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On non-denseness for a method of fundamental solutions with source points fixed in time for parabolic equations

Sur la non-densité dans une méthode de solutions fondamentales avec des points sources indépendants du temps pour la résolution d'équations paraboliques

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Abstract. Linear combinations of fundamental solutions to the parabolic heat equation with source points fixed in time is investigated. The open problem whether these linear combinations generate a dense set in the space of square integrable functions on the lateral boundary of a space-time cylinder, is settled in the negative. Linear independence of the set of fundamental solutions is shown to hold. It is outlined at the end, for a particular example, that such linear combinations constitute a linearly independent and dense set in the space of square integrable functions on the upper top part (where time is fixed) of the boundary of this space-time cylinder.

Résumé. Des combinaisons linéaires de solutions fondamentales avec des points sources indépendants du temps pour la résolution de l'équation de la chaleur sont étudiées. On étudie la question ouverte de savoir si ces combinaisons linéaires génèrent un ensemble dense dans l'espace des fonctions de carrés intégrables sur la limite latérale d'un cylindre espace-temps et on montre que la réponse à cette question est négative. L'indépendance linéaire de l'ensemble des solutions fondamentales est démontrée. Il est souligné à la fin pour un cas particulier que de telles combinaisons linéaires sont linéairement indépendantes et denses dans l'espace des fonctions de carrés intégrables définies sur la partie supérieure (où le temps est fixe) de la limite du cylindre espace-temps.

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1. Introduction

The method of fundamental solutions, being what is termed a meshless numerical method for partial differential equations, has gained popularity in recent years both for direct and inverse problems, see the surveys [7] and [19]. Research have in particular been prolific for stationary problems, where linear independence and denseness of fundamental solutions have been settled, see for example [1]. For time-dependent equations, a general strategy is to employ some transformation in time to reduce to the stationary case [12, Section 5]. Reverting such a transformation can cause numerical problems, see [5, p. 25] (the Laguerre transform can be an alternative, see [2]). In [16], a method of fundamental solutions for the parabolic heat equation was proposed, building on results from [20], with no transformation in time. Instead, following on from the stationary case, linear combinations of the fundamental solution of the heat equation are used with source points placed on a fictitious lateral boundary enclosing the lateral part of a given space-time cylinder in which the heat equation is posed. This method has then been applied for various other direct and inverse heat problems, see, for example, [4, 17].

A key fact to motivate the MFS in [16] is the linear independence and denseness of linear combinations of fundamental solutions of the heat equation. Proofs thereof are collected in [15] together with convergence of an MFS approximation.

There are alternative versions of an MFS for the parabolic heat equation not involving any time-transformation. For example, in [23] it was suggested to generate linear combinations of fundamental solutions with the source points not along a lateral boundary but to distribute them at the base of the space-time cylinder shifted down to a fixed negative time. It has been an open problem in the community of meshless methods whether placing source points for a fixed time do generate the requested denseness properties.

In the present work, we show that distributing source points at a fixed time does not generate a dense set of approximations on the lateral boundary of the space-time solution cylinder in the space of square integrable functions. We also demonstrate that the approximations restricted to the top part of the cylinder do form a dense set in the corresponding space of square integrable functions.

Settling this open problem in the negative of denseness on the boundary of the approximations for a fixed time and clarifying the denseness on the top part of the space-time cylinder, constitute the novelty of the present work together with stating relevant results needed in the presented proofs.

For the outline of the work, in the present section, we formulate the main result. In Section 2, some results needed in the proof are collected. The proof itself is given in Section 3. In Section 4, some remarks are pointed out and linear independence and denseness on the top part of the cylinder is outlined for a particular case.

We consider the parabolic heat equation

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega \times (0,T), \\ u = \psi & \text{on } \Gamma \times (0,T), \\ u(x,0) = \varphi(x) & \text{for } x \in \Omega. \end{cases} \tag{1}$$

Here, Ω is a bounded domain in $\mathbb{R}^n$, $n = 2,3$, with the boundary surface Γ being simple (no self-intersections) closed (the surface has no boundary and is connected) and is at least Lipschitz smooth. Doubly-connected domains and also one-dimensional spatial domains can be adjusted for. The space-time cylinder is the region $\Omega \times (0,T)$.

We introduce some further notation in order to formulate the main result to be proved. Let

$$F(x,t;y,\tau) = \frac{H(t-\tau)e^{-\frac{|x-y|^2}{4(t-\tau)}}}{(4\pi(t-\tau))^{\frac{n}{2}}} \tag{2}$$

be the standard fundamental solution to the heat equation (1) representing the temperature at the location x and time t resulting from an instantaneous release of a unit point source of thermal energy at position y and time τ, with H the Heaviside function. The fundamental solution has the expected physical properties, for example, it is a positive solution to the heat equation for $t > \tau$, for $x \neq y$ the temperature tends to zero at τ that is $\lim_{t\to\tau^+} F(x,t;y,\tau) = 0$, at the point where the heat source is inserted in space the temperature tends to infinity at time τ, thus $F(x,t;x,\tau)$ tends to infinity as $t \to \tau^+$, and the total energy is one, $\int_{\mathbb{R}^n} F(x,t;y,\tau)\,dy = 1$ when $t > \tau$; for further properties, see [13] and [10, Chapter 1.4-6].

Let $\{y_k\}_{k=1,2,\ldots}$ be a dense set of points in Ω, the bottom part of the cylindrical solution domain. By a dense set in L^2, we mean that the span of the set is dense. The main result is the following:

Theorem 1. *The set of functions $\{F(x,t;y_k,0)\}_{k=1,2,\ldots}$ is linearly independent but not a dense set in $L^2(\Gamma\times(0,T))$. The same holds for the set consisting of the normal derivatives,*

$$\left\{\partial_{\nu(x)}F\left(x,t;y_k,0\right)\right\}_{k=1,2,\ldots}.$$

In the next section, we formulate some results needed in the proof. The case of denseness on the top of the solution cylinder instead of along the lateral boundary is discussed in Section 4.

2. Some results on the parabolic heat equation

We recall two well-posedness results for the heat equation and one on controllability, and start with some notation. The standard notation $L^2(0,T;X)$ is used, where X is a Hilbert space, and $u(\cdot,t):(0,T)\to X$ is measurable with $\|u(\cdot,t)\|_X$ having finite norm in $L^2(0,T)$. The space $H^k(\Omega)$, $k > 0$, is the standard Sobolev space of elements having weak and square integrable derivatives up to order k, with trace space $H^{k-1/2}(\Gamma)$.

Taking the initial function $\varphi \in L^2(\Omega)$ and the element ψ to be sufficiently regular, then there exists a unique weak solution $u \in L^2(0,T;H^1(\Omega))$ to (1) with $u_t \in L^2(0,T;L^2(\Omega))$, and this solution depends continuously on the data.

The Cauchy problem in all of $\mathbb{R}^n$,

$$\begin{cases}\partial_t u - \Delta u = 0 & \text{in } \mathbb{R}^n\times(0,T),\\ u(x,0) = \xi(x) & \text{for } x\in\mathbb{R}^n,\end{cases} \tag{3}$$

with data ξ not growing faster than $e^{c|x|^2}$, for some positive constant c, has a unique solution among solutions satisfying the similar exponential growth bound. References for this result and the above are given in [15, Section 2].

Given initial data to the heat equation, it is possible to choose the boundary data to obtain a designated temperature profile at $t = T$, that is $u(x,T)$ is equal to a prescribed function (provided this function matches the outcome of a heat process, see [8, Theorem 4.1] for a class of admissible profiles $u(x,T)$). Results of this form are usually termed as controllability and null-controllability (when the zero profile is to be obtained at $t = T$); an early work is [9]. For an overview of results, see for example [24, Sections 7–8], [11] and in [22, the introduction]. A consequence of classical results on null-controllability is (see [8, Theorem 4.1] and [21, Theorem 2]):

Proposition 2. *Let $\varphi = 0$. Then there exists a non-trivial element $\psi \in L^2(\Gamma\times(0,T))$ such that the solution u to* (1) *satisfies $u(x,T) = 0$.*

Choosing $\psi = 0$ renders the requested profile but it is the choice of a non-trivial function which is of interest. Physically, it makes sense, one can start with a slab of material having zero temperature, then heat it slightly on the boundary and then cooling to obtain zero temperature again.

In the one-dimensional case, considering for simplicity the heat equation in the rectangle $(0,1)\times(0,1)$ with boundary conditions $u(0,t) = 0$ and $u(1,t) = \psi_1(t)$, an explicit expression of a solution to (1) is

$$u(x,t) = \sum_{k=0}^{\infty} w^{(k)}(t)\frac{x^{2k+1}}{(2k+1)!}.$$

Thus, simply choosing $w(t)$ as a function with a peak in $(0,1)$ and compactly supported in this interval, u will be identically zero for $t = 0$ and $t = T$, and $u(1,t) = \psi_1(t)$ is non-trivial. The choice of w can be $e^{-1/(t(1-t))}$; investigations of convergence for this type of element w is given in [3, Chapter 2.4].

We then turn to the proof of the main theorem.

3. Proof of Theorem 1

We shall then make a proof of Theorem 1.

Linear independence on $\Gamma\times(0,T)$**.** Assume that linear independence does not hold. After a possible renumbering of the points $\{y_k\}_{k=1,2,\ldots}$

$$\sum_{k=1}^{M} c_k F\left(x,t;y_k,0\right) = 0 \quad \text{on} \quad \Gamma\times(0,T) \tag{4}$$

for some integer $M > 0$, with the coefficients c_k all being non-zero. The function

$$u_M(x,t) = \sum_{k=1}^{M} c_k F\left(x,t;y_k,0\right) \tag{5}$$

satisfies the heat equation in $\Omega\times(0,T)$ and is zero on $\Gamma\times(0,T)$. Since there are only a finite number of points y_k in the expression of u_M, we can clearly select a point x_0 on the boundary Γ and an index k_0, with $1 \le k_0 \le M$, such that $|x_0 - y_{k_0}| < |x_0 - y_k|$, for $k = 1,\ldots,M$, $k \neq k_0$. Then, as $t \to 0$, $u_M(x_0,t) \simeq c_{k_0}F(x_0,t;y_{k_0},0)$. Since u_M is supposed to be zero on the boundary and $c_{k_0} \neq 0$, we have a contradiction. We thus conclude that $\{F(x,t;y_k,0)\}_{k=1,2,\ldots}$ is a linearly independent set.

The same idea carries over for the normal derivative.

Non-denseness on $\Gamma\times(0,T)$**.** We can, according to Proposition 2, choose non-trivial boundary data such that the solution to the adjoint equation (simple time-reversal) $\partial_t v + \Delta v = 0$ satisfies $v(T) = 0$ and $v(0) = 0$. Represent v as a single-layer,

$$v\left(y,\tau\right) = \int_\tau^T\int_\Gamma f(x,t)F\left(x,t;y,\tau\right)\,dx\,dt.$$

We can find a non-trivial density f to match the given boundary condition (for properties of the single-layer, see [6, Theorem 3.4]).

Then, with this choice of the density f the element v is zero at $\tau = 0$, hence

$$v\left(y_k,0\right) = \int_0^T\int_\Gamma f(x,t)F\left(x,t;y_k,0\right)\,dx\,dt = 0.$$

This means f is orthogonal in $L^2(\Gamma\times(0,T))$ to $F(x,t;y_k,0)$, and this holds for any point y_k in Ω. Thus, the span of $\{F(x,t;y_k,0)\}_{k=1,2,\ldots}$ is not dense in $L^2(\Gamma\times(0,T))$.

We leave it to the reader to check the similar result for the normal derivative of the fundamental solution, by replacing the single-layer operator with the double-layer. □

4. Some remarks

We point out the following:

(i) The boundary data needed in the proof rendering the solution to be zero both at the initial and final time can be chosen as a function with peaks and compact support on the lateral boundary. Thus, moving the peaks along $(0, T)$ it is seen that the set of functions being orthogonal to the given set of linear combinations of fundamental solutions will not be finite dimensional.

(ii) Physically, Theorem 1 is logical since the future of a temperature distribution does not depend only on the past but also on outside influences up to the current time. Thus, approximating with a set of functions only depending on a previous time can not capture a general temperature distribution.

(iii) In terms of properties of the set of fundamental solutions on the top boundary of the solution cylinder, linear independence holds. Assume on the contrary that the given set is not linearly independent for $t = T$. Then, after a possible re-numbering of the points $\{y_k\}_{k=1,2,\ldots}$,

$$u_M(x,t) = \sum_{k=1}^{M} c_k F\left(x, T; y_k, 0\right) = 0 \quad \text{for } x \in \Omega \tag{6}$$

for some integer $M > 0$, with at least one of the coefficients c_k being non-zero.

The element $F(x, t; y_k, 0)$ is real analytic in the spatial variable in all of $\mathbb{R}^n$ for $t > 0$ (see [18, p. 219]); therefore (6) can be extended to $x \in \mathbb{R}^n$. Clearly, u_M satisfies an exponential bound in the spatial variable x. Hence, from the well-posedness result for the Cauchy problem in $\mathbb{R}^n$, the solution u_M is identically zero in $\mathbb{R}^n$ for say $[T, 2T]$. Due to the analyticity of $F(x, t; y_k, 0)$ for $t > 0$, the function u_M in (6) is identically zero on the lateral boundary $\Gamma \times (0, T)$. From the main result, Theorem 1, restrictions to the boundary of fundamental solutions constitute a linearly independent set. Thus, the coefficients in (6) are all identically zero, and linear independence of the given set for a fixed instance in time is established.

(iv) Denseness on the top part of the cylinder can be seen to hold. We show this when Ω is the cube $]-1,1[^n$ in $\mathbb{R}^n$. Assume that there exists an element $g \in L^2(\Omega)$, with $\int_\Omega g(x) F(x, T; y_k, 0)\, dx = 0$ for $k = 1, 2\ldots$ From (2), this is equivalent to the existence of an element g with

$$\int_\Omega g(x) e^{-\frac{|x-y_k|^2}{4T}}\, dx = 0$$

for $k = 1, 2\ldots$ Let

$$U(y) = \int_\Omega g(x) e^{-\frac{|x-y|^2}{4T}}\, dx.$$

Since U is at least continuous and $\{y_k\}_{k=1,2,\ldots}$ a dense set of points in Ω, it follows that U is identically zero in Ω. In fact, U is real analytic, see [18, p. 219], and therefore U is identically zero in $\mathbb{R}^n$. The Fourier transform of U is then also zero. Using this and standard properties of the transform, we find that the Fourier transform $\widehat{h}$ of $h = \chi_{]-1,1[^n} g$ (χ is the characteristic indicator function) satisfies

$$e^{-T|\xi|^2} \widehat{h}(\xi) = 0.$$

Hence, $h = 0$ and also $g = 0$. Thus, the given set is dense on the top part of the cylinder.

(v) There are variants of [23] placing source points translated in time, see [14, 25]. Non-denseness on the boundary can be investigated also for these variants.

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