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Group extensions and marginal series of pair of groups

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Abstract. In this article, using the concept of generalized Baer-invariant of a pair of groups, we establish some related isomorphisms between lower marginal quotient pairs of groups, which are generalized versions of some isomorphisms of Stallings. We also derive a result for the pair $(\mathcal{V}.\mathcal{W},\mathcal{X})$ to be an ultra Hall pair for special varieties of groups. This result generalizes that of Fung in 1977, which has roots in Philip Hall's criterion on nilpotency.

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1. Introduction

Let $\mathcal{V}$ be a variety of groups defined by the set of words (laws) V. Then for a given group G two subgroups $V(G)$ and $V^*(G)$ correspond to this variety are defined as follows:

$$V(G) = \left\langle v(g_1,g_2,\dots,g_r)\,\middle|\, g_i \in G,\ v\in V,\ 1\le i\le r\right\rangle,$$
$$V^*(G) = \left\{a\in G\,\middle|\, v(g_1,g_2,\dots,g_i a,\dots,g_r) = v(g_1,g_2,\dots,g_r);\ g_j\in G,\ v\in V,\ 1\le i,j\le r\right\},$$

which are called the *verbal* and *marginal* subgroups of G, and these are fully invariant and characteristic subgroups of G respectively; see [4, 8] for notion of variety of groups. Let N be a normal subgroup of G. Then we define $V(N,G)$ to be the subgroup of G generated by the following set:

$$\left\{v(g_1,g_2,\dots,g_i n,\dots,g_r)v(g_1,g_2,\dots,g_r)^{-1}\,\middle|\, v\in V,\ g_j\in G, 1\le i,j\le r,\ n\in N\right\}.$$

This is the least normal subgroup T of G contained in N such that N/T is contained in $V^*(G/T)$. Also $V^*(N,G)$ is defined as $N\cap V^*(G)$.

The following preliminary lemma gives the basic properties of these subgroups; see [4] for further information.

Lemma 1. *Let $\mathcal{V}$ be a variety of groups defined by the set of words V and N be a normal subgroup of a given group G. Then*

(i) $G \in \mathcal{V} \iff V(G) = \{1\} \iff V^*(G) = G$,
(ii) $V(G/N) = V(G)N/N$ *and* $V^*(G/N) \supseteq V^*(G)N/N$,
(iii) $N \subseteq V^*(G) \iff V(N,G) = \{1\}$,
(iv) $V(N) \subseteq V(N,G) \subseteq N \cap V(G)$. *In particular,* $V(G) = V(G,G)$,
(v) $V(V^*(G)) = \{1\}$ *and* $V^*(G/V(G)) = G/V(G)$.

The following similar lemma is straightforward.

Lemma 2. *Let V be a set of words, K and N be two normal subgroups of a group G such that K is contained in N. Then*

(i) $V(V^*(N,G),G) = 1$, *in particular* $V(N,G) = 1$ *if and only if* $V^*(N,G) = N$,
(ii) $K \le V^*(N,G)$ *if and only if* $V(K,G) = 1$,
(iii) $V(N/K, G/K) = V(N,G)K/K$.

In 1998, Ellis introduced the concept of pair of groups (G,N), where N is normal subgroup of a group G. He also established some related (co)homological and topological properties.

Let (G,N) and (H,K) be two pairs of groups. Then $(f, f|) : (G,N) \to (H,K)$ is a homomorphism if $f : G \to H$ is homomorphism and $f(N) \subseteq K$. The series

$$N \ge N_0 \ge N_1 \ge \cdots \ge N_r \ge \cdots$$

is said to be $\mathcal{V}_G$-marginal series of N, or $\mathcal{V}$- marginal series of the pair (G,N) if $N_i \trianglelefteq G$ and $N_i/N_{i+1} \le V^*(G/N_{i+1})$, for $i \ge 0$. The subgroup N is said to be $\mathcal{V}_G$*-nilpotent* or, the pair (G,N) is said to be $\mathcal{V}$*-nilpotent* if $N_r = 1$ for a positive integer r. The least such r is called the $\mathcal{V}_G$-nilpotency class of N or $\mathcal{V}$-nilpotency class of the pair (G,N).

We have the following two series

$$N = V_0(N,G) \ge V_1(N,G) \ge \cdots \ge V_i(N,G) \ge \cdots,$$

where $V_1(N,G) = V(N,G)$ and $V_i(N,G) = V(V_{i-1}(N,G),G)$, for $i \ge 1$, which is called the *lower $\mathcal{V}$-marginal series* of (G,N). The *upper $\mathcal{V}$-marginal series* of (G,N) is defined as

$$1 = V_0^*(N,G) \le V_1^*(N,G) \le \cdots \le V_i^*(N,G) \le \cdots,$$

where $V_1^*(N,G) = V^*(N,G)$ and

$$V_{i+1}^*(N,G)/V_i^*(N,G) = V^*(N/V_i^*(N,G), G/V_i^*(N,G)), \quad i \ge 1.$$

If one puts $N = G$, then he concept of $\mathcal{V}$-marginal series and $\mathcal{V}$-nilpotency of G is obtained; see [2, 9]. In addition if $V = \{\gamma_2\}$, where $\gamma_2 = [x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2$, i.e. $\mathcal{V}$ is the variety of abelian groups, one obtains the usual concepts of central series and nilpotency; see [12].

We need the following technical lemma.

Lemma 3. *Let $\mathcal{V}$ be a variety of groups defined by the set of words V, (G,N) be a pair of groups and let $N = N_0 \ge N_1 \ge \cdots \ge N_r \ge \cdots$ be a $\mathcal{V}$-marginal series of (G,N). Then*

(i) $V_i(N,G) \le N_i$, $i \ge 0$,
(ii) *If c is the class of $\mathcal{V}$-nilpotecy of (G,N), then $N_{c-i} \le V_i^*(N,G)$ and hence*

$$V_i(N,G) \le N_i \le V_{c-i}^*(N,G), \quad 0 \leqslant i \leqslant c.$$

Let G be an arbitrary group and $1 \to R \to F \xrightarrow{\pi} G \to 1$ be a free presentation of G. Then the Baer-invariant of the group G with respect to the variety $\mathcal{V}$, is defined by

$$\mathcal{V}M(G) = \frac{R \cap V(F)}{V(R,F)}$$

which is abelian and independent of the choice of free presentation of G; see [7].

If $\mathcal{V}$ is the variety of abelian groups, then the Baer-invariant of the group G will be $R \cap F'/[R,F]$, which by Hopf's formula is the Schur multiplier $M(G)$ of the group G and is isomorphic to $H_2(G)$

the second homology group of G; see [5, 13, 14], see also [11] for c-nilpotent multiplier of Lie algebras.

In 1998 Ellis [1], introduced the concept of Schur multiplier of a pair of groups (G,N), where N is a normal subgroup of G, as

$$M(G,N)=\frac{R\cap[S,F]}{[R,F]}$$

in which $N\cong S/R$ for a suitable normal subgroup S of F, i.e. $S=\pi^{-1}(N)$. The Baer-invariant of the pair (G,N) with respect to the variety $\mathcal{V}$ is defined by

$$\mathcal{V}M(G,N)=\frac{R\cap V(S,F)}{V(R,F)}.$$

Clearly if $N=G$, then $M(G,G)=M(G)$ and $\mathcal{V}M(G,G)=\mathcal{V}M(G)$.

In 1976 Leedham-Green and McKay [7], introduced the concept of the generalized Baer-invariant of a group with respect to two varieties as follows. Let $\mathcal{W}$ be another variety of groups defined by the set of words W and $G\in\mathcal{W}$. Then by Lemma 1, $\{1\}=W(G)=W(F)R/R$ and hence $W(F)\subseteq R$. Therefore,

$$1\longrightarrow R/W(F)\longrightarrow F/W(F)\longrightarrow G\longrightarrow 1$$

is a $\mathcal{W}$-free presentation of the group G. The *generalized Baer-invariant* of the group G with respect to the variety $\mathcal{V}$ is denoted by

$$\mathcal{W}\mathcal{V}M(G)=\frac{R/W(F)\cap V(F/W(F))}{V(R/W(F),F/W(F))}\cong\frac{(R\cap V(F))W(F)}{V(R,F)W(F)}$$

which is also abelian and independent of the choice of the free presentation of G. Similar to the Baer-invariant of the pair, the generalized Baer-invariant of the pair (G,N), where $G\in\mathcal{W}$, with respect to the variety $\mathcal{V}$ is defined by

$$\mathcal{W}\mathcal{V}M(G,N)=\frac{(R\cap V(S,F))W(F)}{V(R,F)W(F)}.$$

If one puts $\mathcal{W}$ variety of all groups, then $W(F)=\{1\}$. Thus $\mathcal{W}\mathcal{V}M(G)=\mathcal{V}M(G)$ and $\mathcal{W}\mathcal{V}M(G,N)=\mathcal{V}M(G,N)$; see [7, 10].

In Section 2 we get a generalized version of the well-known 5-term exact sequence of homology groups and then obtain some isomorphisms between lower marginal factors of pairs of groups, under special conditions. In Section 3, we study $\mathcal{V}$-nilpotency of the pair (G,N) and then derive a result which has roots in the Philip Hall's criterion on nilpotency.

2. Homological methods and generalized Baer-invariant of pair of groups

In this section using the concept of generalized Baer-invariant of a pair of groups, we obtain a generalization of well-known 5-term exact sequence and then we establish some isomorphisms which are wide generalization of some results of Stallings [15]. The following main result generalizes [9, Theorem 3.2] extensively; see also [5].

Theorem 4. *Let $\mathcal{V}$ and $\mathcal{W}$ be a varieties of groups defined by the set of laws V and W, respectively, and $E\in\mathcal{W}$. If $1\to N\xrightarrow{\iota}E\xrightarrow{\pi}G\to 1$ is a group extension and L is a normal subgroup of E such that $1\to N\xrightarrow{\iota}L\xrightarrow{\pi|}M\to 1$ is a group extension which ι is the inclusion map, then the following sequence is exact:*

$$\mathcal{W}\mathcal{V}M(E,L)\xrightarrow{\psi}\mathcal{W}\mathcal{V}M(G,M)\xrightarrow{\varphi}\frac{N}{V(N,E)}\xrightarrow{\sigma}\frac{L}{V(L,E)}\xrightarrow{\pi'}\frac{M}{V(M,G)}\longrightarrow 1.$$

Proof. We define the following maps

$$\pi' : \frac{L}{V(L,E)} \longrightarrow \frac{M}{V(M,G)} \qquad\qquad \sigma : \frac{N}{V(N,E)} \longrightarrow \frac{L}{V(L,E)}$$
$$xV(L,E) \longmapsto \pi(x)V(M,G) \qquad\qquad nV(N,E) \longmapsto nV(L,E).$$

Clearly, π' is an epimorphism with the kernel $\frac{NV(L,E)}{V(L,E)}$. The image and the kernel of σ are $\frac{NV(L,E)}{V(L,E)}$ and $\frac{N\cap V(L,E)}{V(N,E)}$, respectively. So, the exactness at $\frac{L}{V(L,E)}$ and $\frac{M}{V(M,G)}$ follows immediately. Now, let $1 \to R \to F \overset{\pi_1}{\to} E \to 1$ be a free presentation of E and $L \cong T/R$ for a normal subgroup T of the free group F. Then $\pi\circ\pi_1 : F \to G$ is a free presentation of G. Put $\ker \pi\circ\pi_1 = S$. Therefore, S is the inverse image of N under π_1. Hence, $R \subseteq S \subseteq T$, $N \cong S/R$ and $M \cong T/S$. Also,

$$\mathscr{W}\mathscr{V}M(E,L) = \frac{(R\cap V(T,F))W(F)}{V(R,F)W(F)} \qquad\qquad \mathscr{W}\mathscr{V}M(G,M) = \frac{(S\cap V(T,F))W(F)}{V(S,F)W(F)}.$$

Now, we define the maps

$$\varphi : \mathscr{W}\mathscr{V}M(G,M) \longrightarrow \frac{N}{V(N,E)} \qquad\qquad \psi : \mathscr{W}\mathscr{V}M(E,L) \longrightarrow \mathscr{W}\mathscr{V}M(G,M)$$
$$xV(S,F)W(F) \longmapsto \pi_1(x)V(N,E) \qquad\qquad xV(R,F)W(F) \longmapsto \pi(x)V(S,F)W(F).$$

It can be easily checked that the image of φ is $\frac{N\cap V(L,E)}{V(N,E)}$ which is the same as the kernel of σ. Also, the kernel of φ is $\frac{(R\cap V(T,F))V(S,F)W(F)}{V(S,F)W(F)}$ which is the same as the image of ψ. Thus, the sequence is exact and the proof is completed. □

The above lemma has the following important corollary, which generalizes [15, Theorem 2.1].

Corollary 5. *Let G be a group with two normal subgroups K and N such that $K \subseteq N$. Then the following sequence is exact:*

$$\mathscr{W}\mathscr{V}M(G,N) \longrightarrow \mathscr{W}\mathscr{V}M(G/K,N/K) \longrightarrow \frac{K}{V(K,G)} \longrightarrow \frac{N}{V(N,G)} \longrightarrow \frac{N}{V(N,G)K} \longrightarrow 1.$$

By using Corollary 5, we have the following theorem, which generalizes [5, Theorem 7.9.1]; see also [15, Theorem 3.4].

Theorem 6. *Let $(f,f|):(G,N)\to(H,K)$ be a homomorphism, where $G,H\in\mathscr{W}$. Suppose f induces isomorphisms $f_0 : G/N \to H/K$ and $f_1| : N/V(N,G) \to K/V(K,H)$, and that $f_* : \mathscr{W}\mathscr{V}M(G,N) \to \mathscr{W}\mathscr{V}M(H,K)$ is an epimorphism. Then f induces isomorphisms*

$$(f_n,f_n|):(G/V_n(N,G),N/V_n(N,G)) \overset{\simeq}{\longrightarrow} (H/V_n(K,H),K/V_n(K,H)), \quad \forall\, n\geq 0.$$

Proof. Let us define $P_n = V_n(N,G)$ and $Q_n = V_n(K,H)$. We proceed by induction. For $n = 0$, the assertion is trivial. For $n = 1$, consider the following diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N/V(N,G) & \longrightarrow & G/V(N,G) & \longrightarrow & G/N & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle f_1|} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \\ 1 & \longrightarrow & K/V(K,H) & \longrightarrow & H/V(K,H) & \longrightarrow & H/K & \longrightarrow & 1. \end{array}$$

By the hypothesis, $f_1|$ and f_0 are isomorphism. Hence, f_1 is an isomorphism. Assume that $n\geq 2$. By considering Corollary 5, we can conclude the following diagram:

$$\begin{array}{ccccccccccc} \mathscr{W}\mathscr{V}M(G,N) & \to & \mathscr{W}\mathscr{V}M(G/P_{n-1},N/P_{n-1}) & \to & P_{n-1}/Pn & \to & N/V(N,G) & \to & N/V(N,G)P_{n-1} & \to & 1 \\ \downarrow{\scriptstyle\alpha_1} & & \downarrow{\scriptstyle\alpha_2} & & \downarrow{\scriptstyle\alpha_3} & & \downarrow{\scriptstyle\alpha_4} & & \downarrow{\scriptstyle\alpha_5} & & \\ \mathscr{W}\mathscr{V}M(H,K) & \to & \mathscr{W}\mathscr{V}M(H/Q_{n-1},K/Q_{n-1}) & \to & Q_{n-1}/Q_n & \to & K/V(K,H) & \to & K/V(K,H)Q_{n-1} & \to & 1. \end{array}$$

Note that the naturallity of the map f induces homomorphisms α_i, $i = 1,2,\ldots,5$ such that the above diagram is commutative. By hypothesis, α_1 is an epimorphism, α_4 and α_5 are isomorphisms. Also, by considering the induction hypothesis and definition of the Baer-invariant of the pair of groups, α_2 is an isomorphism. Hence, by the well-known five lemma, α_3 is an isomorphism. Now, consider the following diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & P_{n-1}/P_n & \longrightarrow & N/P_n & \longrightarrow & N/P_{n-1} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \alpha_3} & & \downarrow{\scriptstyle f_n|} & & \downarrow{\scriptstyle f_{n-1}|} & & \\ 1 & \longrightarrow & Q_{n-1}/Q_n & \longrightarrow & K/Q_n & \longrightarrow & K/Q_{n-1} & \longrightarrow & 1. \end{array}$$

By the above discussion, α_3 is an isomorphism and by induction hypothesis, $f_{n-1}|$ is an isomorphism. Therefore, $f_n|$ is an isomorphism. Finally, by the following diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N/P_n & \longrightarrow & G/P_n & \longrightarrow & G/N & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle f_n|} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_1} & & \\ 1 & \longrightarrow & K/Q_n & \longrightarrow & H/Q_n & \longrightarrow & H/K & \longrightarrow & 1 \end{array}$$

and in the same way, f_n is an isomorphism. □

Now we obtain the following corollary, which generalizes [15, Corollary 3.5] and [9, Corollary 3.4].

Corollary 7. *Let $(f, f|) : (G,N) \to (H,K)$ be a homomorphism which satisfies the hypotheses of Theorem 6. Suppose further that (G,N) and (H,K) are $\mathcal{V}$-nilpotent. Then $(f, f|)$ is an isomorphism.*

Proof. There exists $n \geq 0$ such that $V_n(N,G) = \{1\}$ and $V_n(K,H) = \{1\}$. So, the assertion follows from Theorem 6. □

As a final result we have the following theorem, which is of interest in its own account.

Theorem 8. *Let $(f, f|) : (G,N) \to (H,K)$ be an epimorphism of pairs of groups, where $G, H \in \mathcal{W}$. Let (G,N) be a $\mathcal{V}$-nilpotent pair. If $\ker f \subseteq V(N,G)$ and $\mathcal{W}\mathcal{V}M(H,K)$ is trivial, then $(f, f|)$ is an isomorphism.*

Proof. Put $M = \ker f$. Then $\frac{N}{V(N,G)} \cong \frac{K}{V(K,H)}$, $\frac{G}{N} \cong \frac{H}{K}$ and $\frac{V_n(N,G)M}{M} = V_n(K,H)$ for all $n \geq 0$. Now, the result follows from Corollary 7. □

3. Ultra Hall pair

The concept of a Schur pair was first introduced by Philip Hall [3] in 1940. Then in 1976, Hulse and Lennox [6] studied more properties of this pair and introduced the notion of an ultra Schur pair, a persistent pair and an ultra persistent pair. In 1977, Fung introduced the notion of a Hall pair as the following.

Definition 9. *Let $\mathcal{X}$ be a class of groups and $\mathcal{V}$ be a variety of groups. If for every group G and normal $\mathcal{V}$-nilpotent subgroup N of G, $G/V(N) \in \mathcal{X}$ implies that $G \in \mathcal{X}$, then the pair $(\mathcal{V}, \mathcal{X})$ is said to be a* Hall pair.

In the special case if $\mathcal{V}$ is the variety of abelian groups and $\mathcal{X}$ is the class of nilpotent groups, we observe that this notion has roots in the well-known nilpotency criterion of Philip Hall; see [12, Theorem 5.2.10].

Let F_∞ be the free group with the set of free generators $\{x_1, x_2, x_3, \ldots\}$. The *outer commutator* words (henceforth o.c. words) are defined inductively as follows. The word x_i is an o.c. word of weight one. If $U = U(x_1, \ldots, x_m)$ and $V = V(x_{m+1}, \ldots, x_{m+n})$ are o.c. words of weight m and n, respectively, then

$$W(x_1, \ldots, x_{m+n}) = [U(x_1, \ldots, x_m), V(x_{m+1}, \ldots, x_{m+n})],$$

the commutator of U and V, is an o.c. word of weight $m+n$. Let $V = V(x_1,\ldots,x_m)$ and $W = W(x_1,\ldots,x_n)$ be two arbitrary words. Then VoW, the composite of V and W, is defined as $VoW = V(y_1,\ldots,y_m)$, where $y_i = W(x_{(i-1)n+1},\ldots,x_{in})$, $1 \le i \le m$. In the sequel, $\mathscr{V}.\mathscr{W}$ is the variety of groups defined by the word VoW.

Theorem 10 (cf. [2, Theorem 3]). *Let $\mathscr{V}$ be variety of groups defined by an o.c. word V of weight at least two and let $\mathscr{W}$ be a variety of groups defined by a single word W. Then the assumption that $(\mathscr{V},\mathscr{X})$ is a Hall pair always implies that $(\mathscr{V}.\mathscr{W},\mathscr{X})$ is also a Hall pair.*

In the following we state the definition of ultra Hall pair and derive a result which is a generalization of [2, Theorem 3].

Definition 11. *Let $\mathscr{X}$ be a class of groups and $\mathscr{V}$ be a variety of groups defined by the set of words V. If for every normal subgroups K and N of a given group G such that K is $\mathscr{V}_N$-nilpotent, the assumption $G/V(K,N) \in \mathscr{X}$ implies that $G \in \mathscr{X}$, then $(\mathscr{V},\mathscr{X})$ is called an* ultra Hall pair.

The following lemma will be useful for the proof of our results; see [2, Lemma 2.6].

Lemma 12. *Let V and W be two words of distinct variables and $U = [V,W]$. Then for every normal subgroup N of a given group G, the following statements hold*

(i) $U(N,G) = [V(N,G),W(G)][W(N,G),V(G)]$,
(ii) *If V is an o.c. word, then*

$$VoW(N,G) = V(W(N,G),W(G)).$$

The following easy lemma is useful in the next result.

Lemma 13. *Let V be an o.c. word of weight at least two. Then for every normal subgroup N of a given group G, $V(N,G) \le [N,G]$.*

Proof. Let c be the weight of V. For $c=2$, $V=\gamma_2$, then $V(N,G)=[N,G]$. Let the result holds for o.c. words of weight less than c. Then $V = [V_1,V_2]$, where V_1 and V_2 are o.c. words of weight less than c. By Lemma 12 (i)

$$\begin{aligned} V(N,G) &= [V_1(N,G),V_2(G)][V_2(N,G),V_1(G)] \\ &\le [[N,G],V_2(G)][[N,G],V_1(G)] \\ &\le [N,G]. \end{aligned}$$ □

The following theorem gives a necessary and sufficient condition for a normal subgroup N of a group G to be $\mathscr{U}_G$-nilpotent, where $\mathscr{U}$ is the variety of groups defined by the word VoW.

Theorem 14. *Let V and W be two words of distinct variables such that V is an o.c. word of weight at least two. Then for any normal subgroup N of a group G, the subgroup N is $\mathscr{U}_G$-nilpotent if and only if $W(N,G)$ is $\mathscr{V}_{W(G)}$-nilpotent.*

Proof. Let $W(N,G)$ be $\mathscr{V}_{W(G)}$-nilpotent. By considering $U = VoW$, since V is an o.c. word, then $U(N,G) = VoW(N,G) = V(W(N,G),W(G))$. Using induction on k, we prove that for any positive integer k, $U_k(N,G) \le V_k(W(N,G),W(G))$. The result is true for $k=1$. Suppose that for $k=i$ the statement holds. Then

$$\begin{aligned} U_{i+1}(N,G) &= U(U_i(N,G),G) \\ &= V(W(U_i(N,G),G),W(G)) \\ &\le V(W(V_i(W(N,G),W(G)),G),W(G)) \\ &\le V(V_i(W(N,G),W(G)),W(G)) \\ &= V_{i+1}(W(N,G),W(G)), \end{aligned} \tag{1}$$

where (1) follows from Lemma 1 (iv). Since $W(N,G)$ is $\mathcal{V}_{W(G)}$-nilpotent, then

$$V_r(W(N,G),W(G)) = 1$$

for a positive integer r. Thus $U_r(N,G) = 1$, which implies that N is $\mathcal{U}_G$-nilpotent.

Now, let N be $\mathcal{U}_G$-nilpotent. By induction we will prove that

$$V_k(W(N,G),W(G)) \le U_{\left[\frac{k+1}{2}\right]}(N,G), \tag{2}$$

for any positive integer k, where $\left[\frac{k+1}{2}\right]$ is the integer part of $\frac{k+1}{2}$. Clearly the result holds for $k = 1$. Suppose the statement holds for every i, where $i \leqslant k$. Then

$$V_k(W(N,G),W(G)) = V(V_{k-1}(W(N,G),W(G)),W(G)).$$

Since V is an o.c. word, by the above lemma the right hand of equality is contained in $[V_{k-1}(W(N,G),W(G)),W(G)]$. This subgroup is contained in

$$V(V_{k-1}(W(N,G),W(G)),G) \le W(U_{[\frac{k}{2}]}(N,G),G),$$

by Lemma 1 (v). So

$$\begin{aligned} V_{k+1}(W(N,G),W(G)) &= V(V_k(W(N,G),W(G)),W(G)) \\ &\le V(W(U_{[\frac{k}{2}]}(N,G),G),W(G)) \\ &= U_{[\frac{k}{2}]+1}(N,G) \\ &= U_{[\frac{k+3}{2}]}(N,G). \end{aligned}$$

Thus for every positive integer k, (2) holds. As N is $\mathcal{U}_G$-nilpotent, $U_r(N,G) = 1$ for a positive integer r. So, $V_{2r-1}(W(N,G),W(G)) = 1$, i.e. $W(N,G)$ is $\mathcal{V}_{W(G)}$-nilpotent. □

The following result generalizes [2, Theorem 3].

Theorem 15. *Let $\mathcal{V}$ and $\mathcal{W}$ be two varieties of groups as in the above theorem. Then the assumption that $(\mathcal{V},\mathcal{X})$ is an ultra Hall pair, implies that $(\mathcal{V}.\mathcal{W},\mathcal{X})$ is also an ultra Hall pair.*

Proof. Let K and N be two normal subgroups of G such that K is $\mathcal{U}_N$-nilpotent, where $\mathcal{U} = \mathcal{V}.\mathcal{W}$, and $G/U(K,N) \in \mathcal{X}$. So, $G/V(W(K,N),W(N)) \in \mathcal{X}$. By the above theorem, $W(K,N)$ is $\mathcal{V}_{W(N)}$-nilpotent. Since $(\mathcal{V},\mathcal{X})$ is an ultra Hall pair, then $G \in \mathcal{X}$. □

If one puts $K = N$, then the result that of Fung yields. The following result generalizes [9, Theorem 2.4].

Theorem 16. *Let $\mathcal{V}$ be a variety of groups and N be a $\mathcal{V}_G$-nilpotent subgroup of G. If K is nontrivial normal subgroup of G, contained in N, then $K \cap V^*(N,G) \neq 1$.*

Proof. Let the $\mathcal{V}_G$-nilpotency class of N be c. Then by Lemma 3 (ii), $V_c^*(N,G) = N$. So, there exists a least integer i such that $K \cap V_i^*(N,G) \neq 1$. Clearly

$$V(K \cap V_i^*(N,G),G) \le K \cap V(V_i^*(N,G),G).$$

On the other hand by Lemma 1 (iv) and Lemma 2,

$$\begin{aligned} V\left(\frac{V_i^*(N,G)}{V_{i-1}^*(N,G)},\frac{G}{V_{i-1}^*(N,G)}\right) &= V\left(V^*\left(\frac{N}{V_{i-1}^*(N,G)},\frac{G}{V_{i-1}^*(N,G)}\right),\frac{G}{V_{i-1}^*(N,G)}\right) \\ &= 1_{G/V_{i-1}^*(N,G)}. \end{aligned}$$

Therefore, $V(K \cap V_i^*(N.G),G) \le K \cap V_{i-1}^*(N,G) = 1$. Hence,

$$K \cap V_i^*(N,G) \le K \cap V^*(N,G),$$

our required result. □

If one puts $N = G$ and considers $\mathscr{V}$ as the variety of abelian groups, then the well-known result of Philip Hall is obtained; see [8, Theorem 31.26].

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