)\times\Omega)} \leqslant C\left(\left\|\frac{f}{\rho_3}\right\|_{L^2((0,T)\times\Omega)} + \left\|z^0\right\|_{L^2(\Omega)}\right) \tag{79}$$
and that
$$\iint_{(0,T)\times\Omega} z\cdot\left(L^*\check{\varphi}+\nabla\check{\pi}\right)\,dx\,dt = \iint_{(0,T)\times\omega} u\cdot\check{\varphi}\,dx\,dt + \iint_{(0,T)\times\Omega} f\cdot\check{\varphi}\,dx\,dt + \int_\Omega z^0\cdot\check{\varphi}(0,\cdot)\,dx.$$
The last relation yields that z is a weak solution of (6). We recall that ρ_0 is defined by (27). We can check that
$$\begin{cases} \partial_t\left(\dfrac{z}{\rho_0}\right) - \mu\Delta\left(\dfrac{z}{\rho_0}\right) + \left(\displaystyle\int_\Omega b\cdot\left(\dfrac{z}{\rho_0}\right)\,dy\right)a + \left(\dfrac{z}{\rho_0}\right)\cdot\nabla\overline{v} & \\ +\overline{v}\cdot\nabla\left(\dfrac{z}{\rho_0}\right) + \nabla\left(\dfrac{q}{\rho_0}\right) = \left(\dfrac{f}{\rho_0}\right) + 1_\omega\left(\dfrac{u}{\rho_0}\right) - \left(\dfrac{\rho_0' z}{\rho_0^2}\right) & \text{in } (0,T)\times\Omega, \\ \operatorname{div}\left(\dfrac{z}{\rho_0}\right) = 0 & \text{in } (0,T)\times\Omega, \\ \left(\dfrac{z}{\rho_0}\right) = 0 & \text{on } (0,T)\times\partial\Omega, \\ \left(\dfrac{z}{\rho_0}\right)(0,\cdot) = z^0 & \text{in } \Omega, \end{cases} \tag{80}$$
and that
$$\frac{\rho_2}{\rho_0},\ \frac{\rho_3}{\rho_0},\ \frac{\rho_0'\rho_1}{\rho_0^2} \in L^\infty(0,T),$$
and thus using Lemma 5 and (79), we deduce that
$$\left\|\frac{z}{\rho_0}\right\|_{X_1} \leqslant C\left(\left\|\frac{f}{\rho_3}\right\|_{L^2((0,T)\times\Omega)} + \left\|z^0\right\|_{H^1(\Omega)}\right). \tag{81}$$
This implies in particular that $z(T,\cdot) = 0$.

In order to prove the local null controllability of (4), we define
$$\mathcal{F}_3 := \left\{f\,;\ \frac{f}{\rho_3}\in L^2\left((0,T)\times\Omega\right)\right\}$$

and the mapping

$$\mathcal{N} : \mathcal{F}_3 \to \mathcal{F}_3, \quad f \mapsto F(z)$$

where z is the above solution (that is given by (78)) and where $F(z)$ is defined by (5).

Using that

$$\frac{\rho_0^2}{\rho_3} \in L^\infty(0,T),$$

we can check that the map $\mathcal{N}$ is well-defined and from (81), we can also show that if $\|z^0\|_{H^1(\Omega)} \leqslant r$ and if r is small enough, the closed ball

$$\mathcal{B}_3 := \left\{ f \in \mathcal{F}_3 \, ; \, \left\| \frac{f}{\rho_3} \right\|_{L^2((0,T)\times\Omega)} \leqslant r \right\}$$

is invariant by $\mathcal{N}$ and is a strict contraction on this set. This yields the existence of a fixed point for $\mathcal{N}$. The corresponding solution z satisfies (4), and since $\frac{z}{\rho_0} \in X_1$, we deduce that $z(T,\cdot) = 0$.

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