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The heredity and bimeromorphic invariance of the $\partial\bar{\partial}$-lemma property

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Abstract. We give a simple proof of a result on the $\partial\bar{\partial}$-lemma property under a blow-up transformation by Deligne–Griffiths–Morgan–Sullivan's criterion. Here, we use an explicit blow-up formula for Dolbeault cohomology given in our previous work, which can be induced by a morphism expressed on the level of spaces of forms and currents. At last, we discuss the heredity and bimeromorphic invariance of the $\partial\bar{\partial}$-lemma property.

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1. Introduction

In non-Kähler geometry, the heredity and bimeromorphic invariance of the $\partial\bar{\partial}$-lemma property are two interesting problems, extensively studied in [2, 3, 6, 7, 12, 15–17] especially in the recent days. The *$\partial\bar{\partial}$-lemma* on a compact complex manifold X refers to that for every pure-type d-closed form on X, the properties of d-exactness, ∂-exactness, $\bar{\partial}$-exactness and $\partial\bar{\partial}$-exactness are equivalent while a compact complex manifold is called a *$\partial\bar{\partial}$-manifold* if the *$\partial\bar{\partial}$-lemma* holds on it.

Question 1 (Heredity). *Does any closed complex submanifold of an n-dimensional $\partial\bar{\partial}$-manifold still satisfy the $\partial\bar{\partial}$-lemma?*

Question 2 (Bimeromorphic invariance). *Does any compact complex manifold being bimeromorphic to an n-dimensional $\partial\bar{\partial}$-manifold satisfy the $\partial\bar{\partial}$-lemma?*

Clearly, the heredity is true for the $\partial\bar{\partial}$-manifolds of dimensions ≤ 2. Suppose that $\widetilde{X}$ is a modification of a compact complex manifold X. A. Parshin [11] and P. Deligne, Ph. Griffiths, J. Morgan, D. Sullivan [6] proved that if $\widetilde{X}$ is a $\partial\bar{\partial}$-manifold, then so is X. L. Alessandrini [2] posed a question in its inverse direction: if X satisfies the $\partial\bar{\partial}$-lemma, so does $\widetilde{X}$? We can easily prove that, Question 2 is equivalent to Alessandrini's one. It is true on complex surfaces by the classical results that each compact complex surface with even first Betti number is Kähler (see [5, 8] for a uniform proof) and the first Betti number is a bimeromorphic invariant, while the case of threefolds was first proved by S. Rao, S. Yang, X.-D. Yang [12] using a Dolbeault blow-up formula and S. Yang, X.-D. Yang [17] using a Bott–Chern blow-up formula. The general case is still open. For any nonnegative integer $k \le n$, we weaken Question 1 as

Question 3 (Heredity for codimension $\ge k$). *Does any closed complex submanifold of codimension $\ge k$ of an n-dimensional $\partial\bar\partial$-manifolds still satisfy the $\partial\bar\partial$-lemma?*

For convenience, Questions 1-3 are denoted by (H_n), (B_n) and $(H_{n,k})$, respectively. Obviously, $(H_n) = (H_{n,0}) \Leftrightarrow (H_{n,1})$ and if $k_1 \le k_2$, then $(H_{n,k_1}) \Rightarrow (H_{n,k_2})$.

P. Deligne et al. [6, (5.21)] gave an important result, which related the $\partial\bar\partial$-lemma property with Hodge filtration and the degeneracy of the Frölicher spectral sequence at E_1-page. S. Rao, S. Yang and X.-D. Yang [12, Theorem 1.6] investigated the bimeromorphic invariance of the degeneracy of Frölicher spectral sequence at E_1 by their Dolbeault blow-up formula and pointed out that these results are applicable to Question 2 in the remarks after [12, Question 1.2]. Subsequently, their [13, Theorem 1.2] gave an explicit expression of the isomorphism between Dolbeault cohomologies in the blow-up formula to implicitly obtain $(B_n) \Leftrightarrow (H_{n,2})$ via Proposition 9 indeed. D. Angella, T. Suwa, N. Tardini and A. Tomassini [3, Theorem 13, Questions 22-24] also studied this equivalence by the Čech–Dolbeault cohomology with additional hypotheses and generalized their results to compact complex orbifolds. In his PhD thesis, by Angella–Tomassini's characterization [4, Theorems A and B], J. Stelzig [15, Corollary F] claimed that the $\partial\bar\partial$-lemma property is a bimeromorphic invariant of compact complex manifolds if and only if every submanifold of a $\partial\bar\partial$-manifold is again a $\partial\bar\partial$-manifold. Inspired by them, we will prove the following theorem.

Theorem 4. *For any integer $k \in \{1,2,\dots,n\}$, there holds the implication hierarchy*

$$(B_{n+k}) \Rightarrow (H_{n+k,k+1}) \Rightarrow (H_n).$$

Moreover, $(H_{n,2}) \Rightarrow (B_n)$.

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2. Preliminaries

2.1. *A criterion on the $\partial\bar\partial$-lemma*

For a compact complex manifold X, a natural filtration on the complex $A^\bullet(X)_{\mathbb{C}}$ of $\mathbb{C}$-valued smooth forms on X is defined as

$$F^p A^k(X)_{\mathbb{C}} = \bigoplus_{\substack{r+s=k\\ r\ge p}} A^{r,s}(X),$$

for all k, p, which give a spectral sequence $(E_r^{p,q}, F^p H^k(X,\mathbb{C}))$, namely, the *Frölicher spectral sequence* of X. Then $E_1^{p,q} = H_{\bar\partial}^{p,q}(X)$ and

$$F^p H^k(X,\mathbb{C}) = \left\{[\alpha] \in H^k(X,\mathbb{C}) \,\middle|\, \alpha \in F^p A^k(X) \text{ and } d\alpha = 0\right\}. \tag{1}$$

Clearly, $F^p H^k(X,\mathbb{C}) = 0$ for $p < 0$ or $p > k$. For convenience, we call $F^\bullet H^k(X,\mathbb{C})$ the *Hodge filtration* on $H^k(X,\mathbb{C})$. Set $V^{p,q}(X) = F^p H^k(X,\mathbb{C}) \cap \overline{F}^q H^k(X,\mathbb{C})$ for $p+q=k$, where $\overline{F}^q H^k(X,\mathbb{C})$ is the complex conjugation of the complex subspace $F^qH^k(X,\mathbb{C})$ in $H^k(X,\mathbb{C})$. We say that *the Hodge filtration gives a Hodge structure of weight k on $H^k(X,\mathbb{C})$*, if

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} V^{p,q}(X), \tag{2}$$

and

$$\overline{V^{p,q}(X)} = V^{q,p}(X), \text{ for any } p+q=k. \tag{3}$$

P. Deligne, Ph. Griffiths, J. Morgan and D. Sullivan established the well-known criterion on the $\partial\bar\partial$-lemma as follows.

Theorem 5 (cf. [6, (5.21)]). *For a compact complex manifold X, the following statements are equivalent:*

(1) *X satisfies the $\partial\bar\partial$-lemma.*

(2) (a) *The Frölicher spectral sequence of X degenerates at E_1, and*
    (b) *the Hodge filtration gives a Hodge structure of weight k on $H^k(X,\mathbb{C})$, for every $k \geq 0$.*

Remark 6. For a compact complex manifold X, denote by $b_k(X)$, $h^{p,q}(X)$ the k-th Betti, (p,q)-th Hodge numbers respectively.

(1) In general, $b_k(X) \leq \sum_{p+q=k} h^{p,q}(X)$ for all k.

(2) The statement of Theorem 5 (2a) is equivalent to that $F^pH^k(X,\mathbb{C})/F^{p+1}H^k(X,\mathbb{C}) \cong H^{p,k-p}_{\bar\partial}(X)$ for all k, p, and hence is equivalent to that $b_k(X) = \sum_{p+q=k} h^{p,q}(X)$ for all k.

We refer to [3, Section 1.5] and [14, Section 2.3] for more discussions on the Frölicher spectral sequence and the Hodge structure.

2.2. *Some notations*

Assume that X is a complex manifold with complex dimension n. Denote by $\mathscr{D}'^{p,q}(X)$ the space of (p,q)-currents on X, which is defined as the dual of the topological vector space $A^{n-q,n-q}(X)$ equipped with its natural topology. The operators ∂ and $\bar\partial$ on $A^{\bullet,\bullet}(X)$ naturally induce two differentials ∂ and $\bar\partial$ on $\mathscr{D}'^{\bullet,\bullet}(X)$. Evidently, $(A^{\bullet,\bullet}(X),\partial,\bar\partial)$ and $(\mathscr{D}'^{\bullet,\bullet}(X),\partial,\bar\partial)$ are both double complexes. Denote by $H^q(\mathscr{D}'^{p,\bullet}(X))$ the q-th cohomology of the complex $(\mathscr{D}'^{p,\bullet}(X),\bar\partial)$. The natural inclusion $A^{p,\bullet}(X) \hookrightarrow \mathscr{D}'^{p,\bullet}(X)$ induces an isomorphism $\rho_X : H^{p,q}_{\bar\partial}(X) \tilde{\to} H^q(\mathscr{D}'^{p,\bullet}(X))$.

Let $f : X \to Y$ be a proper holomorphic map between complex manifolds. Set $r = \dim_{\mathbb{C}} X - \dim_{\mathbb{C}} Y$. The pushforward $f_* : \mathscr{D}'^{\bullet,\bullet}(X) \to \mathscr{D}'^{\bullet-r,\bullet-r}(Y)$ of the currents defines a morphism $f_* : H^q(\mathscr{D}'^{p,\bullet}(X)) \to H^{q-r}(\mathscr{D}'^{p-r,\bullet}(Y))$ for any p, q. For convenience, we also denote by f_* the morphism $\rho_Y \circ f_* \circ \rho_X^{-1} : H^{p,q}_{\bar\partial}(X) \to H^{p-r,q-r}_{\bar\partial}(Y)$.

3. The Hodge structures on blow-ups and projective bundles

3.1. *Blow-up cases*

Let $\pi : \widetilde{X} \to X$ be the blow-up of a compact complex manifold X along a complex submanifold Y and E the exceptional divisor. Set $r = \operatorname{codim}_{\mathbb{C}} Y \geq 2$ and assume that $i_E : E \to \widetilde{X}$ is the inclusion. Let $t \in \mathscr{A}^{1,1}(E)$ be a Chern form of the universal line bundle $\mathscr{O}_E(-1)$ on $E = \mathbb{P}(N_{Y/X})$. Define a double complex

$$K^{\bullet,\bullet} = A^{\bullet,\bullet}(X) \oplus \bigoplus_{i=1}^{r-1} A^{\bullet-i,\bullet-i}(Y).$$

and a morphism of bounded double complexes

$$\psi : K^{\bullet,\bullet} \to \mathscr{D}'^{\bullet,\bullet}(\widetilde{X})$$

as

$$(\alpha,\beta^1,\ldots,\beta^{r-1})\mapsto \pi^*\alpha+\sum_{i=1}^{r-1} i_{E*}\left(t^{i-1}\wedge(\pi|_E)^*\beta^i\right),$$

where $\alpha\in A^{\bullet,\bullet}(X)$ and $\beta^i\in A^{\bullet-i,\bullet-i}(Y)$. By [10, Theorem 1.2], ψ induces an isomorphism

$$H_{\bar\partial}^{\bullet,\bullet}(X)\oplus\bigoplus_{i=1}^{r-1} H_{\bar\partial}^{\bullet-i,\bullet-i}(Y)\tilde\to H_{\bar\partial}^{\bullet,\bullet}(\widetilde X), \tag{4}$$

i.e., the isomorphism on E_1-pages between the spectral sequences associated to $K^{\bullet,\bullet}$ and $\mathscr{D}'^{\bullet,\bullet}(\widetilde X)$. Hence ψ induces an isomorphism $H^k(X,\mathbb{C})\oplus\bigoplus_{i=1}^{r-1}H^{k-2i}(Y,\mathbb{C})\tilde\to H^k(\widetilde X,\mathbb{C})$ with the isomorphism on the Hodge filtrations

$$F^\bullet H^k(X,\mathbb{C})\oplus\bigoplus_{i=1}^{r-1}F^{\bullet-i}H^{k-2i}(Y,\mathbb{C})\tilde\to F^\bullet H^k(\widetilde X,\mathbb{C}) \tag{5}$$

for any k. Moreover, ψ induces an isomorphism

$$V^{p,q}(X)\oplus\bigoplus_{i=1}^{r-1}V^{p-i,q-i}(Y)\tilde\to V^{p,q}(\widetilde X)$$

for any p, q.

Lemma 7. *For a given k, the Hodge filtration gives a Hodge structure of weight k on $H^k(\widetilde X,\mathbb{C})$, if and only if, the Hodge filtrations give a Hodge structure of weight k on $H^k(X,\mathbb{C})$ and a Hodge structure of weight $k-2i$ on $H^{k-2i}(Y,\mathbb{C})$ for all $1\le i\le r-1$.*

By (4), (5) and Remark 6, we easily obtain

Lemma 8 ([12, Theorem 1.6]). *The Frölicher spectral sequence of $\widetilde X$ degenerates at E_1, if and only if, so do those of X and Y.*

Combining Lemmas 7, 8 and Theorem 5, we get

Proposition 9. *Let $\widetilde X$ be the blow-up of a compact complex manifold X along a complex submanifold Y of complex codimension ≥ 2. Then $\widetilde X$ satisfies the $\partial\bar\partial$-lemma, if and only if, X and Y do.*

Remark 10. S. Rao, S. Yang, X.-D. Yang [12, Theorem 1.6] [13, Theorem 1.2] first understood Proposition 9 from the viewpoint of Deligne–Griffiths–Morgan–Sullivan's criterion for the $\partial\bar\partial$-lemma and S. Yang, X.-D. Yang [17, Theorem 1.3] studied it from the viewpoint of Angella–Tomassini's characterization for the case of threefolds. Shortly, D. Angella, T. Suwa, N. Tardini, A. Tomassini [3, Theorem 13] also considered it by use of the Čech–Dolbeault cohomology under some additional assumptions. Eventually, J. Stelzig obtianed a blow-up formula for Bott–Chern cohomology and wrote this result out explicitly in [15, Corollary 1.40] [4, Theorems A and B].

Remark 11. S. Rao, S. Yang, X.-D. Yang [13, Theorem 1.2] gave an isomorphism for blow-up in the inverse direction of ψ as

$$\phi: H_{\bar\partial}^{\bullet,\bullet}(\widetilde X)\tilde\to H_{\bar\partial}^{\bullet,\bullet}(X)\oplus\bigoplus_{i=1}^{r-1}H_{\bar\partial}^{\bullet-i,\bullet-i}(Y),$$
$$\alpha\mapsto(\pi_*\alpha,\beta^1,\ldots,\beta^{r-1}),$$

where $i_E^*\alpha=\sum_{i=0}^{r-1}h^i\cup(\pi|_E)^*\beta^i$ for unique $\beta^i\in H_{\bar\partial}^{\bullet-i,\bullet-i}(Y)$, $0\le i\le r-1$ and $h=[t]_{\bar\partial}\in H_{\bar\partial}^{1,1}(E)$. Actually, ϕ can also be lifted to a morphism between complexes of the spaces of forms and currents, see [9, Lemma 6.5]. Using this morphism, we can also give the relationship between $V^{p,q}(X)$, $V^{p,q}(Y)$ and $V^{p,q}(\widetilde X)$ by above progress.

As we know, the exceptional divisor for the blow-up $\widetilde{X}$ of X along Y is biholomorphic to the projective bundle of the normal bundle over Y in X. By Proposition 9 and the following Proposition 15, we easily get

Corollary 12. *Let $\widetilde{X}$ be a blow-up of a complex manifold X along a smooth center with the exceptional divisor E. Then $\widetilde{X}$ is a $\partial\bar\partial$-manifold, if and only if, X and E are both $\partial\bar\partial$-manifolds.*

3.2. *Projective bundle cases*

Let $\pi : \mathbb{P}(E) \to X$ be the projective bundle associated to a holomorphic vector bundle E of rank r over a compact complex manifold X. Denote by $t \in \mathscr{A}^{1,1}(\mathbb{P}(E))$ a Chern form of $\mathscr{O}_{\mathbb{P}(E)}(-1)$. Define a morphism

$$\mu = \sum_{i=0}^{r-1} t^i \wedge \pi^*(\bullet) : \bigoplus_{i=0}^{r-1} A^{\bullet-i,\bullet-i}(X) \to A^{\bullet,\bullet}(\mathbb{P}(E))$$

of bounded double complexes. Then μ induces an isomorphism on E_1-pages of the spectral sequences, see [12, Proposition 3.3], [3, Proposition 11] or [10, Corollary 3.2]. With the similar arguments as Section 3.1, we can prove following results

Lemma 13. *For a given k, the Hodge filtration gives a Hodge structure of weight k on $H^k(\mathbb{P}(E),\mathbb{C})$, if and only if, the Hodge filtration gives a Hodge structure of weight $k-2i$ on $H^{k-2i}(X,\mathbb{C})$.*

Lemma 14. *The Frölicher spectral sequence of $\mathbb{P}(E)$ degenerates at E_1, if and only if, so does that of X.*

Proposition 15. *Let $\mathbb{P}(E)$ be the projective bundle associated to a holomorphic vector bundle E on a compact complex manifold X. Then $\mathbb{P}(E)$ is a $\partial\bar\partial$-manifold, if and only if, X is a $\partial\bar\partial$-manifold.*

Remark 16. The part of "if" in Proposition 15 was also proved by D. Angella et al. [3, Corollary 12] in a different way.

4. A proof of Theorem 4

Proof. Here we just prove $(H_{n+k,k+1}) \Rightarrow (H_n)$ and the others are the direct corollary of Proposition 9 and the weak factorization theorem [1, Theorem 0.3.1].

Let X be a $\partial\bar\partial$-manifold and Y arbitrary closed complex submanifold of codimension ≥ 1 in X. Note that $X \times \mathbb{C}P^k$ is the projective bundle associated to the trivial bundle $X \times \mathbb{C}^{k+1}$ over X and thus satisfies the $\partial\bar\partial$-lemma by Proposition 15. Denote by $\{\mathrm{pt}\}$ a set consisting of a single point in $\mathbb{C}P^k$. Then $Y \cong Y \times \{\mathrm{pt}\}$ has the codimension $\geq k+1$ in $X \times \mathbb{C}P^k$ and satisfies the $\partial\bar\partial$-lemma by $(H_{n+k,k+1})$. □

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