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Boundedness of second-order Riesz transforms on weighted Hardy and BMO spaces associated with Schrödinger operators

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Abstract. Let $d \in \{3, 4, 5, \dots\}$ and a weight $w \in A_{\infty}^p$. We consider the second-order Riesz transform $T = \nabla^2 L^{-1}$ associated with the Schrödinger operator $L = -\Delta + V$, where $V \in RH_{\sigma}$ with $\sigma > \frac{d}{2}$. We present three main results. First T is bounded on the weighted Hardy space $H_{w,L}^1(\mathbb{R}^d)$ associated with L if w enjoys a certain stable property. Secondly T is bounded on the weighted BMO space $BMO_{w,\rho}(\mathbb{R}^d)$ associated with L if w also belongs to an appropriate doubling class. Thirdly $BMO_{w,\rho}(\mathbb{R}^d)$ is the dual of $H_{w,L}^1(\mathbb{R}^d)$ when $w \in A_1^p$.

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1. Introduction

It is well-known that classical Calderón–Zygmund operators are bounded on L^p spaces for $p \in (1, \infty)$. Such a property is no longer true for the two endpoints $p = 1$ and $p = \infty$. This invokes a need to replace L^1 and L^∞ spaces by other spaces so that the boundedness property are reinstated. Hardy and BMO spaces as well as their variances have found their places in this context as substitutions for L^1 and L^∞ spaces respectively. Various results on the boundedness of (classical or generalized) Calderón–Zygmund operators on these two spaces and their variances can be found in the vast literature.

In this paper we investigate second-order Riesz transform associated with Schrödinger operators and their boundedness on weighted Hardy and BMO spaces which are also associated with the Schrödinger operators. To formulate our problems precisely we need to introduce some definitions.

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Let $d \in \{3, 4, 5, \dots\}$ and $\sigma > \frac{d}{2}$. Let $V \in RH_\sigma$, i.e., V is a non-negative locally integrable function and there exists a $C = C(\sigma, V) > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B|} \int_B V^\sigma \right)^{\frac{1}{\sigma}} \leq \frac{C}{|B|} \int_B V$$

holds for every ball $B \subset \mathbb{R}^d$. Consider the Schrödinger operator

$$L = -\Delta + V$$

on its maximal domain in $L^2(\mathbb{R}^d)$. It is well-known that L is non-negative and self-adjoint. Furthermore L generates a C_0 -semigroup $(e^{-tL})_{t>0}$ on $L^2(\mathbb{R}^d)$.

We first define the weighted Hardy and BMO spaces associated with L . Let $w \in L^1_{loc}(\mathbb{R}^d, [0, \infty))$ be a weight.

Definition 1. *The weighted Hardy space associated with L is defined as*

$$H^1_{w,L}(\mathbb{R}^d) = \{f \in L^1_w(\mathbb{R}^d) : \mathcal{M}_L f \in L^1_w(\mathbb{R}^d)\}$$

with a norm given by

$$\|\cdot\|_{H^1_{w,L}(\mathbb{R}^d)} := \|\mathcal{M}_L \cdot\|_{L^1_w(\mathbb{R}^d)},$$

where $\mathcal{M}_L f(\cdot) = \sup_{t>0} |e^{-tL} f(\cdot)|$ for all $f \in L^1_w(\mathbb{R}^d)$.

Next let $\rho : \mathbb{R}^d \rightarrow [0, \infty)$ be defined by

$$\rho(\cdot) = \sup \left\{ r > 0 : \frac{1}{r^{d-2}} \int_{B(\cdot, r)} V \leq 1 \right\}.$$

The function ρ is usually called a *critical radius function*.

The following weighted BMO -space was first introduced in [4].

Definition 2. *A function $f \in L^1_{loc}(\mathbb{R}^d)$ is said to belong to the space $BMO_{w,\rho}(\mathbb{R}^d)$ if there exists a $C > 0$ such that*

$$\frac{1}{w(B_s)} \int_{B_s} |f - f_{B_s}| \leq C \quad \text{and} \quad \frac{1}{w(B_r)} \int_{B_r} |f| \leq C \tag{1}$$

for all balls $B_s = B(x, s)$ and $B_r = B(x, r)$ such that $0 < s < \rho(x) \leq r$. Here we denote $f_B := \frac{1}{|B|} \int_B f$ and $w(B) = \int_B w$ for all ball $B \subset \mathbb{R}^d$.

The norm of $BMO_{w,\rho}(\mathbb{R}^d)$ is defined by

$$\|\cdot\|_{BMO_{w,\rho}(\mathbb{R}^d)} = \inf\{C > 0 : (1) \text{ holds}\}.$$

The object to study in this paper is the second-order Riesz transform of L defined by

$$T = \nabla^2 L^{-1}.$$

It is known that T is a classical Calderón–Zygmund operator when V is a non-negative polynomial (cf. [23]). However this need not be the case under our current assumption on V . Still T is observed to belong to a class of generalized Calderón–Zygmund operator (cf. [8, Subsection 5.2] or Proposition 37 below).

Our main goals are to provide the boundedness of T on $H^1_{w,L}(\mathbb{R}^d)$ and $BMO_{w,\rho}(\mathbb{R}^d)$ when w belongs to the A^ρ_∞ -class of weights (see Definition 3 below).

Prior to our work the boundedness of T on various spaces has been considered. Particularly Shen in [20, Theorem 0.3] showed that T is bounded on $L^p(\mathbb{R}^d)$ with p depending on d and the reverse Hölder index σ of V . Ly then expanded Shen’s result to weighted Lebesgue spaces $L^p_w(\mathbb{R}^d)$, where $w \in A^\rho_\infty$ and p relies on w (cf. [15, Theorem 1.1]). He also showed that T is bounded on the unweighted Hardy space $H^1_L(\mathbb{R}^d)$ and from the unweighted Hardy space $H^1_L(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$ (cf. [14, Theorem 1.2]). A generalized version of [14, Theorem 1.2] is given in [8, Theorems 1.2 and 1.6]. Regarding the boundedness results on (generalized) BMO spaces, characterizations

via $T1$ criteria are available for generalized Calderón–Zygmund operators which include the family of classical Calderón–Zygmund operators as a special case (cf. [16, Theorems 1.1 and 1.2], [4, Theorem 2], [8, Theorem 1.4] and references therein). Concerning the operator T in this paper, although T is observed to be a generalized Calderón–Zygmund operator previously, we emphasize that it remains open whether or not T satisfies the $T1$ criteria.

Back to the main discussion, for the boundedness of T on $H^1_{w,L}(\mathbb{R}^d)$ to be possible, we are particularly interested in the following class of weights.

Definition 3. Let $\theta \geq 0$ and $s \in [1, \infty)$. Let $w \in L^1_{\text{loc}}(\mathbb{R}^d, [0, \infty))$.

If $s > 1$, then we say that $w \in A_s^{\rho, \theta}$ when there exists a $C > 0$ such that

$$\left(\int_B w \right)^{\frac{1}{s}} \left(\int_B w^{\frac{-1}{(s-1)}} \right)^{\frac{1}{s'}} \leq C |B| \left(1 + \frac{r_B}{\rho(x_B)} \right)^\theta$$

for every ball $B = B(x_B, r_B)$.

If $s = 1$, then we say that $w \in A_1^{\rho, \theta}$ when there exists a $C > 0$ such that

$$\int_B w \leq C |B| \left(1 + \frac{r_B}{\rho(x_B)} \right)^\theta \inf_B w$$

for every ball $B = B(x_B, r_B)$.

We also write $A_s^\rho = \bigcup_{\theta \geq 0} A_s^{\rho, \theta}$ and $A_\infty^\rho = \bigcup_{s \geq 1} A_s^\rho$.

This new class of weights was first introduced in [6]. As a special case, when $\theta = 0$ we regain the well-known Muckenhoupt weights.

It is observed that the condition $w \in A_\infty^\rho$ alone does not guarantee enough richness in the structures of the corresponding weighted spaces such as weighted Hardy spaces and weighted Lebesgue spaces for the boundedness on these spaces of Calderón–Zygmund operators in general (cf. [15, 19]). Therefore it is natural that in our circumstance we also want the weights to enjoy a certain stable property in the following sense.

Definition 4. Let $\epsilon > 0$ and $w \in L^1_{\text{loc}}(\mathbb{R}^d, [0, \infty))$. Then w is called ϵ -Lebesgue stable if there exist constants $C, c > 0$ such that

$$\sum_{l=1}^{\infty} 2^{-le} \frac{|2^l B|}{|B|} \frac{w(B)}{w(2^l B)} \leq C \tag{2}$$

for all ball $B = B(x, r)$ with $r \leq c \rho(x)$.

Note that (2) is trivially true when $w = 1$ and $\epsilon > 0$. Therefore the class of ϵ -Lebesgue stable weights is non-void for all $\epsilon > 0$. More interesting weights which also enjoy this property are given in Section 2.

With these notions in mind, we are now able to state the first main result of this paper.

Theorem 5. Let $\sigma > \frac{d}{2}$ and $V \in RH_\sigma$. Let $\sigma_0 = 2 - \frac{d}{\sigma}$, $\theta \in [0, \sigma_0)$ and $\epsilon \in (0, \sigma_0 - \theta)$. Suppose that $w \in A_1^{\rho, \theta}$ is ϵ -Lebesgue stable. Suppose further that there exists a $\nu > 1$ satisfying $\nu' \in (1, \sigma)$ and $w^\nu \in A_1^\rho$, where ν' is the conjugate index of ν . Then T is bounded on $H^1_{L,w}(\mathbb{R}^d)$.

It is interesting to see that if a weight $w \in A_1^\rho$ is known as well to belong to a reverse Hölder class, then w automatically satisfies the conditions in Theorem 5 with appropriate constraints imposed on the reverse Hölder indices (see Proposition 17).

Next we move to consider the boundedness of T on $BMO_{w,\rho}(\mathbb{R}^d)$. We note that $BMO_{w,\rho}(\mathbb{R}^d)$ is the dual space of $H^1_{w,L}(\mathbb{R}^d)$.

Proposition 6. Let $\sigma > \frac{d}{2}$ and $V \in RH_\sigma$. Let $w \in A_1^\rho$. Then

$$BMO_{w,\rho}(\mathbb{R}^d) = \left(H^1_{w,L}(\mathbb{R}^d) \right)^* .$$

The following corollary is immediate from Theorem 5 and Proposition 6.

Corollary 7. Let $\sigma > \frac{d}{2}$ and $V \in RH_\sigma$. Let $\sigma_0 = 2 - \frac{d}{\sigma}$, $\theta \in (0, \sigma_0)$ and $\epsilon \in (0, \sigma_0 - \theta)$. Suppose that $w \in A_1^{\rho, \theta}$ is ϵ -Lebesgue stable. Suppose further that there exists a $\nu > 1$ satisfying $\nu' \in (1, \sigma)$ and $w^\nu \in A_1^\rho$, where ν' is the conjugate index of ν . Then $T^* = L^{-1} \nabla^2$ is bounded on $BMO_{w, \rho}(\mathbb{R}^d)$.

To derive the boundedness of T on $BMO_{w, \rho}(\mathbb{R}^d)$, we need to introduce one more definition.

Definition 8. Let $\theta \geq 0$ and $s \in [1, \infty)$. Let $w \in L_{\text{loc}}^1(\mathbb{R}^d, [0, \infty))$. Then we say that $w \in D_s^{\rho, \theta}$ if there exists a constant $C > 0$ such that

$$w(tB) \leq C t^{ds} w(B) \left(1 + \frac{r_B}{\rho(x_B)}\right)^\theta$$

for all $t > 1$ and for all ball $B = B(x_B, r_B) \subset \mathbb{R}^d$.

We also write $D_s^\rho = \bigcup_{\theta \geq 0} D_s^{\rho, \theta}$ and $D_\infty^\rho = \bigcup_{s \geq 1} D_s^\rho$.

The second main theorem of this paper is as follows.

Theorem 9. Let $\sigma > \frac{d}{2}$ and $V \in RH_\sigma$. Let $\delta \in (0, 1]$ and $\alpha \in [1, 1 + \frac{\delta}{d}]$. Suppose $w \in A_\infty^\rho \cap D_\alpha^\rho$. Suppose further that there exists a $C > 0$ such that

$$|\nabla V(x)| \leq C \rho(x)^{-3} \quad \text{and} \quad |\nabla^2 V(x)| \leq C \rho(x)^{-4} \quad (3)$$

for all $x \in \mathbb{R}^d$. Then T is bounded on $BMO_{w, \rho}(\mathbb{R}^d)$.

Note that for notational simplicity we have identified in (3) an arbitrary first-order derivative ∂_j with the gradient ∇ and an arbitrary second-order derivative ∂_{ij} with the Hessian matrix ∇^2 , where $i, j \in \{1, \dots, d\}$. We will follow this practice consistently in the whole paper.

As a summary, our main contributions in this paper include:

- (1) Proving the boundedness of T on the *weighted Hardy space* $H_{w, L}^1(\mathbb{R}^d)$ when $w \in A_\infty^\rho$ (see Theorem 5). This extends the unweighted versions in [14, Theorem 1.2] and [8, Theorem 1.6].
- (2) Proving the boundedness of T on the *weighted BMO space* $BMO_{w, \rho}(\mathbb{R}^d)$ when $w \in A_\infty^\rho$ (see Theorem 9). The result is new even in the special case when w is a Muckenhoupt weight. This also provides a very first concrete example that verifies the characterizations of boundedness on $BMO_{w, \rho}(\mathbb{R}^d)$ for the generalized Calderón–Zygmund operators in [4, Theorem 2].
- (3) Proving the dual space of $H_{w, L}^1(\mathbb{R}^d)$ is $BMO_{w, \rho}(\mathbb{R}^d)$ when $w \in A_\infty^\rho$ (see Proposition 6). This extends the unweighted version in [9, Theorem 4] and the weighted version when the weights belong to the classical Muckenhoupt class in [13, Theorem 4.8].

The outline of the paper is as follows. Section 2 contains preliminaries which collects basic facts about the critical functions, A_∞^ρ -weights and maximal functions associated with them. In Section 3 we discuss about the space $H_{w, L}^1(\mathbb{R}^d)$. Theorem 5 is proved in Section 4. The space $BMO_{w, \rho}(\mathbb{R}^d)$ and its dual result - Proposition 6 are dealt with in Section 5. Section 6 provides estimates on fundamental solution of $Lu = 0$ and the kernel of the operator T . Lastly we prove Theorem 9 in Section 7.

Notation. Throughout the paper the following set of notation is used without mentioning. Set $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ and $\mathbb{N}^* = \{1, 2, 3, \dots\}$. Given a $j \in \mathbb{N}$ and a ball $B = B(x, r)$, we let $2^j B = B(x, 2^j r)$, $U_0(B) = B$ and $U_j(B) = 2^j B \setminus 2^{j-1} B$ if $j \geq 1$. For all ball $B \subset \mathbb{R}^d$ we write $w(B) := \int_B w$. The constants C and c are always assumed to be positive and independent of the main parameters whose values change from line to line. For any two functions f and g , we write $f \lesssim g$ and $f \sim g$ to mean $f \leq Cg$ and $cg \leq f \leq Cg$ respectively. Given a $p \in [1, \infty)$, the conjugate index of p is denoted by p' . Lastly $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$ for all $a, b \in \mathbb{R}$.

2. Preliminaries

This section includes several facts about the critical functions, the new weights and maximal functions associated with the new weights.

Regarding the critical functions the following results are extremely useful which will be used frequently later on.

Proposition 10 (cf. [20, Lemma 1.4]). *There exist a $C_\rho > 0$ and a $k_0 \geq 1$ such that*

$$C_\rho^{-1} \rho(x) \left(1 + \frac{|x - y|}{\rho(x)}\right)^{-k_0} \leq \rho(y) \leq C_\rho \rho(x) \left(1 + \frac{|x - y|}{\rho(x)}\right)^{\frac{k_0}{k_0+1}} \tag{4}$$

for all $x, y \in \mathbb{R}^d$.

As a corollary of Proposition 10 we obtain the following.

Corollary 11 (cf. [20, Corollary 1.5]). *There exist a $C_\rho > 0$ and a $k_0 \geq 1$ such that*

$$C_\rho^{-1} \left(1 + \frac{|x - y|}{\rho(y)}\right)^{\frac{1}{k_0+1}} \leq 1 + \frac{|x - y|}{\rho(x)} \leq C_\rho \left(1 + \frac{|x - y|}{\rho(y)}\right)^{k_0+1}$$

for all $x, y \in \mathbb{R}^d$.

Next we discuss some properties of the A_∞^ρ -weights.

Lemma 12 (cf. [7, Lemma 3]). *Let $w \in A_1^\rho$. Then there exists a $\nu > 1$ such that $w^\nu \in A_1^\rho$.*

Lemma 13 (cf. [22, Lemma 2.4]). *Let $\theta \geq 0, s \in [1, \infty)$ and $w \in A_s^{\rho, \theta}$. Then $w \in D_{s(1+\frac{\theta}{s})}^{\rho, s\theta}$.*

Recall that a weight w is called ϵ -Lebesgue stable for some $\epsilon > 0$ if there exist constants $C, c > 0$ such that

$$\sum_{l \in \mathbb{N}^*} 2^{-le} \frac{|2^l B|}{|B|} \frac{w(B)}{w(2^l B)} \leq C$$

for all ball $B = B(x, r)$ with $r \leq c \rho(x)$. It is observed above that Lebesgue measure is automatically ϵ -Lebesgue stable for all $\epsilon > 0$. Next we will construct other classes of weights which are ϵ -Lebesgue stable for some $\epsilon > 0$.

Definition 14. *Let $\theta \geq 0$ and $q \in (0, \infty)$. Let $w \in L_{loc}^1(\mathbb{R}^d, [0, \infty))$. Then we say that $w \in RH_q^{\rho, \theta}$ if there exists a constant $C > 0$ such that*

$$\left(\int_B w^q\right)^{\frac{1}{q}} \leq C \left(1 + \frac{r}{\rho(x)}\right)^\theta \int_B w$$

for all ball $B = B(x, r)$.

Lemma 15. *Let $\theta \geq 0, q \in [1, \infty)$ and $w \in RH_q^{\rho, \theta}$. Then there exists a $C > 0$ such that*

$$\frac{w(B)}{w(2^l B)} \leq C \left(\frac{|B|}{|2^l B|}\right)^{\frac{1}{q'}} \left(1 + \frac{2^l r}{\rho(x)}\right)^\theta$$

for all ball B and $l \in \mathbb{N}$.

Proof. Let B be a ball and $l \in \mathbb{N}$. Then

$$\begin{aligned} \frac{w(B)}{w(2^l B)} &\leq \frac{1}{w(2^l B)} \left(\int_B w^q\right)^{\frac{1}{q}} |B|^{\frac{1}{q'}} \leq \frac{1}{w(2^l B)} \left(\int_{2^l B} w^q\right)^{\frac{1}{q}} |B|^{\frac{1}{q'}} = \frac{|2^l B|}{w(2^l B)} \left(\int_{2^l B} w^q\right)^{\frac{1}{q}} \left(\frac{|B|}{|2^l B|}\right)^{\frac{1}{q'}} \\ &\lesssim \frac{|2^l B|}{w(2^l B)} \left(\frac{|B|}{|2^l B|}\right)^{\frac{1}{q'}} \left(1 + \frac{2^l r}{\rho(x)}\right)^\theta \int_{2^l B} w = \left(\frac{|B|}{|2^l B|}\right)^{\frac{1}{q'}} \left(1 + \frac{2^l r}{\rho(x)}\right)^\theta \end{aligned}$$

as required. □

Given an arbitrary $\epsilon > 0$ we can always choose a weight in the reverse Hölder class which is ϵ -Lebesgue stable.

Proposition 16. *Let $\theta \geq 0$, $q \in [1, \infty)$ and $w \in RH_q^{\rho, \theta}$. Let $\epsilon > \frac{d}{q} + \theta$. Then w is ϵ -Lebesgue stable.*

Proof. Let $B = B(x, r)$ be a ball with $r < c\rho(x)$ for some $c > 0$. Then

$$\frac{w(B)}{w(2^l B)} \lesssim \left(\frac{|B|}{|2^l B|}\right)^{\frac{1}{q'}} \left(1 + \frac{2^l r}{\rho(x)}\right)^\theta \lesssim 2^{l\theta - l\frac{d}{q'}}$$

by Lemma 15. It follows that

$$2^{-l\epsilon} \frac{|2^l B|}{|B|} \frac{w(B)}{w(2^l B)} \lesssim 2^{-l\epsilon} 2^{ld} 2^{l\theta - l\frac{d}{q'}} = 2^{-l(\epsilon - \frac{d}{q} - \theta)}$$

for all $l \in \mathbb{N}^*$. This guarantees the convergence of the corresponding series as $\epsilon > \frac{d}{q} + \theta$. Hence the claim follows. \square

The A_1^ρ weights in a certain reverse Hölder class assure the conditions stated in Theorem 5.

Proposition 17. *Let $\epsilon > 0$. Let $\theta \geq 0$ and $\nu > 1$ be such that $\nu' \in (1, \sigma)$ and $\frac{d}{\nu} + \theta < \epsilon$. Let $w \in A_1^\rho \cap RH_\nu^{\rho, \theta}$. Then w is ϵ -Lebesgue stable and $w^\nu \in A_1^\rho$.*

Proof. The stability follows from Proposition 16 and $w \in A_1^\rho \cap RH_\nu^\rho$ implies $w^\nu \in A_1^\rho$. \square

Next we investigate when an A_∞^ρ weight is ϵ -Lebesgue stable for some $\epsilon > 0$.

Proposition 18 (cf. [6, Lemma 5]). *Let $\theta \geq 0$, $s \in [1, \infty)$ and $w \in A_s^{\rho, \theta}$. Then there exists a $\kappa > 1$ and $\eta \geq 0$ such that $w \in RH_\kappa^{\rho, \eta}$.*

Lemma 19. *Let $\theta \geq 0$, $s \in [1, \infty)$ and $w \in A_s^{\rho, \theta}$. Let κ and η be determined by Proposition 18. Then there exist constants $C, c > 0$ such that*

$$c 2^{-ls(d+\theta)} \left(1 + \frac{r_B}{\rho(x_B)}\right)^{-s\theta} \leq \frac{w(B)}{w(2^l B)} \leq C \left(\frac{|B|}{|2^l B|}\right)^{\frac{1}{\kappa'}} \left(1 + \frac{2^l r_B}{\rho(x_B)}\right)^\eta$$

for all ball $B = B(x_B, r_B)$ and $l \in \mathbb{N}$.

Proof. The first inequality is a consequence of Lemma 13. The second inequality follows from Lemma 15 and Proposition 18. \square

The next proposition shows that a mere A_∞^ρ weight is ϵ -Lebesgue stable when ϵ is large enough.

Proposition 20. *Let $\theta \geq 0$, $s \in [1, \infty)$ and $w \in A_s^{\rho, \theta}$. Let κ and η be determined by Proposition 18. Choose $\epsilon > \frac{d}{\kappa} + \eta$. Then w is ϵ -Lebesgue stable.*

Proof. Let B be a ball. Using Lemma 19 and arguing as in the proof of Proposition 16 we obtain

$$\frac{w(B)}{w(2^l B)} \lesssim 2^{l\eta - l\frac{d}{\kappa'}} \quad \text{and} \quad 2^{-l\epsilon} \frac{|2^l B|}{|B|} \frac{w(B)}{w(2^l B)} \lesssim 2^{-l(\epsilon - \frac{d}{\kappa} - \eta)}$$

for all $l \in \mathbb{N}^*$. Since we chose $\epsilon > \frac{d}{\kappa} + \eta$, this implies the claim. \square

We end this section with a result about maximal operators associated with the critical functions. For each $\theta \geq 0$ we define the maximal operator M_ρ^θ by

$$M_\rho^\theta f(x) = \sup_{B(x_0, r) \ni x} \left(1 + \frac{r}{\rho(x_0)}\right)^{-\theta} \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} |f| \tag{5}$$

for all $f \in L_{\text{loc}}^1(\mathbb{R}^d)$.

Lemma 21 (cf. [1, Proposition 3]). *Let $s \in (1, \infty)$ and $w \in L_{\text{loc}}^1(\mathbb{R}^d)$. Then $w \in A_s^\rho$ if and only if there exists a $\theta \geq 0$ such that M_ρ^θ is bounded on $L_w^s(\mathbb{R}^d)$.*

3. Weighted Hardy spaces associated with L

Recall that the weighted Hardy space associated with L is defined as

$$H_{w,L}^1(\mathbb{R}^d) = \{f \in L_w^1(\mathbb{R}^d) : \mathcal{M}_L f \in L_w^1(\mathbb{R}^d)\}$$

with a norm given by

$$\|\cdot\|_{H_{w,L}^1(\mathbb{R}^d)} := \|\mathcal{M}_L \cdot\|_{L_w^1(\mathbb{R}^d)},$$

where $\mathcal{M}_L f(\cdot) = \sup_{t>0} |e^{-tL} f(\cdot)|$ for all $f \in L_w^1(\mathbb{R}^d)$.

The following properties of \mathcal{M}_L will be used frequently.

Lemma 22. *Let $f \in L_{\text{loc}}^1(\mathbb{R}^d)$. Then for all $\theta \geq 0$ there exists a $C > 0$ such that*

$$\mathcal{M}_L |f| \leq C M_\rho^\theta |f|,$$

where M_ρ^θ is defined by (5).

Proof. Let $\theta \geq 0$. Define the maximal operator m_ρ^θ by

$$m_\rho^\theta g(x) = \sup_{r>0} \left(1 + \frac{r}{\rho(x)}\right)^{-\theta} \frac{1}{|B(x,r)|} \int_{B(x,r)} |g|$$

for all $g \in L_{\text{loc}}^1(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. Then clearly $m_\rho^\theta g \leq M_\rho^\theta g$ for all $g \in L_{\text{loc}}^1(\mathbb{R}^d)$. It follows from [2, Proposition 4.1] that

$$\mathcal{M}_L |f| \leq C m_\rho^\theta |f|.$$

Hence the claim follows. □

Lemma 23. *Let $s \in (1, \infty)$ and $w \in A_s^\rho$. Then \mathcal{M}_L is bounded on $L_w^s(\mathbb{R}^d)$.*

Proof. Let $f \in L_w^s(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Since $w \in A_s^\rho$, there exists a $\theta \geq 0$ such that M_ρ^θ is bounded on $L_w^s(\mathbb{R}^d)$ by Lemma 21. Note that

$$\mathcal{M}_L f \leq \mathcal{M}_L |f| \leq C M_\rho^\theta |f|,$$

where we used the positivity of $\{e^{-tL}\}_{t>0}$ in the first step (cf. [17, Proof of Proposition 2.20]) and Lemma 22 in the second step. This justifies the claim. □

In what follows, for each $t > 0$, we denote $k_t(\cdot, \cdot)$ the kernel associated with e^{-tL} . In a special case when $V = 0$, the kernel associated with $e^{-t\Delta}$ is denoted by $k_t^0(\cdot, \cdot)$.

Following [2] we also consider local Hardy spaces as follows.

Definition 24. *Define $H_{w,\rho,0}^1(\mathbb{R}^d)$ as*

$$\{f \in L_w^1(\mathbb{R}^d) : W_\rho^{*,0} f \in L_w^1(\mathbb{R}^d)\}$$

with a norm given by

$$\|\cdot\|_{H_{w,\rho,0}^1(\mathbb{R}^d)} := \|W_\rho^{*,0} \cdot\|_{L_w^1(\mathbb{R}^d)},$$

where

$$W_\rho^{*,0} f(x) = \sup_{0 < t < \rho(x)^2} |e_{\text{loc}}^{-t\Delta} f(x)| \quad \text{and} \quad e_{\text{loc}}^{-t\Delta} f(x) = \int_{B(x,\rho(x))} k_t^0(x,y) f(y) dy$$

for all $x \in \mathbb{R}^d$ and $f \in L_w^1(\mathbb{R}^d)$.

Although the two Hardy spaces above are defined differently, they turn out to be the same.

Theorem 25 ([2, Theorem 4.6]). *Let $w \in A_1^\rho$. Then*

$$\left(H_{w,L}^1(\mathbb{R}^d), \|\cdot\|_{H_{w,L}^1(\mathbb{R}^d)}\right) = \left(H_{w,\rho,0}^1(\mathbb{R}^d), \|\cdot\|_{H_{w,\rho,0}^1(\mathbb{R}^d)}\right).$$

In the rest of the paper, we denote

$$\tau = 2C_\rho C_0(1 + 2C_0)^{k_0}, \quad (6)$$

where C_ρ and k_0 are defined in (4) and $C_0 = 1 + 6\sqrt{d}$. By choosing a $C_\rho > 1$ in (4), we may assume that $\tau > 1$ without loss of generality.

Next we define atoms and molecules of $H_{w,L}^1(\mathbb{R}^d)$.

Definition 26. Let $q \in (1, \infty]$. Let $x_0 \in \mathbb{R}^d$ and $r > 0$. A function a is an $(1, q)_w$ -atom associated with a ball $B = B(x_0, r)$ if

- (i) $r \leq \tau\rho(x_0)$,
- (ii) $\text{supp } a \subset B$,
- (iii) $\|a\|_{L^q(\mathbb{R}^d)} \leq |B|^{\frac{1}{q}} w(B)^{-1}$ and
- (iv) $\int_{\mathbb{R}^d} a = 0$ if $r < \frac{\rho(x_0)}{\tau}$.

Definition 27. Let $q \in (1, \infty]$. Let $x_0 \in \mathbb{R}^d$ and $r > 0$. Let $\epsilon > 0$. A function m is an $(1, q, \epsilon)_w$ -molecule associated with a ball $B = B(x_0, r)$ if

- (i) $r \leq \tau\rho(x_0)$,
- (ii) $\|m\|_{L^q(U_j(B))} \leq 2^{-j\epsilon} |2^j B|^{\frac{1}{q}} w(2^j B)^{-1}$ for all $j \in \{0, 1, 2, \dots\}$ and
- (iii) $\int_{\mathbb{R}^d} m = 0$.

It is known that each function in $H_{w,L}^1(\mathbb{R}^d)$ can be decomposed into a linear combination of $(1, \infty)_w$ -atoms.

Proposition 28 (cf. [2, Proposition 5.5]). A function $f \in H_{w,L}^1(\mathbb{R}^d)$ if and only if there exist a sequence of $(1, \infty)_w$ -atoms $\{a_i\}$ and a sequence of scalars $\{\lambda_i\}$ such that

$$\sum_{i \in \mathbb{N}} |\lambda_i| < \infty \quad \text{and} \quad f = \sum_{i \in \mathbb{N}} \lambda_i a_i \text{ in } L_w^1(\mathbb{R}^d).$$

Moreover,

$$\|f\|_{H_{w,L}^1(\mathbb{R}^d)} \sim \inf \left\{ \sum_{i \in \mathbb{N}} |\lambda_i| : f = \sum_{i \in \mathbb{N}} \lambda_i a_i \text{ and } a_i \text{'s are } (1, \infty)_w \text{-atoms} \right\}$$

for all $f \in H_{w,L}^1(\mathbb{R}^d)$.

Next we aim to show that molecules in Definition 27 belong to $H_{w,L}^1(\mathbb{R}^d)$ with a suitable choice of the index q . To do so we need some preliminary estimates of the C_0 -semigroup $(e^{-tL})_{t>0}$.

First we list some basic facts regarding the kernel $k_t(\cdot, \cdot)$ of e^{-tL} for each $t > 0$.

Proposition 29 (cf. [10, Theorem 4.10 and Proposition 4.11]). The following hold.

- (i) The kernel $k_t(\cdot, \cdot)$ admits a Gaussian upper bound, i.e., for every $N > 0$, there exist constants $C, c > 0$ such that

$$0 \leq k_t(x, y) \leq \frac{C}{t^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{ct}} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)} \right)^{-N} \quad (7)$$

for all $x, y \in \mathbb{R}^d$ and $t > 0$.

- (ii) For every $N > 0$, there are constants $C, c > 0$ such that

$$|k_t(x+h, y) - k_t(x, y)| \leq C \left(\frac{|h|}{\sqrt{t}} \right)^\delta \frac{1}{t^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{ct}} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)} \right)^{-N} \quad (8)$$

for all $\delta \in (0, 1 \wedge \sigma_0)$ and for all $x, y \in \mathbb{R}^d$ and $|h| < \sqrt{t}$.

Generally speaking the restriction in $|h| < \sqrt{t}$ in (8) can be removed, as shown by the following proposition.

Proposition 30. *There exist constants $C, c > 0$ such that*

$$|k_t(x, y) - k_t(x, z)| \leq C \left(\frac{|y - z|}{|x - y|} \right)^\delta \frac{1}{t^{\frac{d}{2}}} e^{-\frac{|x-z|^2}{ct}} \tag{9}$$

for all $\delta \in (0, 1 \wedge \sigma_0)$, $t > 0$ and for all $x, y, z \in \mathbb{R}^d$ such that $|x - y| \sim |x - z|$.

Proof. Let $t > 0$ and $x, y, z \in \mathbb{R}^d$ such that $|x - y| \sim |x - z|$. We consider two cases.

Case 1. Suppose $|y - z| < \sqrt{t}$. Then it follows from (8) and the symmetry of k_t in the first and second variables that

$$\begin{aligned} |k_t(x, y) - k_t(x, z)| &\lesssim \left(\frac{|y - z|}{\sqrt{t}} \right)^\delta \frac{1}{t^{\frac{d}{2}}} e^{-\frac{|x-z|^2}{ct}} \lesssim \left(\frac{|y - z|}{\sqrt{t}} \right)^\delta \frac{1}{t^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{2ct}} e^{-\frac{|x-z|^2}{2ct}} \\ &\lesssim \left(\frac{|y - z|}{\sqrt{t}} \right)^\delta \frac{1}{t^{\frac{d}{2}}} \left(\frac{\sqrt{t}}{|x - y|} \right)^\delta e^{-\frac{|x-z|^2}{2ct}} \sim \left(\frac{|y - z|}{|x - y|} \right)^\delta \frac{1}{t^{\frac{d}{2}}} e^{-\frac{|x-z|^2}{ct}} \end{aligned}$$

for all $\delta \in (0, 1 \wedge \sigma_0)$, where we used the inequality $e^{-cs} \lesssim s^{-\alpha}$ for all $\alpha > 0$ in the third step.

Case 2. Suppose $|y - z| \geq \sqrt{t}$. Then we deduce from (7) and the symmetry of k_t in the first and second variables that

$$\begin{aligned} |k_t(x, y) - k_t(x, z)| &\leq k_t(x, y) + k_t(x, z) \lesssim \frac{1}{t^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{ct}} \sim \frac{1}{t^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{ct}} e^{-\frac{|x-z|^2}{ct}} \\ &\lesssim \frac{1}{t^{\frac{d}{2}}} \left(\frac{\sqrt{t}}{|x - y|} \right)^\delta e^{-\frac{|x-z|^2}{ct}} \lesssim \frac{1}{t^{\frac{d}{2}}} \left(\frac{|y - z|}{|x - y|} \right)^\delta e^{-\frac{|x-z|^2}{ct}} \end{aligned}$$

for all $\delta > 0$, where again we used the inequality $e^{-cs} \lesssim s^{-\alpha}$ for all $\alpha > 0$ in the fourth step.

The claim now follows. □

Based on the above kernel estimates, we can obtain the following point estimate of the C_0 -semigroup $(e^{-t\Delta})_{t>0}$.

Lemma 31. *Let $j \in \mathbb{N}$, $\beta \in (1, \infty]$ and $\delta \in (0, 1 \wedge \sigma_0)$ be determined by (8). Let $B = B(x_B, r_B)$ be a ball. Let $u \in L^{\beta}_{loc}(\mathbb{R}^d)$ be such that $\text{supp } u \subset 2^j B$ and $\int_{\mathbb{R}^d} u = 0$. Then*

$$|e^{-t\Delta}_{loc} u(x)| \lesssim \frac{(2^j r_B)^\delta}{|x - x_B|^{d+\delta}} |2^j B|^{\frac{1}{\beta'}} \|u\|_{L^\beta(\mathbb{R}^d)}$$

for all $t > 0$ and $x \notin 2^{j+2} B$.

Proof. Let $t > 0$ and $x \notin 2^{j+2} B$. Then $|x - y| \sim |x - x_B|$ for all $y \in 2^j B$ and

$$\begin{aligned} |e^{-t\Delta}_{loc} u(x)| &= \left| \int_{2^j B \cap B(x_B, \rho(x_B))} (k_t^0(x, y) - k_t^0(x, x_B)) u(y) dy \right| \\ &\lesssim \int_{2^j B} \left(\frac{|y - x_B|}{|x - y|} \right)^\delta \frac{1}{t^{\frac{d}{2}}} e^{-\frac{|x-x_B|^2}{ct}} |u(y)| dy \\ &\lesssim \left(\frac{2^j r_B}{|x - x_B|} \right)^\delta \frac{1}{t^{\frac{d}{2}}} e^{-\frac{|x-x_B|^2}{ct}} \int_{2^j B} |u(y)| dy \lesssim \frac{(2^j r_B)^\delta}{|x - x_B|^{d+\delta}} \int_{2^j B} |u(y)| dy \\ &\lesssim \frac{(2^j r_B)^\delta}{|x - x_B|^{d+\delta}} |2^j B|^{\frac{1}{\beta'}} \|u\|_{L^\beta(2^j B)} \end{aligned}$$

for all $\delta > 0$, where we used (9) in the second step and Hölder inequality in the last step. □

The following technical lemma is also in need when we prove that certain $(1, q, \epsilon)_w$ -molecules belong to $H^1_{w,L}(\mathbb{R}^d)$.

Lemma 32. Let $j \in \mathbb{N}$, $w \in A_1^p$ and $v > 1$ be determined by Lemma 12. Suppose $u_j \in L_{loc}^{v'}(\mathbb{R}^d)$ satisfies

$$\text{supp } u_j \subset 2^j B, \quad \int_{\mathbb{R}^d} u_j = 0 \quad \text{and} \quad \|u_j\|_{L^{v'}(\mathbb{R}^d)} \lesssim 2^{-j\epsilon} |2^j B|^{\frac{1}{v'}} w(2^j B)^{-1} \tag{10}$$

for all ball $B = B(x_B, r_B)$ with $r_B < \tau \rho(x_B)$. Then

$$\sum_{j \in \mathbb{N}} \|\mathcal{M}_L u_j\|_{L_w^1(\mathbb{R}^d)} \lesssim 1.$$

Proof. Using Theorem 25 it suffices to show

$$\sum_{j \in \mathbb{N}} \|W_\rho^{*,0} u_j\|_{L_w^1(\mathbb{R}^d)} \lesssim 1.$$

Let $j \in \mathbb{N}$. Then

$$\|W_\rho^{*,0} u_j\|_{L_w^1(\mathbb{R}^d)} = \|W_\rho^{*,0} u_j\|_{L_w^1(2^{j+2}B)} + \|W_\rho^{*,0} u_j\|_{L_w^1((2^{j+2}B)^c)} =: I + II.$$

We will estimate the two terms separately.

Term I. Since $r_B < \tau \rho(x_B)$, Lemma 13 implies that $w(2B) \lesssim w(B)$. Using this and the boundedness of $W_\rho^{*,0}$ on $L^{v'}(\mathbb{R}^d)$, we derive

$$\begin{aligned} \|W_\rho^{*,0} u_j\|_{L_w^1(2^{j+2}B)} &\leq \|w\|_{L^v(2^{j+2}B)} \|W_\rho^{*,0} u_j\|_{L^{v'}(2^{j+2}B)} \lesssim |2^{j+2}B|^{\frac{1}{v}} \left(\inf_{2^{j+2}B} w \right) \|u_j\|_{L^{v'}(2^{j+2}B)} \\ &\lesssim |2^j B|^{\frac{1}{v}} \left(\inf_{2^j B} w \right) 2^{-j\epsilon} |2^j B|^{\frac{1}{v'}} w(2^j B)^{-1} \\ &= |2^j B| \left(\inf_{2^j B} w \right) w(2^j B)^{-1} 2^{-j\epsilon} \leq 2^{-j\epsilon}, \end{aligned}$$

where we used Hölder inequality in the first step, Lemma 12 in the second step and Lemma 13 in the last step.

Term II. Let $\delta \in (0, 1 \wedge \sigma_0)$ be determined by (8). We use Lemma 31 and (10) to derive

$$\begin{aligned} |e^{-t\Delta} u_j(x)| &\lesssim \frac{(2^j r_B)^\delta}{|x - x_B|^{d+\delta}} |2^j B|^{\frac{1}{v}} \|u_j\|_{L^{v'}(2^j B)} \lesssim \frac{(2^j r_B)^\delta}{|x - x_B|^{d+\delta}} |2^j B|^{\frac{1}{v}} 2^{-j\epsilon} |2^j B|^{\frac{1}{v'}} w^{-1}(2^j B) \\ &= 2^{-j\epsilon} |2^j B| w^{-1}(2^j B) \frac{(2^j r_B)^\delta}{|x - x_B|^{d+\delta}} \end{aligned}$$

for all $t > 0$ and $x \in (2^{j+2}B)^c$. Furthermore it follows from the definition of $W_\rho^{*,0}$ that $W_\rho^{*,0} u_j(x) > 0$ implies

$$|x - x_B| < r + \rho(x) \leq (\tau + (2 + \tau)^{k_0}) \rho(x_0) =: \tilde{\tau} \rho(x_B),$$

where k_0 is given by (4). Choose the smallest $a \in \{3, 4, 5, \dots\}$ such that $\tilde{\tau} \rho(x_B) < 2^{j+a} r_B$. Consequently

$$\begin{aligned} \|W_\rho^{*,0} u_j\|_{L_w^1((2^{j+2}B)^c)} &\lesssim 2^{-j\epsilon} |2^j B| w(2^j B)^{-1} \int_{(2^{j+2}B)^c \cap B(x_B, \tilde{\tau} \rho(x_B))} \frac{(2^j r_B)^\delta}{|x - x_B|^{d+\delta}} w(x) \, dx \\ &= 2^{-j\epsilon} |2^j B| w(2^j B)^{-1} \int_{2^{j+2}r_B < |x-x_B| < \tilde{\tau} \rho(x_B)} \frac{(2^j r_B)^\delta}{|x - x_B|^{d+\delta}} w(x) \, dx \\ &\lesssim 2^{-j\epsilon} |2^j B| w(2^j B)^{-1} \int_{2^{j+2}r_B < |x-x_B| < 2^{j+a}r_B} \frac{(2^j r_B)^\delta}{|x - x_B|^{d+\delta}} w(x) \, dx \\ &\lesssim 2^{-j\epsilon} |2^j B| w(2^j B)^{-1} \sum_{k=j+3}^{j+a} \frac{1}{2^{(k-j)(d+\delta)}} \int_{U_k(B)} w \\ &\lesssim 2^{-j\epsilon} |2^j B| w(2^j B)^{-1} \inf_{2^j B} w \lesssim 2^{-j\epsilon}. \end{aligned}$$

Combining the estimates for the two terms *I* and *II* together yields

$$\sum_{j \in \mathbb{N}} \|W_p^{*,0} u_j\|_{L^1_w(\mathbb{R}^d)} \lesssim \sum_{j \in \mathbb{N}} 2^{-j\epsilon} \lesssim 1.$$

This completes the proof. □

We are now ready to show that certain $(1, q, \epsilon)_w$ -molecules in Definition 27 belong to $H^1_{w,L}(\mathbb{R}^d)$.

Proposition 33. *Let $B = B(x_B, r_B)$ be a ball and $\epsilon > 0$. Let $w \in A^p_1$ be ϵ -Lebesgue stable and $\nu > 1$ be determined by Lemma 12. Let m be a $(1, \nu', \epsilon)_w$ -molecule associated with B . Then $m \in H^1_{w,L}(\mathbb{R}^d)$. Moreover, $\|m\|_{H^1_{w,L}(\mathbb{R}^d)} \lesssim 1$.*

Proof. We decompose m as

$$m = \sum_{j \in \mathbb{N}} a_j + \sum_{j \in \mathbb{N}} b_j,$$

where

$$a_j = \left(m - \frac{1}{|U_j(B)|} \int_{U_j(B)} m \right) \mathbb{1}_{U_j(B)} \quad \text{and} \quad b_j = \left(\frac{\mathbb{1}_{U_{j+1}(B)}}{|U_{j+1}(B)|} - \frac{\mathbb{1}_{U_j(B)}}{|U_j(B)|} \right) \int_{(2^j B)^c} m$$

for all $j \in \mathbb{N}$. From this we deduce that

$$\|\mathcal{M}_L(m)\|_{L^1_w(\mathbb{R}^d)} \leq \sum_{j \in \mathbb{N}} \|\mathcal{M}_L(a_j)\|_{L^1_w(\mathbb{R}^d)} + \sum_{j \in \mathbb{N}} \|\mathcal{M}_L(b_j)\|_{L^1_w(\mathbb{R}^d)} =: I + II.$$

Next we estimate each term separately.

Term I. It is straightforward to check that

$$\text{supp } a_j \subset 2^j B, \quad \int_{\mathbb{R}^d} a_j = 0 \quad \text{and} \quad \|a_j\|_{L^{\nu'}(\mathbb{R}^d)} \lesssim 2^{-j\epsilon} |2^j B|^{\frac{1}{\nu'}} w^{-1}(2^j B)$$

for all $j \in \mathbb{N}$. Therefore $I \lesssim 1$ by Lemma 32.

Term II. Let $j \in \mathbb{N}$. It follows from the definition of b_j that

$$\text{supp } b_j \subset 2^{j+1} B \quad \text{and} \quad \int_{\mathbb{R}^d} b_j = 0.$$

Next we aim to show that

$$\|b_j\|_{L^{\nu'}(\mathbb{R}^d)} \lesssim 2^{-j\epsilon} |2^j B|^{\frac{1}{\nu'}} w(2^j B)^{-1}. \tag{11}$$

To prove this we first note that

$$\|b_j\|_{L^{\nu'}(\mathbb{R}^d)} \lesssim |2^j B|^{\frac{1}{\nu'}-1} \sum_{k=j+1}^{\infty} \int_{U_k(B)} |m|.$$

Using Hölder inequality and the size condition of m , we also have

$$\int_{U_k(B)} |m| \leq |2^k B|^{\frac{1}{\nu}} \|m\|_{L^{\nu'}(U_k(B))} \leq |2^k B|^{\frac{1}{\nu}} 2^{-k\epsilon} |2^k B|^{\frac{1}{\nu'}} w(2^k B)^{-1} = 2^{-k\epsilon} |2^k B| w(2^k B)^{-1}.$$

The two estimates together with the ϵ -Lebesgue stability of w give

$$\begin{aligned} \|b_j\|_{L^{\nu'}(\mathbb{R}^d)} &\lesssim |2^j B|^{\frac{1}{\nu'}-1} \sum_{k=j+1}^{\infty} 2^{-k\epsilon} |2^k B| w(2^k B)^{-1} \\ &= 2^{-j\epsilon} |2^j B|^{\frac{1}{\nu'}} w(2^j B)^{-1} \sum_{k=j+1}^{\infty} 2^{(j-k)\epsilon} \frac{|2^k B|}{|2^j B|} \frac{w(2^j B)}{w(2^k B)} \\ &\lesssim 2^{-j\epsilon} |2^j B|^{\frac{1}{\nu'}} w(2^j B)^{-1}. \end{aligned}$$

Hence (11) follows from Lemma 32.

The claim is now justified by combining the estimates for *I* and *II* together. □

The following proposition provides a molecular decomposition of the Hardy space $H^1_{w,L}(\mathbb{R}^d)$.

Proposition 34. *Let v be determined by Lemma 12. Let $\epsilon \in (0, \infty)$ and $f \in L^1_{loc}(\mathbb{R}^d)$ be such that there exist a sequence of $(1, v', \epsilon)_w$ -molecules $\{m_i\}$ and a sequence of scalars $\{\lambda_i\}_i$ such that*

$$\sum_{i \in \mathbb{N}} |\lambda_i| < \infty \quad \text{and} \quad f = \sum_{i \in \mathbb{N}} \lambda_i m_i \text{ in } L^1_w(\mathbb{R}^d).$$

Then $f \in H^1_{w,L}(\mathbb{R}^d)$ and

$$\|f\|_{H^1_{w,L}(\mathbb{R}^d)} \lesssim \sum_{i \in \mathbb{N}} |\lambda_i|.$$

Proof. This is a direct consequence of Proposition 33. □

4. Boundedness on weighted Hardy spaces associated with L

In this section we will show that the operator $T = \nabla^2 L^{-1}$ is bounded on $H^1_{L,w}(\mathbb{R}^d)$, which is the content of our first main theorem - Theorem 5.

Recall that we let $k_t(\cdot, \cdot)$ denote the kernel of e^{-tL} for each $t > 0$. In what follows we always denote $\sigma_0 = 2 - \frac{d}{\sigma}$.

Proposition 35 (cf. [8, Proposition 5.2]). *Let $\theta \in [1, \sigma]$. Then there exist constants $C, c > 0$ such that the following holds for all $N > 0$.*

(i) *For all $y \in \mathbb{R}^d$ and $t > 0$ one has*

$$\left\| \nabla^2 k_t(\cdot, y) e^{\frac{|\cdot-y|^2}{ct}} \right\|_{L^\theta(\mathbb{R}^d)} \leq C t^{-1-\frac{d}{2\theta'}} \left(1 + \frac{\sqrt{t}}{\rho(y)} \right)^{-N}.$$

(ii) *For all $|y - z| \leq \sqrt{t}$ and $\sigma_1 \in (0, \sigma_0)$ one has*

$$\left\| \left(\nabla^2 k_t(\cdot, y) - \nabla^2 k_t(\cdot, z) \right) e^{\frac{|\cdot-y|^2}{ct}} \right\|_{L^\theta(\mathbb{R}^d)} \leq C \left(\frac{|y-z|}{\sqrt{t}} \right)^{\sigma_1} t^{-1-\frac{d}{2\theta'}} \left(1 + \frac{\sqrt{t}}{\rho(y)} \right)^{-N}.$$

Next we aim to remove the condition $|y - z| \leq \sqrt{t}$ in Proposition 35(ii).

Proposition 36. *Let $\theta \in [1, \sigma]$. Then there exist constants $C, c > 0$ such that for all $t > 0, x, y, z \in \mathbb{R}^d$ such that $|x - y| \sim |x - z|$ and $\sigma_1 \in (0, \sigma_0)$ one has*

$$\left(\int_{\mathbb{R}^d} \left| \left(\nabla^2 k_t(x, y) - \nabla^2 k_t(x, z) \right) e^{\frac{|x-y|^2}{ct}} \right|^\theta dx \right)^{\frac{1}{\theta}} \leq C \left(\frac{|y-z|}{\sqrt{t}} \right)^{\sigma_1} t^{-1-\frac{d}{2\theta'}}.$$

Proof. Let $t > 0$ and $x, y, z \in \mathbb{R}^d$ such that $|x - y| \sim |x - z|$. We consider two cases.

Case 1. Suppose $|y - z| \leq \sqrt{t}$. Then the claim follows from Proposition 35(ii).

Case 2. Suppose $|y - z| \geq \sqrt{t}$. Then

$$\begin{aligned} & \left(\int_{\mathbb{R}^d} \left| \left(\nabla^2 k_t(x, y) - \nabla^2 k_t(x, z) \right) e^{\frac{|x-y|^2}{ct}} \right|^\theta dx \right)^{\frac{1}{\theta}} \\ & \leq \left(\int_{\mathbb{R}^d} \left| \nabla^2 k_t(x, y) e^{\frac{|x-y|^2}{ct}} \right|^\theta dx \right)^{\frac{1}{\theta}} + \left(\int_{\mathbb{R}^d} \left| \nabla^2 k_t(x, z) e^{\frac{|x-y|^2}{ct}} \right|^\theta dx \right)^{\frac{1}{\theta}} \\ & \lesssim \left(\int_{\mathbb{R}^d} \left| \nabla^2 k_t(x, y) e^{\frac{|x-y|^2}{ct}} \right|^\theta dx \right)^{\frac{1}{\theta}} + \left(\int_{\mathbb{R}^d} \left| \nabla^2 k_t(x, z) e^{\frac{|x-z|^2}{ct}} \right|^\theta dx \right)^{\frac{1}{\theta}} \\ & \lesssim t^{-1-\frac{d}{2\theta'}} \leq \left(\frac{|y-z|}{\sqrt{t}} \right)^{\sigma_1} t^{-1-\frac{d}{2\theta'}}, \end{aligned}$$

for all $\delta_1 > 0$, where we used $|x - y| \sim |x - z|$ in the second step and Proposition 35(i) in the third step.

The claim now follows. □

Let $K(\cdot, \cdot)$ be the kernel of T . We note that

$$K(\cdot, \cdot) = \int_0^\infty \nabla_1^2 k_t(\cdot, \cdot) dt,$$

where $\nabla_1 k_t(\cdot, \cdot)$ is the gradient with respect to the first variable of $k_t(\cdot, \cdot)$ for all $t > 0$.

The following proposition states that T is a generalized Calderón–Zygmund operator in the sense of [8, Definition 1.1].

Proposition 37. *The following hold.*

(i) *Let $q \in (1, \sigma)$. For all $N > 0$ there exists a $C > 0$ such that*

$$\left(\int_{R < |x - x_B| < 2R} |K(x, y)|^q dx \right)^{\frac{1}{q}} \leq C R^{-\frac{d}{q'}} \left(\frac{\rho(x_B)}{R} \right)^N \tag{12}$$

for all $y \in B(x_B, \rho(x_B))$ and $R > \rho(x_B)$.

(ii) *Let $q \in (1, \sigma)$, $\sigma_0 = 2 - \frac{d}{\sigma}$ and $\gamma \in (0, \sigma_0)$. There exists a constant $C > 0$ such that*

$$\left(\int_{2^k r_B < |x - x_B| < 2^{k+1} r_B} |K(x, y) - K(x, x_B)|^q dx \right)^{\frac{1}{q}} \leq C 2^{-k\gamma} |2^k B|^{-\frac{1}{q'}} \tag{13}$$

for all balls $B = B(x_B, r_B)$, $y \in B$ and $k \in \mathbb{N}$.

Proof. We divide the proof into two parts.

Part I. We prove (12). Let $N > 0$ and $B = B(x_B, R)$ with $R > \rho(x_B)$. Let $y \in B(x_B, \rho(x_B))$. We deduce that $\rho(y) \sim \rho(x_B)$ and

$$\begin{aligned} \left(\int_{R < |x - x_B| < 2R} |K(x, y)|^q dx \right)^{\frac{1}{q}} &= \left(\int_{R < |x - x_B| < 2R} \left| \int_0^\infty \nabla_1^2 k_t(x, y) dt \right|^q dx \right)^{\frac{1}{q}} \\ &\leq \int_0^\infty \left(\int_{R < |x - x_B| < 2R} |\nabla_1^2 k_t(x, y)|^q dx \right)^{\frac{1}{q}} dt \\ &\lesssim \int_0^\infty e^{-\frac{R^2}{ct}} t^{-1 - \frac{d}{2q'}} \left(1 + \frac{\sqrt{t}}{\rho(x_B)} \right)^{-N} dt \\ &= \int_0^{R^2} e^{-\frac{R^2}{ct}} t^{-1 - \frac{d}{2q'}} \left(1 + \frac{\sqrt{t}}{\rho(x_B)} \right)^{-N} dt + \int_{R^2}^\infty e^{-\frac{R^2}{ct}} t^{-1 - \frac{d}{2q'}} \left(1 + \frac{\sqrt{t}}{\rho(x_B)} \right)^{-N} dt \\ &=: Ia + Ib, \end{aligned}$$

where we used Minkowski’s inequality in the second step and Proposition 35(i) in the third step. For Ia we have

$$Ia = \int_0^{R^2} e^{-\frac{R^2}{ct}} t^{-1 - \frac{d}{2q'}} \left(1 + \frac{\sqrt{t}}{\rho(x_B)} \right)^{-N} dt \lesssim R^{-\frac{d}{q'}} \left(\frac{\rho(x_B)}{R} \right)^N,$$

where we used $e^{-\frac{R^2}{ct}} \lesssim \left(\frac{\sqrt{t}}{R} \right)^{N + \frac{d}{q'} + 2}$ in the last step. For Ib we have

$$\begin{aligned} Ib &= \int_{R^2}^\infty e^{-\frac{R^2}{ct}} t^{-1 - \frac{d}{2q'}} \left(1 + \frac{\sqrt{t}}{\rho(x_B)} \right)^{-N} dt \leq \int_{R^2}^\infty t^{-1 - \frac{d}{2q'}} \left(1 + \frac{\sqrt{t}}{\rho(x_B)} \right)^{-N} dt \\ &\lesssim R^{-\frac{d}{q'}} \left(\frac{\rho(x_B)}{R} \right)^N. \end{aligned}$$

Hence (12) follows.

Part II. We prove (13). Let $B = B(x_B, r_B)$ be a ball and $y \in B$. Let $k \in \mathbb{N}$. Then

$$\begin{aligned} & \left(\int_{2^k r_B < |x-x_B| < 2^{k+1} r_B} |K(x, y) - K(x, x_B)|^q dx \right)^{\frac{1}{q}} \\ &= \left(\int_{2^k r_B < |x-x_B| < 2^{k+1} r_B} \left| \int_0^\infty \nabla_1^2 k_t(x, y) - \nabla_1^2 k_t(x, x_B) dt \right|^q dx \right)^{\frac{1}{q}} \\ &\leq \int_0^\infty \left(\int_{2^k r_B < |x-x_B| < 2^{k+1} r_B} |\nabla_1^2 k_t(x, y) - \nabla_1^2 k_t(x, x_B)|^q dx \right)^{\frac{1}{q}} dt \\ &\lesssim \int_0^\infty \left(\frac{|y-x_B|}{\sqrt{t}} \right)^{\sigma_1} e^{-\frac{4^k r_B^2}{ct}} t^{-1-\frac{d}{2q'}} dt \\ &= \int_0^{r_B^2} \left(\frac{|y-x_B|}{\sqrt{t}} \right)^{\sigma_1} e^{-\frac{4^k r_B^2}{ct}} t^{-1-\frac{d}{2q'}} dt + \int_{r_B^2}^\infty \left(\frac{|y-x_B|}{\sqrt{t}} \right)^{\sigma_1} e^{-\frac{4^k r_B^2}{ct}} t^{-1-\frac{d}{2q'}} dt \\ &=: IIa + IIb, \end{aligned}$$

where we used Minkowski’s inequality in the second step and Proposition 35(ii) in the third step.

To estimate IIa , we first set $\epsilon = \frac{1}{2}(\gamma + \frac{d}{q'})$. It follows that

$$\begin{aligned} IIa &= \int_0^{r_B^2} \left(\frac{|y-x_B|}{\sqrt{t}} \right)^{\sigma_1} e^{-\frac{4^k r_B^2}{ct}} t^{-1-\frac{d}{2q'}} dt \lesssim \int_0^{r_B^2} e^{-\frac{4^k r_B^2}{ct}} t^{-1-\frac{d}{2q'}} dt \\ &\lesssim 4^{-k\epsilon} r_B^{-2\epsilon} \int_0^{r_B^2} t^{-1-\frac{d}{2q'}+\epsilon} dt \lesssim 4^{-k\epsilon} r_B^{-\frac{d}{q'}} = 2^{-k\gamma} |2^k B|^{-\frac{1}{q'}}. \end{aligned}$$

For IIb let $\sigma_1 \in (\gamma, \sigma_0)$. Then

$$\begin{aligned} IIb &= \int_{r_B^2}^\infty \left(\frac{|y-x_B|}{\sqrt{t}} \right)^{\sigma_1} e^{-\frac{4^k r_B^2}{ct}} t^{-1-\frac{d}{2q'}} dt \leq \int_{r_B^2}^\infty \left(\frac{|y-x_B|}{\sqrt{t}} \right)^{\sigma_1} e^{-\frac{4^k r_B^2}{ct}} t^{-1-\frac{d}{2q'}} dt \\ &\lesssim 4^{-k\epsilon} r_B^{\sigma_1-2\epsilon} \int_{r_B^2}^\infty t^{-1-\frac{d}{2q'}+\epsilon-\frac{\sigma_1}{2}} dt \lesssim 4^{-k\epsilon} r_B^{-\frac{d}{q'}} = 2^{-k\gamma} |2^k B|^{-\frac{1}{q'}}. \end{aligned}$$

This completes the proof of (13). □

We are now ready to prove Theorem 5.

Proof of Theorem 5. Let $q \in (1, \sigma)$. By Propositions 28 and 34 it suffices to show that T maps $(1, \infty)_w$ -atoms into $(1, q, \epsilon)_w$ -molecules.

Let $B = B(x_B, r_B)$ be a ball and a a $(1, \infty)_w$ -atom associated with B . We will show that Ta is a $(1, q, \epsilon)_w$ -molecule associated with B , where $q \in (1, \sigma)$.

Condition (i) in Definition 27 is automatic.

The cancellation condition (iii) in Definition 27 is also clear as

$$\int_{\mathbb{R}^d} Ta = \int_B a T^* 1 = 0.$$

It remains to show that size condition (ii) in Definition 27 also holds for Ta , i.e.,

$$\|Ta\|_{L^q(U_j(B))} \leq 2^{-j\epsilon} |2^j B|^{\frac{1}{q}} w(2^j B)^{-1} \tag{14}$$

for all $j \in \mathbb{N}$.

Choose a $j_0 \in \mathbb{N}$ with $j_0 > \log_2(4\tau)$, where τ is given in (6). Then

$$\|Ta\|_{L^q(2^j B)} \lesssim \|a\|_{L^q(2^j B)} \leq |2^j B|^{\frac{1}{q}} \|a\|_{L^\infty(2^j B)} = \frac{|2^j B|^{\frac{1}{q}}}{w(B)} \lesssim |2^j B|^{\frac{1}{q}} w(2^j B)^{-1}$$

for all $j \in \{0, \dots, j_0\}$, where we used the boundedness of T on $L^q(\mathbb{R}^d)$ (cf. [14, Theorem 1.2]) in the first step and the size condition of a $(1, \infty)_w$ -atom in the second step and Lemma 19 in the last step. Hence (14) holds for all $j \in \{0, \dots, j_0\}$.

Next let $j > j_0$. Set $\gamma = \theta + \epsilon \in (\theta, \sigma_0)$ and $\rho_B = \rho(x_B)$. We consider two cases.

Case 1. Suppose $r_B < \frac{\rho_B}{\tau}$. Then Lemma 19 implies $w(B)^{-1} \lesssim 2^{j(d+\theta)} w(2^j B)^{-1}$. Furthermore

$$\begin{aligned} \|Ta\|_{L^q(U_j(B))} &\leq \left(\int_{U_j(B)} \left(\int_B |K(x, y) - K(x, x_B)| |a(y)| dy \right)^q dx \right)^{\frac{1}{q}} \\ &\leq \int_B \left(\int_{U_j(B)} |K(x, y) - K(x, x_B)|^q dx \right)^{\frac{1}{q}} |a(y)| dy \\ &\lesssim 2^{-j\gamma} |2^j B|^{-\frac{1}{q'}} \|a\|_{L^\infty(B)} |B| \lesssim 2^{-j\gamma} |2^j B|^{-\frac{1}{q'}} w(B)^{-1} |B| \\ &= 2^{-j(\gamma+d)} |2^j B|^{\frac{1}{q}} w(B)^{-1} \lesssim 2^{-j(\gamma+d)} |2^j B|^{\frac{1}{q}} 2^{j(d+\theta)} w(2^j B)^{-1} \\ &= 2^{-j(\gamma-\theta)} |2^j B|^{\frac{1}{q}} w(2^j B)^{-1} = 2^{-j\epsilon} |2^j B|^{\frac{1}{q}} w(2^j B)^{-1}, \end{aligned}$$

where we used the cancellation property of a in the first step, Minkowski's inequality in the second step, Proposition 37(ii) in the third step, the size condition of a in the fourth step and Lemma 19 in the sixth step.

Case 2. Suppose $\frac{\rho_B}{\tau} \leq r_B \leq \tau \rho_B$. Then

$$\begin{aligned} \|Ta\|_{L^q(U_j(B))} &= \left(\int_{U_j(B)} \left| \int_B K(x, y) a(y) dy \right|^q dx \right)^{\frac{1}{q}} \leq \int_B \left(\int_{U_j(B)} |K(x, y)|^q dx \right)^{\frac{1}{q}} |a(y)| dy \\ &\leq |2^j B|^{-\frac{1}{q'}} \left(\frac{\rho_B}{2^j r_B} \right)^\gamma \|a\|_{L^1(B)} \leq |2^j B|^{-\frac{1}{q'}} \left(\frac{\rho_B}{2^j r_B} \right)^\gamma w(B)^{-1} |B| \\ &\lesssim 2^{-j(\gamma+d)} |2^j B|^{\frac{1}{q}} w(B)^{-1} \lesssim 2^{-j(\gamma+d)} |2^j B|^{\frac{1}{q}} 2^{j(d+\theta)} w(2^j B)^{-1} \\ &= 2^{-j(\gamma-\theta)} |2^j B|^{\frac{1}{q}} w(2^j B)^{-1} = 2^{-j\epsilon} |2^j B|^{\frac{1}{q}} w(2^j B)^{-1}, \end{aligned}$$

where Minkowski's inequality in the second step, Proposition 37(i) with $N = \gamma$ in the third step, the size condition of a in the fourth step and Lemma 19 in the sixth step.

Hence (14) holds for all $j \in \mathbb{N}$. The theorem now follows. □

5. Weighted BMO spaces associated with L

This section deals with the space $BMO_{w,\rho}(\mathbb{R}^d)$. By the end of this section we prove Proposition 6: the space $BMO_{w,\rho}(\mathbb{R}^d)$ is the dual of $H^1_{w,L}(\mathbb{R}^d)$.

In what follows we denote

$$f_B := \frac{1}{|B|} \int_B f$$

for all ball $B \subset \mathbb{R}^d$.

Recall that a function $f \in L^1_{loc}(\mathbb{R}^d)$ is said to belong to the space $BMO_{w,\rho}(\mathbb{R}^d)$ if there exists a $C > 0$ such that

$$\frac{1}{w(B_s)} \int_{B_s} |f - f_{B_s}| \leq C \quad \text{and} \quad \frac{1}{w(B_r)} \int_{B_r} |f| \leq C \tag{15}$$

for all balls $B_s = B(x, s)$ and $B_r = B(x, r)$ such that $0 < s < \rho(x) \leq r$.

The norm of $BMO_{w,\rho}(\mathbb{R}^d)$ is defined by

$$\| \cdot \|_{BMO_{w,\rho}(\mathbb{R}^d)} = \inf\{C > 0 : (15) \text{ holds}\}.$$

The next proposition gives a simpler equivalent norm of $BMO_{w,\rho}(\mathbb{R}^d)$.

Proposition 38 (cf. [3, Proposition 3.2]). *Let $w \in A_1^\rho$ and $f \in L^1_{\text{loc}}(\mathbb{R}^d)$. Then the following are equivalent.*

- (a) *There exists a $C > 0$ such that $\frac{1}{w(B_r)} \int_{B_r} |f| \leq C$ for all balls $B_r = B(x, r)$ with $r \geq \rho(x)$.*
- (b) $\sup_{x \in \mathbb{R}^d} \frac{1}{w(B(x, \rho(x)))} \int_{B(x, \rho(x))} |f| < \infty$.

Now we state two technical lemmas required in the proof of the duality from Hardy to BMO spaces.

Lemma 39. *Let $\theta \geq 0$ and $w \in A_1^{\rho, \theta}$. Then there exists a $C > 0$ such that*

$$\left(\int_B \frac{|u|^2}{w} \right)^{\frac{1}{2}} \leq C \|u\|_{BMO_{w, \rho}(\mathbb{R}^d)} w(B)^{\frac{1}{2}} \left(1 + \frac{r}{\rho(x)} \right)^{2\theta}$$

for every ball $B = B(x, r)$ and $u \in BMO_{w, \rho}(\mathbb{R}^d)$.

Proof. This follows from [22, Lemma 2.4 and Proposition 3.5]. □

Lemma 40. *Let $\theta \geq 0$ and $w \in A_1^{\rho, \theta}$. Let $B = B(x_0, R)$ be a ball with $R > \rho(x_0)$. Then there exists a $C > 0$ such that*

$$\|g\|_{H^1_{w, L}(\mathbb{R}^d)} \leq C \left(1 + \frac{R}{\rho(x_0)} \right)^{2\theta} w(B)^{\frac{1}{2}} \|g\|_{L^2_w(B)}$$

for all $g \in L^2_w(B)$ with $\text{supp } g \subset B$.

Proof. Let $g \in L^2_w(B)$ with $\text{supp } g \subset B$. Consider

$$\|g\|_{H^1_{w, L}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} (\mathcal{M}_L g) w = \int_{4B} (\mathcal{M}_L g) w + \int_{(4B)^C} (\mathcal{M}_L g) w =: I + II.$$

Next we estimate each term separately.

Term I. Since $w \in A_1^{\rho, \theta} \subset A_2^{\rho, \theta}$, the maximal function \mathcal{M}_L is bounded on $L^2_w(\mathbb{R}^d)$ by Lemma 23. It follows that

$$I \leq w(4B)^{\frac{1}{2}} \left(\int_{4B} (\mathcal{M}_L g)^2 w \right)^{\frac{1}{2}} \lesssim \left(1 + \frac{R}{\rho(x_0)} \right)^{2\theta} w(B)^{\frac{1}{2}} \|g\|_{L^2_w(B)},$$

where we used Hölder's inequality in the first step and Lemma 19 in the second step.

Term II. Since $w \in A_1^{\rho, \theta} \subset A_2^{\rho, \theta}$, we deduce from the definition of $A_2^{\rho, \theta}$ -weights that $w^{-1} \in A_2^{\rho, \theta}$ and

$$w^{-1}(B) \lesssim \left(1 + \frac{R}{\rho(x_0)} \right)^{2\theta} \frac{|B|^2}{w(B)}. \tag{16}$$

Let $x \in (4B)^C$. Then $|x - x_0| \sim |x - y|$ for all $y \in B$ and

$$\begin{aligned} |e^{-tL} g(x)| &\lesssim \int_B \left(\frac{\rho(y)}{\sqrt{t}} \right)^N \frac{1}{t^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{ct}} |g(y)| dy \\ &\lesssim \left(\frac{\rho(x_0)}{\sqrt{t}} \right)^N \frac{1}{t^{\frac{d}{2}}} e^{-\frac{|x-x_0|^2}{ct}} \int_B \left(1 + \frac{|x_0 - y|}{\rho(x_0)} \right)^{\frac{Nk_0}{k_0+1}} |g(y)| dy \\ &\lesssim \frac{\rho(x_0)^N}{|x - x_0|^{d+N}} \left(\frac{|x_0 - x|}{\rho(x_0)} \right)^{\frac{Nk_0}{k_0+1}} \int_B |g(y)| dy \lesssim \frac{\rho(x_0)^{\frac{N}{k_0+1}}}{|x - x_0|^{d+\frac{N}{k_0+1}}} \|g\|_{L^2_w(B)} (w^{-1}(B))^{\frac{1}{2}} \\ &\lesssim \frac{\rho(x_0)^{\frac{N}{k_0+1}}}{|x - x_0|^{d+\frac{N}{k_0+1}}} \|g\|_{L^2_w(B)} \left(1 + \frac{R}{\rho(x_0)} \right)^\theta \frac{|B|}{w(B)^{\frac{1}{2}}} \end{aligned}$$

for all $t > 0$ and $N > 0$, where we used Proposition 29(i) in the first step, Proposition 10 in the second step, the inequality $e^{-cs} \lesssim s^{-\alpha}$ for all $\alpha > 0$ in the third step, Hölder's inequality in the fourth step and (16) in the last step. As a consequence we have

$$\mathcal{M}_L g(x) \lesssim \frac{\rho(x_0)^{\frac{N}{k_0+1}}}{|x-x_0|^{d+\frac{N}{k_0+1}}} \|g\|_{L^2_w(B)} \left(1 + \frac{R}{\rho(x_0)}\right)^\theta \frac{|B|}{w(B)^{\frac{1}{2}}}$$

for all $N > 0$. Therefore by choosing $N = (k_0 + 1)(\theta + 1)$ we derive

$$\begin{aligned} II &\lesssim \|g\|_{L^2_w(B)} \left(1 + \frac{R}{\rho(x_0)}\right)^\theta \frac{|B|}{w(B)^{\frac{1}{2}}} \int_{(4B)^c} \frac{\rho(x_0)^{\frac{N}{k_0+1}}}{|x-x_0|^{d+\frac{N}{k_0+1}}} w(x) \, dx \\ &\leq \|g\|_{L^2_w(B)} \left(1 + \frac{R}{\rho(x_0)}\right)^\theta \frac{|B|}{w(B)^{\frac{1}{2}}} \sum_{j=5}^\infty \int_{U_j(B)} \frac{\rho(x_0)^{\frac{N}{k_0+1}}}{|x-x_0|^{d+\frac{N}{k_0+1}}} w(x) \, dx \\ &\leq \|g\|_{L^2_w(B)} \left(1 + \frac{R}{\rho(x_0)}\right)^\theta \frac{|B|}{w(B)^{\frac{1}{2}}} \sum_{j=5}^\infty \frac{\rho(x_0)^{\frac{N}{k_0+1}}}{(2^j R)^{d+\frac{N}{k_0+1}}} w(2^j B) \\ &\leq \|g\|_{L^2_w(B)} \left(1 + \frac{R}{\rho(x_0)}\right)^{2\theta} w(B)^{\frac{1}{2}} \sum_{j=5}^\infty \frac{1}{2^{j(\frac{N}{k_0+1}-\theta)}} \\ &\lesssim \|g\|_{L^2_w(B)} \left(1 + \frac{R}{\rho(x_0)}\right)^{2\theta} w(B)^{\frac{1}{2}}, \end{aligned}$$

where we used the fact that $R > \rho(x_0)$ and Lemma 19 in the second-to-last step.

Combining the estimates for I and II together yields the lemma. □

Lastly we prove that $BMO_{w,\rho}(\mathbb{R}^d)$ is the dual of $H^1_{w,L}(\mathbb{R}^d)$.

Proof of Proposition 6. \subset . Let $u \in BMO_{w,\rho}(\mathbb{R}^d)$. Consider

$$\Phi_u(f) = \int_{\mathbb{R}^d} u f$$

for all $f \in H^1_{w,L}(\mathbb{R}^d)$. We aim to show that $\Phi_u \in (H^1_{w,L}(\mathbb{R}^d))^*$. By Proposition 28, it suffices to show that

$$|\Phi_u(a)| \lesssim \|u\|_{BMO_{w,\rho}(\mathbb{R}^d)} \tag{17}$$

for all $(1, \infty)_w$ -atom a of $H^1_{w,L}(\mathbb{R}^d)$.

Let $B = B(x, r)$ be a ball and a be a $(1, \infty)_w$ -atom of $H^1_{w,L}(\mathbb{R}^d)$ associated with B . There are two cases.

Case 1. Suppose $r < \frac{\rho(x)}{\tau}$. Then the cancellation condition applies and we yield

$$|\Phi_u(a)| = \left| \int_{\mathbb{R}^d} u a \right| = \left| \int_B u a \right| = \left| \int_B (u - u_B) a \right| \leq \frac{1}{w(B)} \int_B |u - u_B| \leq \|u\|_{BMO_{w,\rho}(\mathbb{R}^d)}.$$

Case 2. Suppose $\frac{\rho(x)}{\tau} \leq r < \tau \rho(x)$. In this case $\frac{r}{\rho(x)} \sim 1$. It follows that

$$\begin{aligned} |\Phi_u(a)| &= \left| \int_{\mathbb{R}^d} u a \right| = \left| \int_B u a \right| \leq \left(\int_B \frac{u^2}{w} \right)^{\frac{1}{2}} \left(\int_B |a|^2 w \right)^{\frac{1}{2}} \\ &\lesssim \|u\|_{BMO_{w,\rho}(\mathbb{R}^d)} w(B)^{\frac{1}{2}} \left(1 + \frac{r}{\rho(x)}\right)^{2\theta} w(B)^{-\frac{1}{2}} \\ &\lesssim \|u\|_{BMO_{w,\rho}(\mathbb{R}^d)}, \end{aligned}$$

where we used Lemma 39 and the size condition of a in the second-to-last step.

Hence (17) follows and we showed that $BMO_{w,\rho}(\mathbb{R}^d) \subset (H^1_{w,L}(\mathbb{R}^d))^*$.

▷. Let $\Phi \in (H^1_{w,L}(\mathbb{R}^d))^*$. Let $N > \rho(0)$ and $B_N = B_N(0)$. Then Lemma 40 implies Φ is bounded on $L^2_w(B_N)$. By a Riesz representation theorem, there exists a $u_N \in L^2_{w^{-1}}(B_N)$ such that

$$\Phi(g) = \int_{B_N} u_N g$$

for all $g \in L^2_w(B_N)$ with $\text{supp } g \subset B_N$. Using Lemma 40 we derive

$$\left| \int_{B_N} u_N g \right| \leq \|\Phi\| \|g\|_{H^1_{w,L}(\mathbb{R}^d)} \lesssim \|\Phi\| \left(1 + \frac{R}{\rho(x_0)}\right)^{2\theta} w(B)^{\frac{1}{2}} \|g\|_{L^2_w(B)}$$

for all $g \in L^2_w(B_N)$ with $\text{supp } g \subset B_N$. Therefore

$$\|u_N\|_{L^2_{w^{-1}}(B_N)} \lesssim \|\Phi\| w(B)^{\frac{1}{2}} \left(1 + \frac{R}{\rho(x_0)}\right)^{2\theta}.$$

Iterating the above arguments in N , we deduce that there exists a $u \in L^2_{w^{-1},\text{loc}}(\mathbb{R}^d)$ such that $u|_{B_N} = u_N$ for all $N > \rho(0)$ and

$$\Phi(g) = \int_{\mathbb{R}^d} u g$$

for all $g \in L^2_{w,c}(\mathbb{R}^d) := \{f \in L^2_w(\mathbb{R}^d) : \text{supp } f \Subset \mathbb{R}^d\}$. Since each atom of $H^1_{w,L}(\mathbb{R}^d)$ belongs to $L^2_{w,c}(\mathbb{R}^d)$, we deduce that

$$\Phi(g) = \int_{\mathbb{R}^d} u g$$

for all $g \in H^1_{w,L}(\mathbb{R}^d)$.

It remains to show that $u \in BMO_{w,\rho}(\mathbb{R}^d)$. Let $B = B(x, r)$ be a ball. With Proposition 38 in mind, we consider two cases.

Case 1. Suppose $r < \rho(x)$. Let $f \in L^\infty(\mathbb{R}^d)$ with $\text{supp } f \subset B$ be such that $\|f\|_{L^\infty(\mathbb{R}^d)} \leq 1$ and

$$a = w(B)^{-1} (f - f_B) \mathbb{1}_B.$$

Then a is a $(1, \infty)_w$ -atom of $H^1_{w,L}(\mathbb{R}^d)$ associated with B . Indeed, a obviously satisfies Conditions (i), (ii) and (iii) of Definition 26. Concerning Condition (iv) of Definition 26, we observe that

$$\int_{\mathbb{R}^d} a = \frac{1}{w(B)} \int_B (f - f_B) = \frac{1}{w(B)} \left(\int_B f - \int_B f \right) = 0.$$

So the cancellation property is available for a by its definition, regardless of the size of r .

Using the cancellation property of a we obtain

$$\|\Phi\| \geq \Phi(a) = \int_B a u = \frac{1}{w(B)} \int_B (f - f_B) (u - u_B) = \frac{1}{w(B)} \int_B f (u - u_B).$$

Since f is arbitrary, this implies

$$\frac{1}{w(B)} \int_B |u - u_B| \leq \|\Phi\|.$$

Case 2. Suppose $r = \rho(x)$. Let $f \in L^\infty(\mathbb{R}^d)$ with $\text{supp } f \subset B$ be such that $\|f\|_{L^\infty(\mathbb{R}^d)} \leq 1$ and

$$a = w(B)^{-1} f \mathbb{1}_B.$$

Then a is a $(1, \infty)_w$ -atom of $H^1_{w,L}(\mathbb{R}^d)$ associated with B . Note that the cancellation property is not available in this case. The boundedness of Φ now implies

$$\|\Phi\| \geq \Phi(a) = \int_B a u = \frac{1}{w(B)} \int_B f u.$$

Since f is arbitrary, this implies

$$\frac{1}{w(B)} \int_B |u| \leq \|\Phi\|.$$

Based on the estimates in both cases together with Proposition 38 we can conclude that $u \in BMO_{w,\rho}(\mathbb{R}^d)$. Thus $(H^1_{w,L}(\mathbb{R}^d))^* \subset BMO_{w,\rho}(\mathbb{R}^d)$.

The proposition now follows. □

6. More estimates on the kernel of T

Recall that we denote $K(\cdot, \cdot)$ to be the kernel of $T = \nabla^2 L^{-1}$. Previously Proposition 37 provides the integral estimates of $K(\cdot, \cdot)$ which are useful for the boundedness of T on $H^1_{w,L}(\mathbb{R}^d)$. Nevertheless for the boundedness on $BMO_{w,\rho}(\mathbb{R}^d)$, pointwise estimates of $K(\cdot, \cdot)$ are required.

In this section and Section 7, the following assumptions are imposed on V :

- (1) $V \in RH_\sigma$, where $\sigma > \frac{d}{2}$;
- (2) There exists a $C > 0$ such that

$$|\nabla V(x)| \leq C \rho(x)^{-3} \tag{18}$$

$$|\nabla^2 V(x)| \leq C \rho(x)^{-4} \tag{19}$$

for all $x \in \mathbb{R}^d$.

Remark 41. It is straightforward to verify that (18) implies

$$V(x) \leq C \rho(x)^{-2}$$

for all $x \in \mathbb{R}^d$. (cf. [12, Remark 5] and [21, Remark 1.8].)

Denote $\Gamma(\cdot, \cdot)$ and $\Gamma_0(\cdot, \cdot)$ to be the fundamental solutions of L and $-\Delta$ respectively. Let

$$K(x, y) = \nabla_1^2 \Gamma(x, y) \quad \text{and} \quad K_0(x, y) = \nabla_1^2 \Gamma_0(x, y)$$

for all $x, y \in \mathbb{R}^d$. Here we use ∇_1 to denote the gradient of $\Gamma(\cdot, \cdot)$ with respect to the first variable. Likewise ∇_2 means the gradient of $\Gamma(\cdot, \cdot)$ with respect to the second variable.

The following estimates on Γ_0 and Γ are well-known.

Lemma 42 (cf. [11, Section 2.4] and [20, Theorem 2.7 and (6.1)]). *Let $j \in \{1, 2\}$. The following statements hold.*

- (a) *There exists a $C > 0$ such that*

$$\begin{aligned} |\Gamma_0(x, y)| &\leq \frac{C}{|x - y|^{d-2}}, \\ |\nabla_j \Gamma_0(x, y)| &\leq \frac{C}{|x - y|^{d-1}}, \\ |\nabla_j^2 \Gamma_0(x, y)| &\leq \frac{C}{|x - y|^d} \end{aligned}$$

for all $x, y \in \mathbb{R}^d$ with $x \neq y$.

- (b) *For all $N > 0$ there exists a $C > 0$ such that*

$$\begin{aligned} |\Gamma(x, y)| &\leq \frac{C}{|x - y|^{d-2}} \left(1 + \frac{|x - y|}{\rho(x)} \right)^{-N}, \\ |\nabla_j \Gamma(x, y)| &\leq \frac{C}{|x - y|^{d-1}} \left(1 + \frac{|x - y|}{\rho(x)} \right)^{-N} \end{aligned}$$

for all $x, y \in \mathbb{R}^d$ with $x \neq y$.

With the present assumptions on V , more estimates are available on Γ as shown in the next two lemmas.

Lemma 43 (cf. [18, Lemma 3.6]). *The following hold.*

(i) *For all $N \geq 0$ there exists a $C > 0$ such that*

$$|K(x, y)| \leq C \frac{1}{|x - y|^d} \left(1 + \frac{|x - y|}{\rho(x)} \right)^{-N}$$

for all $x, y \in \mathbb{R}^d$ with $x \neq y$.

(ii) *For all $N \geq 0$ there exists a $C > 0$ such that*

$$|K(x, y) - K(x_0, y)| \leq C \frac{|x - x_0|}{|x - y|^{d+1}} \left(1 + \frac{|x_0 - y|}{\rho(x_0)} \right)^{-N}$$

for all $x_0, x, y \in \mathbb{R}^d$ pairwise different such that $|x - x_0| < \frac{|x - y|}{2}$.

Lemma 44. *Let $c \in (0, 1)$. For all $N > 0$ there exists a $C > 0$ such that for all $x, y \in \mathbb{R}^d$ satisfying $|x - y| < c|x - u|$ one has the following.*

- (a) $|\Gamma(x, u) - \Gamma(y, u)| \leq C \frac{|x - y|}{|x - u|^{d-1}} \left(1 + \frac{|x - u|}{\rho(x)} \right)^{-N}$.
- (b) $|\nabla_1 \Gamma(x, u) - \nabla_1 \Gamma(y, u)| \leq C \frac{|x - y|}{|x - u|^d} \left(1 + \frac{|x - u|}{\rho(x)} \right)^{-N}$.

Proof. Fix $x, u \in \mathbb{R}^d$. Set $R = c|x - u|$. It follows that $|x - u| \lesssim |z - u|$ for all $z \in B(x, R)$.

(a). By the Mean Value Theorem we have

$$\begin{aligned} |\Gamma(x + h, u) - \Gamma(x, u)| &\leq |h| \sup_{z \in B(x, R)} |\nabla_1 \Gamma(z, u)| \lesssim |h| \sup_{z \in B(x, R)} \left(\frac{1}{|z - u|^{d-1}} \left(1 + \frac{|z - u|}{\rho(u)} \right)^{-N} \right) \\ &\lesssim |h| \frac{1}{|x - u|^{d-1}} \left(1 + \frac{|x - u|}{\rho(u)} \right)^{-N} \\ &\lesssim \frac{|h|}{|x - u|^{d-1}} \left(1 + \frac{|x - u|}{\rho(x)} \right)^{-\frac{N}{(k_0+1)}} \end{aligned}$$

for all $N > 0$ and for all $h \in \mathbb{R}^d$ such that $|h| < c|x - u|$, where we used Lemma 42 in the second step and Corollary 11 in the last step.

(b). Similar to the above, it follows from the Mean Value Theorem that

$$\begin{aligned} |\nabla_1 \Gamma(x + h, u) - \nabla_1 \Gamma(z, u)| &\leq |h| \sup_{z \in B(x, R)} |\nabla_1^2 \Gamma(z, u)| \lesssim |h| \sup_{z \in B(x, R)} \left(\frac{1}{|z - u|^d} \left(1 + \frac{|z - u|}{\rho(u)} \right)^{-N} \right) \\ &\lesssim |h| \frac{1}{|x - u|^d} \left(1 + \frac{|x - u|}{\rho(u)} \right)^{-N} \\ &\lesssim \frac{|h|}{|x - u|^d} \left(1 + \frac{|x - u|}{\rho(x)} \right)^{-\frac{N}{(k_0+1)}} \end{aligned}$$

for all $N > 0$ and for all $h \in \mathbb{R}^d$ such that $|h| < c|x - u|$, where we used Lemma 42 in the second step and Corollary 11 in the last step.

This completes the our proof. □

Next we prove some more smoothness results on the kernel $K(\cdot, \cdot)$ of T . The following lemma serves as a preparation.

Lemma 45. *Let $x_0 \in \mathbb{R}^d$ and $R > 0$. Suppose u is such that $Lu = 0$ in $B(x_0, 2R)$. Then there exist a $C > 0$ and a $\kappa > 0$ such that*

$$\sup_{B(x_0, R)} |\nabla u| \leq \frac{C}{R} \left(1 + \frac{R}{\rho(x_0)}\right)^\kappa \sup_{B(x_0, 2R)} |u|.$$

Proof. Let $\eta \in C_c^\infty(B(x_0, 2R))$ such that

$$\eta|_{B(x_0, 3\frac{R}{2})} = 1, \quad |\nabla \eta| \lesssim \frac{1}{R} \quad \text{and} \quad |\nabla^2 \eta| \lesssim \frac{1}{R^2}.$$

By the definitions of Γ_0 and u , we have

$$\begin{aligned} u(x)\eta(x) &= \int_{\mathbb{R}^d} \Gamma_0(x, y) \left(-\Delta(u\eta)\right)(y) \, dy \\ &= \int_{\mathbb{R}^d} \Gamma_0(x, y) \left(-(\Delta u)(y)\eta(y) - 2(\nabla u)(y) \cdot (\nabla \eta)(y) - u(y)(\Delta \eta)(y)\right) \, dy \\ &= \int_{\mathbb{R}^d} \Gamma_0(x, y) \left(-V(y)u(y)\eta(y) - 2(\nabla u)(y) \cdot (\nabla \eta)(y) - u(y)(\Delta \eta)(y)\right) \, dy \\ &= \int_{\mathbb{R}^d} \Gamma_0(x, y) \left(-V(y)u(y)\eta(y) + u(y)(\Delta \eta)(y)\right) \, dy \\ &\quad + 2 \int_{\mathbb{R}^d} \nabla_2 \Gamma_0(x, y) \cdot (\nabla \eta)(y) u(y) \, dy \end{aligned}$$

for all $x \in B(x_0, R)$.

Taking the derivative in both sides of the above equality with respect to x we obtain

$$\begin{aligned} |\nabla u(x)| &\lesssim \sup_{B(x_0, 2R)} |u| \int_{B(x_0, 2R)} \frac{V(y)|\eta(y)|}{|x-y|^{d-1}} \, dy + \frac{1}{R^{d+1}} \int_{B(x_0, 2R)} |u(y)| \, dy \\ &\lesssim \sup_{B(x_0, 2R)} |u| \int_{B(x_0, 2R)} \frac{\rho(y)^{-2}}{|x-y|^{d-1}} \, dy + \frac{1}{R^{d+1}} \int_{B(x_0, 2R)} |u(y)| \, dy \\ &\lesssim \sup_{B(x_0, 2R)} |u| \int_{B(x_0, 2R)} \frac{\rho(x_0)^{-2} \left(1 + \frac{|x_0-y|}{\rho(x_0)}\right)^{2k_0}}{|x-y|^{d-1}} \, dy + \frac{1}{R^{d+1}} \int_{B(x_0, 2R)} |u(y)| \, dy \\ &\lesssim \sup_{B(x_0, 2R)} |u| \rho(x_0)^{-2} \left(1 + \frac{2R}{\rho(x_0)}\right)^{2k_0} \int_{B(x_0, 2R)} \frac{1}{|x-y|^{d-1}} \, dy + \frac{1}{R^{d+1}} \int_{B(x_0, 2R)} |u(y)| \, dy \\ &\lesssim \frac{1}{R} \left(1 + \frac{R}{\rho(x_0)}\right)^{2k_0+2} \sup_{B(x_0, 2R)} |u| \end{aligned}$$

for all $x \in B(x_0, R)$, where we used Lemma 42 in the first step, Remark 41 in the second step, Proposition 10 in the third step.

This verifies the claim. □

Lemma 46. *For all $N > 0$ there exists a $C > 0$ such that*

$$|K(x, y) - K(x, z)| + |K(y, x) - K(z, x)| \leq C \frac{|y-z|}{|x-y|^{d+1}} \left(1 + \frac{|x-y|}{\rho(y)}\right)^{-N}$$

for all $x, y, z \in \mathbb{R}^d$ such that $|x-y| > 2|y-z|$.

Proof. Let $N \geq 0$ and $x, y, z \in \mathbb{R}^d$ be such that $|x-y| > 2|y-z|$. It follows from Lemma 43 that

$$|K(y, x) - K(z, x)| \leq C \frac{|y-z|}{|x-y|^{d+1}} \left(1 + \frac{|x-y|}{\rho(y)}\right)^{-N}. \tag{20}$$

It remains to show that

$$|K(x, y) - K(x, z)| \leq C \frac{|y-z|}{|x-y|^{d+1}} \left(1 + \frac{|x-y|}{\rho(y)}\right)^{-N}. \tag{21}$$

Let $R = \frac{|x-y|}{4}$. Using the symmetry of $\Gamma(\cdot, \cdot)$ and the Mean Value Theorem we obtain

$$\begin{aligned} |K(x, y) - K(x, z)| &= |\nabla_1^2 \Gamma(x, y) - \nabla_1^2 \Gamma(x, z)| = |\nabla_2^2 \Gamma(y, x) - \nabla_2^2 \Gamma(z, x)| \\ &\leq |y - z| \sup_{\xi \in B(y, 2R)} |\nabla_1 \nabla_2^2 \Gamma(\xi, x)| \\ &\leq |y - z| \frac{1}{R} \left(1 + \frac{R}{\rho(y)}\right)^{2k_0+2} \sup_{\xi \in B(y, 2R)} |\nabla_2^2 \Gamma(\xi, x)| \\ &\lesssim |y - z| \frac{1}{R} \left(1 + \frac{R}{\rho(y)}\right)^{2k_0+2} \sup_{\xi \in B(y, 2R)} |\nabla_1^2 \Gamma(x, \xi)| \\ &\lesssim |y - z| \frac{1}{R} \left(1 + \frac{R}{\rho(y)}\right)^{2k_0+2} \sup_{\xi \in B(y, 2R)} \frac{1}{|x - \xi|^d} \left(1 + \frac{|x - \xi|}{\rho(x)}\right)^{-N} \\ &\lesssim |y - z| \frac{1}{R} \left(1 + \frac{R}{\rho(y)}\right)^{2k_0+2} \frac{1}{|x - y|^d} \left(1 + \frac{|x - y|}{\rho(x)}\right)^{-N} \\ &\lesssim \frac{|y - z|}{|x - y|^{d+1}} \left(1 + \frac{|x - y|}{\rho(y)}\right)^{-\frac{N}{(k_0+1)} + 2k_0+2}, \end{aligned}$$

where we used Corollary 11 in the last step. Note further that in the above estimate we used Lemma 45 and the fact that $\nabla_2^2 \Gamma(\xi, x)$ is a solution to $Lu = 0$ on $\mathbb{R}^d \setminus \{x\}$ in the fourth step and Lemma 43 in the sixth step.

Putting (20) and (21) together yields the requirement. □

Lemma 47. *There exists a $C > 0$ such that*

$$|K(x, y) - K_0(x, y)| \leq \frac{C}{|x - y|^d} \left(\frac{|x - y|}{\rho(y)}\right)^\delta \tag{22}$$

for all $\delta \in (0, 1)$ and $x, y \in \mathbb{R}^d$ such that $x \neq y$.

Proof. Let $x, y \in \mathbb{R}^d$ such that $x \neq y$. We consider two cases.

Case 1. Suppose $|x - y| \geq \rho(y)$. Then it follows from Lemma 43(i) that

$$|K(x, y) - K_0(x, y)| \leq |K(x, y)| + |K_0(x, y)| \leq \frac{C}{|x - y|^d} \leq \frac{C}{|x - y|^d} \left(\frac{|x - y|}{\rho(y)}\right)^\delta.$$

Hence (22) holds.

Case 2. Suppose $|x - y| < \rho(y)$. Set $A = \{z \in \mathbb{R}^d : z \neq y\}$. Since $(-\Delta + V)(\Gamma(\cdot, y)) = 0$ on A , we deduce that

$$-\Delta \nabla(\Gamma(\cdot, y)) = -\nabla(V(\cdot)\Gamma(\cdot, y)).$$

We also have $\Delta(\nabla(\Gamma_0(\cdot, y))) = 0$ on A . Consequently

$$\begin{aligned} \nabla(\Gamma(\cdot, y)) - \nabla(\Gamma_0(\cdot, y)) &= -(-\Delta)^{-1} \nabla(V(\cdot)\Gamma(\cdot, y)) \\ &= - \int_{\mathbb{R}^d} \Gamma_0(\cdot, z) (\nabla V)(z) \Gamma(z, y) \, dz - \int_{\mathbb{R}^d} \Gamma_0(\cdot, z) V(z) (\nabla \Gamma)(z, y) \, dz, \end{aligned}$$

where the integrals are understood in the principal value sense. It follows that

$$\begin{aligned} (\nabla_1^2 \Gamma - \nabla_1^2 \Gamma_0)(x, y) &= - \int_{\mathbb{R}^d} \nabla \Gamma_0(x, z) (\nabla V)(z) \Gamma(z, y) \, dz - \int_{\mathbb{R}^d} \nabla \Gamma_0(x, z) V(z) (\nabla \Gamma)(z, y) \, dz \\ &=: I + II. \end{aligned}$$

Next we estimate each term separately.

Term I. Set $R = \frac{|x-y|}{8}$. Then

$$\begin{aligned} I &\leq \int_{|x-z|<R} \nabla\Gamma_0(x, z) (\nabla V)(z) \Gamma(z, y) \, dz + \int_{|y-z|<R} \nabla\Gamma_0(x, z) (\nabla V)(z) \Gamma(z, y) \, dz \\ &\quad + \int_{|x-z|\wedge|y-z|\geq R} \nabla\Gamma_0(x, z) (\nabla V)(z) \Gamma(z, y) \, dz \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

To estimate I_1 we note that $|x - y| < \rho(y)$ and $|z - x| < R$ lead to $|y - z| \leq |x - y| + |x - z| \leq \frac{9\rho(y)}{8}$, which in turn implies $\rho(x) \sim \rho(y) \sim \rho(z)$. Also $|z - y| \geq |x - y| - |x - z| \geq \frac{7|x-y|}{8}$. Therefore

$$\begin{aligned} I_1 &\lesssim \int_{|x-z|<R} \frac{1}{|x-z|^{d-1}} |\nabla V(z)| \frac{1}{|z-y|^{d-2}} \, dz \\ &\lesssim \frac{1}{|x-y|^{d-2}} \rho(y)^{-3} \int_{|x-z|<R} \frac{1}{|x-z|^{d-1}} \, dz \lesssim \frac{1}{|x-y|^{d-2}} \rho(y)^{-3} R \\ &\lesssim \frac{1}{|x-y|^d} \left(\frac{|x-y|}{\rho(y)} \right)^3 \lesssim \frac{1}{|x-y|^d} \left(\frac{|x-y|}{\rho(y)} \right)^\delta \end{aligned}$$

for all $\delta \in (0, 1)$, where we used Lemma 42 in the first step and (18) in the second step.

Next we estimate I_2 . Again $|x - y| < \rho(y)$ and $|y - z| < R$ lead to $|x - z| \leq |x - y| + |y - z| \leq \frac{9\rho(y)}{8}$, which in turn implies $\rho(x) \sim \rho(y) \sim \rho(z)$. Also $|x - z| \geq \frac{7|x-y|}{8}$ and

$$\begin{aligned} I_2 &\leq \int_{|y-z|<R} \frac{1}{|x-z|^{d-1}} |\nabla V(z)| \frac{1}{|z-y|^{d-2}} \, dz \lesssim \frac{1}{|x-y|^{d-1}} \int_{|y-z|<R} |\nabla V(z)| \frac{1}{|z-y|^{d-2}} \, dz \\ &\lesssim \frac{1}{|x-y|^{d-1}} \int_{|y-z|<R} \rho(z)^{-3} \frac{1}{|z-y|^{d-2}} \, dz \lesssim \frac{\rho(y)^{-3}}{|x-y|^{d-1}} \int_{|y-z|<R} \frac{1}{|z-y|^{d-2}} \, dz \\ &\lesssim \frac{\rho(y)^{-3}}{|x-y|^{d-1}} R^2 \lesssim \frac{1}{|x-y|^d} \left(\frac{|x-y|}{\rho(y)} \right)^3 \lesssim \frac{1}{|x-y|^d} \left(\frac{|x-y|}{\rho(y)} \right)^\delta \end{aligned}$$

for all $\delta \in (0, 1)$.

Lastly we estimate I_3 . Let $\delta \in (0, 1)$ and $k = 3(k_0 + 1) - \delta$, where k_0 is determined by Proposition 10. Then

$$\begin{aligned} I_3 &\lesssim \int_{|x-z|\wedge|y-z|\geq R} \frac{1}{|z-x|^{d-1}} |\nabla V(z)| \frac{1}{|z-y|^{d-2} \left(1 + \frac{|z-y|}{\rho(y)}\right)^k} \, dz \\ &\lesssim \frac{1}{|x-y|^{d-1}} \sum_{j \in \mathbb{N}^*} \int_{2^{j-1}R \leq |z-y| \leq 2^jR} \frac{\rho(z)^{-3}}{|z-y|^{d-2} \left(1 + \frac{|z-y|}{\rho(y)}\right)^k} \, dz \\ &\lesssim \frac{1}{|x-y|^{d-1}} \sum_{j \in \mathbb{N}^*} \int_{2^{j-1}R \leq |z-y| \leq 2^jR} \frac{\rho(y)^{-3} \left(1 + \frac{|y-z|}{\rho(y)}\right)^{3k_0}}{|z-y|^{d-2} \left(1 + \frac{|z-y|}{\rho(y)}\right)^k} \, dz \\ &= \frac{\rho(y)^{-3}}{|x-y|^{d-1}} \sum_{j \in \mathbb{N}^*} \int_{2^{j-1}R \leq |z-y| \leq 2^jR} \frac{1}{|z-y|^{d-2} \left(1 + \frac{|z-y|}{\rho(y)}\right)^{3-\delta}} \, dz \\ &\lesssim \frac{\rho(y)^{-3}}{|x-y|^{d-1}} \sum_{j \in \mathbb{N}^*} \frac{(2^jR)^d}{(2^jR)^{d-2} \left(\frac{2^jR}{\rho(y)}\right)^{3-\delta}} \, dz \lesssim \frac{\rho(y)^{-\delta}}{|x-y|^{d-1}} \sum_{j \in \mathbb{N}^*} (2^jR)^{\delta-1} \\ &\lesssim \frac{\rho(y)^{-\delta}}{|x-y|^{d-1}} |x-y|^{\delta-1} \sum_{j \in \mathbb{N}^*} 2^{j(\delta-1)} \lesssim \frac{1}{|x-y|^d} \left(\frac{|x-y|}{\rho(y)} \right)^\delta, \end{aligned}$$

where we used Lemma 42(a) in the first step, (18) in the second step and Proposition 10 in the third step.

Term II. This term is estimated using analogous arguments as those of Term I, in which the estimate on ∇V using (18) in Term I is now replaced by the estimate on $\nabla_1 \Gamma(\cdot, \cdot)$ using Lemma 42(b) in Term II.

Combining the estimates for the two term together justifies the claim. □

Lemma 48. *There exists a positive constant C such that*

$$\left| (K(x, z) - K_0(x, z)) - (K(y, z) - K_0(y, z)) \right| \leq C \frac{|x - y|^\delta}{|x - z|^{d+\delta-2}} \rho(x)^{-2} \tag{23}$$

for all $\delta \in (0, 1)$ and for all $x, y, z \in \mathbb{R}^d$ such that $|x - z| \geq 2|x - y|$.

Proof. Let $\delta \in (0, 1)$ and $x, y, z \in \mathbb{R}^d$ be such that $|x - z| \geq 2|x - y|$. We consider two cases.

Case 1. Suppose that $|x - z| \geq \frac{\rho(x)}{2}$. Then the result follows from Lemma 46. Indeed one has that

$$\begin{aligned} \left| (K(x, z) - K_0(x, z)) - (K(y, z) - K_0(y, z)) \right| &\leq \left| K(x, z) - K(y, z) \right| + \left| K_0(x, z) - K_0(y, z) \right| \\ &\lesssim \frac{|x - y|}{|x - z|^{d+1}} \lesssim \frac{|x - y|}{|x - z|^{d-1}} \rho(x)^{-2} \\ &\leq \frac{|x - y|^\delta}{|x - z|^{d+\delta-2}} \rho(x)^{-2}. \end{aligned}$$

Case 2. Suppose that $|x - z| < \frac{\rho(x)}{2}$. We argue as in Case 2 in the proof of Lemma 47 to derive

$$\begin{aligned} \nabla(\Gamma(\cdot, y)) - \nabla(\Gamma_0(\cdot, y)) &= -(-\Delta)^{-1} \nabla(V(\cdot) \Gamma(\cdot, y)) \\ &= - \int_{\mathbb{R}^d} \Gamma_0(\cdot, z) (\nabla V)(z) \Gamma(z, y) dz - \int_{\mathbb{R}^d} \Gamma_0(\cdot, z) V(z) (\nabla \Gamma)(z, y) dz, \end{aligned}$$

where here and in the rest of the proof the integrals are understood in the principal value sense.

Due to the symmetry of $\Gamma(\cdot, \cdot)$ we obtain

$$\begin{aligned} (K(x, z) - K_0(x, z)) - (K(y, z) - K_0(y, z)) &= (\nabla_1^2 \Gamma(x, z) - \nabla_1^2 \Gamma_0(x, z)) - (\nabla_1^2 \Gamma(y, z) - \nabla_1^2 \Gamma_0(y, z)) \\ &= \int_{\mathbb{R}^d} \nabla_1 \Gamma_0(z, u) (\nabla V(u)) (\Gamma(x, u) - \Gamma(y, u)) du \\ &\quad + \int_{\mathbb{R}^d} \nabla_1 \Gamma_0(z, u) V(u) (\nabla_1 \Gamma(x, u) - \nabla_1 \Gamma(y, u)) du \\ &=: A_1 + A_2. \end{aligned}$$

The two terms A_1 and A_2 are estimated analogously. The idea is to use Lemma 44(a) for the estimate of A_1 and Lemma 44(b) for that of A_2 . Hence we will estimate A_2 only.

We claim that

$$|A_2| \lesssim \frac{|x - y|^\delta}{|x - z|^{d+\delta-2}} \rho(x)^{-2}. \tag{24}$$

To show this, we split \mathbb{R}^d into 4 regions:

$$\begin{aligned} E_1 &= \left\{ u : |x - u| < \frac{3}{2}|x - y| \right\}, \\ E_2 &= \left\{ u : \frac{3}{2}|x - y| \leq |x - u| < \frac{2}{3}|x - z| \right\}, \\ E_3 &= \left\{ u : \frac{2}{3}|x - z| \leq |x - u| < 2|x - z| \right\} \quad \text{and} \\ E_4 &= \{ u : |x - u| \geq 2|x - z| \}. \end{aligned}$$

Write

$$I_j = \int_{E_j} |\nabla_1 \Gamma_0(z, u)| V(u) |\nabla_1 \Gamma(x, u) - \nabla_1 \Gamma(y, u)| \, du$$

for $j \in \{1, 2, 3, 4\}$. Then $A_2 \leq I_1 + I_2 + I_3 + I_4$. Therefore it suffices to show that each I_j is bounded by the right hand side of (24).

Term I_1 . Since $|x - z| \geq 2|x - y|$ implies $|z - u| \geq \frac{|x - z|}{4}$ for all $u \in E_1$. We have $\rho(u) \sim \rho(x) \sim \rho(z)$ for all $u \in E_1$. We decompose I_1 as follows

$$I_1 \leq \int_{E_1} |\nabla_1 \Gamma_0(z, u) \nabla_1 \Gamma(x, u)| V(u) \, du + \int_{E_1} |\nabla_1 \Gamma_0(z, u) \nabla_1 \Gamma(y, u)| V(u) \, du =: I_{11} + I_{12}.$$

For I_{11} we have

$$\begin{aligned} I_{11} &= \int_{E_1} |\nabla_1 \Gamma_0(z, u) \nabla_1 \Gamma(x, u)| V(u) \, du \lesssim \frac{1}{|x - z|^{d-1}} \int_{E_1} \frac{\rho(u)^{-2}}{|x - u|^{d-1}} \, du \\ &\lesssim \frac{\rho(x)^{-2}}{|x - z|^{d-1}} \int_{B(x, 2|x-y|)} \frac{1}{|x - u|^{d-1}} \, du \lesssim \frac{|x - y|}{|x - z|^{d-1}} \rho(x)^{-2} \lesssim \frac{|x - y|^\delta}{|x - z|^{d+\delta-2}} \rho(x)^{-2}, \end{aligned}$$

where we used Remark 41 and Lemma 42 in the second step and the fact that $\rho(u) \sim \rho(x)$ for all $u \in E_1$.

For I_{12} observe that $|y - u| \leq |x - y| + |x - u| < \frac{5}{2}|x - y|$ for all $u \in E_1$. Consequently,

$$\begin{aligned} I_{12} &= \int_{E_1} |\nabla_1 \Gamma_0(z, u) \nabla_1 \Gamma(y, u)| V(u) \, du \lesssim \frac{1}{|x - z|^{d-1}} \int_{E_1} \frac{\rho(u)^{-2}}{|y - u|^{d-1}} \, du \\ &\lesssim \frac{\rho(x)^{-2}}{|x - z|^{d-1}} \int_{B(y, 3|x-y|)} \frac{1}{|y - u|^{d-1}} \, du \lesssim \frac{|x - y|}{|x - z|^{d-1}} \rho(x)^{-2} \lesssim \frac{|x - y|^\delta}{|x - z|^{d+\delta-2}} \rho(x)^{-2}, \end{aligned}$$

where we used Remark 41 and Lemma 42 in the second step and the fact that $\rho(u) \sim \rho(x)$ for all $u \in E_1$.

Combining the estimates for I_{11} and I_{12} together we infer that

$$I_1 \lesssim \frac{|x - y|^\delta}{|x - z|^{d+\delta-2}} \rho(x)^{-2}.$$

Before moving to the estimates of the remaining I_j terms, we note that $|x - y| < c|x - u|$ for some $c \in (0, 1)$ whenever $u \in E_j$ for $j \in \{2, 3, 4\}$. As such Lemma 44(b) applies to give that

$$|\nabla_1 \Gamma(x, u) - \nabla_1 \Gamma(y, u)| \lesssim \frac{|x - y|}{|x - u|^d} \left(1 + \frac{|x - u|}{\rho(x)}\right)^{-N}$$

for all $N > 0$ and for all $u \in E_j$ with $j \in \{2, 3, 4\}$. This together with Lemma 42 now imply

$$I_j \lesssim \int_{E_j} \frac{V(u)}{|u - z|^{d-1}} \frac{|x - y|}{|x - u|^d} \left(1 + \frac{|x - u|}{\rho(x)}\right)^{-N} \, du \tag{25}$$

for all $j \in \{2, 3, 4\}$ and for all $N > 0$.

Term I_2 . The following inequalities hold

$$3|x - y| \leq \frac{4|x - z|}{3} \leq \frac{2\rho(x)}{3} \quad \text{and} \quad \frac{3|x - y|}{2} \leq |x - u| < \frac{2|x - z|}{3}$$

for all $u \in E_2$. As a consequence $\rho(u) \sim \rho(x) \sim \rho(y) \sim \rho(z)$ for all $u \in E_2$. It is also useful to keep in mind that $|x - z| > \frac{3|x-u|}{2}$ implies $|u - z| \geq |x - z| - |u - x| > \frac{|x-z|}{3}$ for all $u \in E_2$. So by choosing $N = 0$ in (25) and referring to Remark 41 we obtain

$$\begin{aligned} I_2 &\lesssim \frac{|x-y|}{|x-z|^{d-1}} \int_{E_2} \frac{\rho(u)^{-2}}{|x-u|^d} du \lesssim \frac{|x-y|^\delta}{|x-z|^{d-1}} \rho(x)^{-2} \int_{B(x, \frac{2|x-z|}{3})} \frac{1}{|x-u|^{d+\delta-1}} du \\ &\lesssim \frac{|x-y|^\delta}{|x-z|^{d+\delta-2}} \rho(x)^{-2}. \end{aligned}$$

Term I_3 . In this case notice that $E_3 \subset B(z, 3|x-z|)$ and $\rho(u) \sim \rho(x) \sim \rho(z)$ for all $u \in E_3$. It then follows that

$$\begin{aligned} I_3 &\lesssim \frac{|x-y|}{|x-z|^d} \int_{E_3} \frac{\rho(u)^{-2}}{|u-z|^{d-1}} du \lesssim \frac{|x-y|}{|x-z|^d} \rho(x)^{-2} \int_{B(z, 3|x-z|)} \frac{1}{|u-z|^{d-1}} du \\ &\lesssim \frac{|x-y|}{|x-z|^{d-1}} \rho(x)^{-2} \lesssim \frac{|x-y|^\delta}{|x-z|^{d+\delta-2}} \rho(x)^{-2}. \end{aligned}$$

Term I_4 . Set $F_1 = \{u : 2|x-z| \leq |x-u| < \rho(x)\}$ and $F_2 = \{u : |x-u| \geq \rho(x)\}$. Then $E_4 = F_1 \cup F_2$. If $u \in E_4$ then $|x-u| \sim |u-z|$. This ensures that

$$\begin{aligned} I_4 &\lesssim |x-y| \int_{E_4} \frac{V(u)}{|x-u|^{2d-1}} \left(1 + \frac{|x-u|}{\rho(x)}\right)^{-N} du \\ &\leq |x-y| \int_{F_1} \frac{V(u)}{|x-u|^{2d-1}} du + |x-y| \int_{F_2} \frac{V(u)}{|x-u|^{2d-1}} \left(1 + \frac{|x-u|}{\rho(x)}\right)^{-N} du \\ &=: I_{41} + I_{42}. \end{aligned}$$

For I_{41} we have

$$\begin{aligned} I_{41} &\lesssim |x-y| \int_{F_1} \frac{\rho(x)^{-2}}{|x-u|^{2d-1}} du \lesssim |x-y| \rho(x)^{-2} \sum_{i \in \mathbb{N}^*} \int_{2^i|x-z| \leq |x-u| \leq 2^{i+1}|x-z|} \frac{1}{|x-u|^{2d-1}} du \\ &\lesssim |x-y| \rho(x)^{-2} \sum_{i \in \mathbb{N}^*} \frac{(2^i|x-z|)^d}{(2^i|x-z|)^{2d-1}} \lesssim \frac{|x-y|}{|x-z|^{d-1}} \rho(x)^{-2} \lesssim \frac{|x-y|^\delta}{|x-z|^{d+\delta-2}} \rho(x)^{-2}. \end{aligned}$$

Next we consider I_{42} . It is useful to observe that $V \in RH_\sigma$ implies that there exists constants $C > 0$ and $\mu \geq 1$ such that

$$\int_{\lambda B} V \leq C \lambda^{d\mu} \int_B V \tag{26}$$

for every ball $B = B(x, r) \subset \mathbb{R}^d$ and $\lambda > 1$, where $\lambda B := B(x, \lambda r)$. This well-known fact can be found in [20, (1.1)] and [5, p. 117].

Then for all $N > d\mu$ we have

$$\begin{aligned}
 I_{42} &\lesssim |x - y| \int_{|x-u| \geq \rho(x)} \frac{V(u)}{|x-u|^{2d-1}} \left(1 + \frac{|x-u|}{\rho(x)}\right)^{-N} du \\
 &\leq |x - y| \rho(x)^N \sum_{j \in \mathbb{N}^*} \int_{2^{j-1}\rho(x) < |x-u| < 2^j \rho(x)} \frac{V(u)}{|x-u|^{N+2d-1}} du \\
 &\leq |x - y| \rho(x)^N \sum_{j \in \mathbb{N}^*} \frac{1}{(2^{j-1}\rho(x))^{N+2d-1}} \int_{|x-u| < 2^j \rho(x)} V(u) du \\
 &\lesssim |x - y| \rho(x)^{-2d+1} \left(\sum_{j \in \mathbb{N}^*} \frac{2^{d\mu}}{2^{(j-1)(N+2d-1-d\mu)}} \right) \int_{|x-u| < \rho(x)} V(u) du \\
 &\lesssim |x - y| \rho(x)^{-2d+1} \int_{|x-u| < \rho(x)} V(u) du \\
 &\lesssim |x - y| \rho(x)^{-2d+1} \int_{|x-u| < \rho(x)} \rho(u)^{-2} du \\
 &\sim |x - y| \rho(x)^{-2d+1} \rho(x)^{-2} \int_{|x-u| < \rho(x)} du \\
 &\sim |x - y| \rho(x)^{-2d+1} \rho(x)^{d-2} = |x - y| \rho(x)^{-d-1} \\
 &= \frac{|x - y|}{|x - z|^{d-1}} \left(\frac{|x - z|}{\rho(x)} \right)^{d-1} \rho(x)^{-2} \leq \frac{|x - y|}{|x - z|^{d-1}} \rho(x)^{-2} \lesssim \frac{|x - y|^\delta}{|x - z|^{d+\delta-2}} \rho(x)^{-2},
 \end{aligned}$$

where we used (26) in the fourth step, Remark 41 in the sixth step, the fact that $\rho(u) \sim \rho(x)$ in the seventh step and the inequality $|x - z| < \frac{\rho(x)}{2}$ in the last step. Thus

$$I_4 \lesssim \frac{|x - y|^\delta}{|x - z|^{d+\delta-2}} \rho(x)^{-2}.$$

Combining the estimates for I_1, I_2, I_3 and I_4 together yields the required result. □

7. Boundedness on weighted BMO spaces associated with L

In this section we aim to prove the second main theorem of this paper: Theorem 9.

Again we emphasize that throughout this section the following conditions are assumed on V :

- (1) $V \in RH_\sigma$, where $\sigma > \frac{d}{2}$;
- (2) There exists a $C > 0$ such that

$$|\nabla V(x)| \leq C \rho(x)^{-3} \quad \text{and} \quad |\nabla^2 V(x)| \leq C \rho(x)^{-4}$$

for all $x \in \mathbb{R}^d$.

For all $a, b > 0$ and $x \in \mathbb{R}^d$ set

$$A(x, a, b) := \{y \in \mathbb{R}^d : a < |x - y| < b\}.$$

Following [4] we introduce the following definition.

Definition 49. Let $s \in [1, \infty]$ and $\delta \in (0, 1]$. A linear operator T is a Schrödinger–Calderón–Zygmund operator of type (s, δ) if it satisfies the following properties.

- (i) T is bounded from $L^s(\mathbb{R}^d)$ into $L^{s', \infty}(\mathbb{R}^d)$.
- (ii) T has an associated kernel $K : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ such that

$$Tf(x) = \int_{\mathbb{R}^d} K(x, y) f(y) dy$$

for all $f \in L^s_c(\mathbb{R}^d)$ and $x \notin \text{supp } f$.

(iii) For all $N > 0$ there exists a $C > 0$ such that

$$\|K(x, \cdot)\|_{L^s(A(x,R,2R))} \leq C R^{-\frac{d}{s'}} \left(1 + \frac{R}{\rho(x)}\right)^{-N} \tag{27}$$

for all $R > 0$ and $x \in \mathbb{R}^d$.

(iv) There exists a $C > 0$ such that

$$\|K(x, \cdot) - K(x_0, \cdot)\|_{L^s(A(x_0,R,2R))} \leq C R^{-\frac{d}{s'}} \left(\frac{r}{R}\right)^\delta \tag{28}$$

for all $R > 0, 0 < r < \frac{R}{8}$ and $|x - x_0| < r \leq \rho(x_0)$.

The next proposition provides a convenient way to examine the boundedness of a Schrödinger–Calderón–Zygmund operator on $BMO_{w,\rho}(\mathbb{R}^d)$.

Proposition 50. Let $\delta \in (0, 1]$ and $\alpha \in [1, 1 + \frac{\delta}{d}]$. Let T be a Schrödinger–Calderón–Zygmund operator of type (∞, δ) . Then the following conditions are equivalent.

(a) There exists a $C > 0$ such that

$$\int_B |T1(y) - (T1)_B| dy \leq C |B| \left(\frac{r}{\rho(x_0)}\right)^{d(\alpha-1)}$$

for all $B = B(x_0, r)$ with $0 < r < \frac{\rho(x_0)}{2}$ if $\alpha > 1$ or

$$\int_B |T1(y) - (T1)_B| dy \leq C |B| \log^{-1}\left(\frac{\rho(x_0)}{r}\right)$$

for all $B = B(x_0, r)$ with $0 < r < \frac{\rho(x_0)}{2}$ if $\alpha = 1$.

(b) T is bounded on $BMO_{w,\rho}(\mathbb{R}^d)$ for all $w \in A_\infty^\rho \cap D_\alpha^\rho$.

Proof. This is a direct consequence of [4, Theorem 2 and Corollary 2]. □

Next we will show that $T = \nabla^2 L^{-1}$ is a Schrödinger–Calderón–Zygmund operator of type (∞, δ) for some $\delta \in (0, 1]$ which satisfies Condition (a) in Proposition 50. From this we will derive the boundedness of T on $BMO_{w,\rho}(\mathbb{R}^d)$.

In what follows we denote $K_0(\cdot, \cdot)$ to be the kernel of $\nabla^2 \Delta^{-1}$.

Lemma 51. Let $a, b > 0$ and $y, z \in \mathbb{R}^d$. Then

$$\int_{A(y,a,b)} K_0(y, x) dx = \int_{A(z,a,b)} K_0(z, x) dx.$$

Proof. Direct calculations give

$$\partial_{ij} \Gamma_0(u, v) = c \left(\frac{2\delta_{ij}}{|u-v|^d} + \frac{4(u_i - v_i)(u_j - v_j)}{|u-v|^{d+2}} \right)$$

for some $c > 0$ depending on d and for all $u, v \in \mathbb{R}^d$ and $i, j \in \{1, \dots, d\}$, where δ_{ij} is the Kronecker's delta and $\Gamma_0(\cdot, \cdot)$ is the fundamental solution of $-\Delta$. Then

$$\begin{aligned} \int_{A(y,a,b)} \partial_{ij} \Gamma_0(y, x) dx &= \int_{A(y,a,b)} c \left(\frac{2\delta_{ij}}{|y-x|^d} + \frac{4(y_i - x_i)(y_j - x_j)}{|y-x|^{d+2}} \right) dx \\ &= \int_{A(z,a,b)} c \left(\frac{2\delta_{ij}}{|z-x'|^d} + \frac{4(z_i - x'_i)(z_j - x'_j)}{|z-x'|^{d+2}} \right) dx' \\ &= \int_{A(z,a,b)} \partial_{ij} \Gamma_0(z, x') dx' \end{aligned}$$

for all $i, j \in \{1, \dots, d\}$, where we used the substitution $x' = x - y + z$ in the second step.

The claim follows from this observation. □

Proposition 52. *Let $\delta \in (0, 1]$ and $\alpha \in [1, 1 + \frac{\delta}{d})$. Suppose that $w \in A_\infty^\rho \cap D_\alpha^\rho$. Then T is a Schrödinger–Calderón–Zygmund operator of type (∞, δ) . Furthermore T satisfies Condition (a) in Proposition 50.*

Proof. That T is a Schrödinger–Calderón–Zygmund operator of type (∞, δ) follows from Lemma 43.

It remains to show that T satisfies Condition (a) in Proposition 50. For this let $x_0 \in \mathbb{R}^d$, $0 < r \leq \frac{\rho(x_0)}{2}$ and $y, z \in B = B(x_0, r)$. Then $\rho(y) \sim \rho(z) \sim \rho(x_0)$. We note that

$$\int_B |T1(y) - (T1)_B| dy \leq \frac{1}{|B|} \int_B \int_B |T1(y) - T1(z)| dy dz.$$

Therefore it suffices to show that

$$|T1(y) - T1(z)| \lesssim \left(\frac{r}{\rho(x_0)} \right)^\delta.$$

Indeed one has

$$\begin{aligned} |T1(y) - T1(z)| &\leq \left| \int_{|x-y| < 4\rho(x_0)} K(y, x) dx - \int_{|x-z| < 4\rho(x_0)} K(z, x) dx \right| \\ &\quad + \left| \int_{|x-y| \geq 4\rho(x_0)} K(y, x) dx - \int_{|x-z| \geq 4\rho(x_0)} K(z, x) dx \right| \\ &=: I + II. \end{aligned}$$

Next we estimate each term separately.

Term I. We start by noticing that

$$T1(y) = \int_{|x-y| < 4\rho(x_0)} K(y, x) dx := \lim_{\epsilon \rightarrow 0^+} \int_{\epsilon < |x-y| < 4\rho(x_0)} K(y, x) dx < \infty$$

for a.e. $y \in \mathbb{R}^d$. In the second step, we emphasize that the integral is implicitly understood in the principal value sense. The third step follows as T is bounded on $L^p(\mathbb{R}^d)$ for all $p \in (1, \sigma]$ due to [20, Theorem 0.3].

With the above observation in mind, it now follows from Lemma 51 that

$$\begin{aligned} I &= \left| \lim_{\epsilon \rightarrow 0^+} \left(\int_{\epsilon < |x-y| < 4\rho(x_0)} K(y, x) dx - \int_{\epsilon < |x-z| < 4\rho(x_0)} K(z, x) dx \right) \right| \\ &= \left| \lim_{\epsilon \rightarrow 0^+} \left(\int_{\epsilon < |x-y| < 4\rho(x_0)} K(y, x) - K_0(y, x) dx - \int_{\epsilon < |x-z| < 4\rho(x_0)} K(z, x) - K_0(z, x) dx \right) \right| \\ &= \left| \int_{|x-y| < 4\rho(x_0)} K(y, x) - K_0(y, x) dx - \int_{|x-z| < 4\rho(x_0)} K(z, x) - K_0(z, x) dx \right| \\ &\leq \left| \int_{\mathbb{R}^d} (K(y, x) - K_0(y, x)) (\mathbb{1}_{|x-y| < 4\rho(x_0)}(x) - \mathbb{1}_{|x-z| < 4\rho(x_0)}(x)) dx \right| \\ &\quad + \left| \int_{\mathbb{R}^d} ((K(y, x) - K_0(y, x)) - (K(z, x) - K_0(z, x))) \mathbb{1}_{|x-z| < 4\rho(x_0)}(x) dx \right| \\ &=: Ia + Ib. \end{aligned}$$

To estimate Ia we consider four cases as follows.

- (i) $|x - z| < |x - y| \leq 4\rho(x_0)$.
- (ii) $|x - y| \leq 4\rho(x_0) < |x - z|$.
- (iii) $|x - y| < |x - z| \leq 4\rho(x_0)$.
- (iv) $|x - z| \leq 4\rho(x_0) < |x - y|$.

By symmetry we only need to consider Case (i) and Case (ii). But Case (i) is trivial since in this situation $Ia = 0$. Next suppose (ii) holds. Then $4\rho(x_0) - 2r \leq |x - z| - |z - y| \leq |x - y| \leq 4\rho(x_0)$. Again Lemma 47 implies that

$$\begin{aligned} Ia &\lesssim \int_{4\rho(x_0) - 2r < |x-y| < 4\rho(x_0)} \frac{1}{|x-y|^d} \left(\frac{|x-y|}{\rho(y)} \right)^\delta dx \\ &\lesssim \frac{1}{\rho(x_0)^\delta} \int_{4\rho(x_0) - 2r < |x-y| < 4\rho(x_0)} \frac{1}{|x-y|^{d-\delta}} dx \lesssim \frac{r}{\rho(x_0)} \lesssim \left(\frac{r}{\rho(x_0)} \right)^\delta, \end{aligned}$$

where the second-to-last step follows from the Mean Value Theorem. Hence

$$Ia \lesssim \left(\frac{r}{\rho(x_0)} \right)^\delta. \quad (29)$$

To estimate Ib we write

$$\begin{aligned} Ib &\leq \left| \int_{|x-z| > 2|y-z|} \left((K(y, x) - K_0(y, x)) - (K(z, x) - K_0(z, x)) \right) \mathbb{1}_{|x-z| < 4\rho(x_0)}(x) dx \right| \\ &\quad + \left| \int_{|x-z| \leq 2|y-z|} \left((K(y, x) - K_0(y, x)) - (K(z, x) - K_0(z, x)) \right) \mathbb{1}_{|x-z| < 4\rho(x_0)}(x) dx \right| \\ &=: Ib_1 + Ib_2. \end{aligned}$$

For Ib_1 we use Lemma 48 to deduce that

$$Ib_1 \lesssim \frac{|y-z|^\delta}{\rho(z)^2} \int_{|x-z| < 4\rho(x_0)} |x-z|^{2-d-\delta} dx \lesssim \left(\frac{r}{\rho(x_0)} \right)^\delta.$$

For Ib_2 Lemma 47 gives

$$\begin{aligned} Ib_2 &\lesssim \int_{|x-z| \leq 2|y-z|} \frac{1}{|x-y|^d} \left(\frac{|x-y|}{\rho(y)} \right)^\delta dx + \int_{|x-z| \leq 2|y-z|} \frac{1}{|x-z|^d} \left(\frac{|x-z|}{\rho(z)} \right)^\delta dx \\ &\lesssim \frac{1}{\rho(x_0)^\delta} \left(\int_{|x-y| \leq 3|y-z|} \frac{1}{|x-y|^{d-\delta}} dx + \int_{|x-z| \leq 2|y-z|} \frac{1}{|x-z|^{d-\delta}} dx \right) \lesssim \left(\frac{r}{\rho(x_0)} \right)^\delta. \end{aligned}$$

Hence

$$Ib \lesssim \left(\frac{r}{\rho(x_0)} \right)^\delta. \quad (30)$$

Combining (29) and (30) together we obtain

$$I \lesssim \left(\frac{r}{\rho(x_0)} \right)^\delta. \quad (31)$$

Term II. We decompose II as

$$\begin{aligned} II &\leq \int_{|x-y| > 4\rho(x_0)} |K(y, x) - K(z, x)| dx + \int_{\mathbb{R}^d} |K(z, x) (\mathbb{1}_{|x-z| > 4\rho(x_0)} - \mathbb{1}_{|x-y| > 4\rho(x_0)})| dx \\ &=: IIa + IIb. \end{aligned}$$

For IIa we note that $|x - y| > 4\rho(x_0) \geq 8r > 2|y - z|$. Now we apply Lemma 46 to yield

$$IIa \lesssim \int_{|x-y| > 4\rho(x_0)} \frac{|y-z|^\delta}{|x-y|^{d+\delta}} dx \lesssim \left(\frac{r}{\rho(x_0)} \right)^\delta.$$

The term IIb is estimated in a similar manner as that of Ia to obtain

$$IIb \lesssim \left(\frac{r}{\rho(x_0)} \right)^\delta.$$

Hence we have shown that

$$II \lesssim \left(\frac{r}{\rho(x_0)} \right)^\delta. \tag{32}$$

In sum (31) and (32) together justify the claim. □

We are now in the position to prove Theorem 9.

Proof of Theorem 9. This follows at once from Propositions 50 and 52. □

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