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Generic simplicity of quantum Hamiltonian reductions

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Abstract. Let a reductive group G act on a smooth affine complex algebraic variety X. Let $\mathfrak{g}$ be the Lie algebra of G and $\mu : T^*(X) \to \mathfrak{g}^*$ be the moment map. If the moment map is flat, and for a generic character $\chi : \mathfrak{g} \to \mathbb{C}$, the action of G on $\mu^{-1}(\chi)$ is free, then we show that for very generic characters χ the corresponding quantum Hamiltonian reduction of the ring of differential operators $D(X)$ is simple.

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Let a reductive algebraic group G act on a smooth affine algebraic variety X over $\mathbb{C}$. Let $\mathfrak{g}$ be the Lie algebra of G. Let $\mu : T^*(X) \to \mathfrak{g}^*$ be the corresponding moment map. We will assume that this map is flat, and for generic G-invariant character $\chi \in \mathfrak{g}^*$ the action of G on $\mu^{-1}(\chi)$ is free.

Given a G-invariant character $\chi \in \mathfrak{g}^*$, denote by $U_\chi(G, X)$ the quantum Hamiltonian reduction of $D(X)$ with respect to χ. So,

$$U_\chi(G, X) = \left(D(X)/D(X)\mathfrak{g}^\chi\right)^G,$$

where $\mathfrak{g}^\chi = \{g - \chi(g) \in D(X), g \in \mathfrak{g}\}$. The usual filtration on $D(X)$ by the order of differential operators induces the corresponding filtration on $U_\chi(G, X)$. Then it follows from the flatness of the moment map that

$$\operatorname{gr} U_\chi(G, X) = \mathcal{O}\left(\mu^{-1}(0)//G\right).$$

In what follows by a very generic subset we mean a complement of a union of countably many proper closed Zariski subsets. Under these assumptions we have the following result.

Theorem 1. *For very generic values of a G-invariant character $\chi \in \mathfrak{g}^*$, the corresponding quantum Hamiltonian reduction $U_\chi(G, X)$ is simple. Moreover, if $f \in \mathfrak{g}_{\mathbb{Z}}/[\mathfrak{g}_{\mathbb{Z}}, \mathfrak{g}_{\mathbb{Z}}]$ is so that G acts freely on $\mu^{-1}(\chi)$ whenever $\chi(f) \neq 0$, then $U_\chi(G, X)$ is simple for all χ such that $\chi(f) \notin \mathbb{Q}$.*

The proof is be based on the reduction modulo p^n technique for a large prime p.

At first, we recall that given a ring R such that p is not a zero divisor, then the center of its reduction modulo $p, R_p = R/pR$ acquires a natural Poisson bracket, to be referred to as the reduction modulo p Poisson bracket, defined as follows. Given central elements $x, y \in Z(R_p)$, let $x', y' \in R$ be their lifts. Then

$$\{x, y\} = \left(\frac{1}{p}\left[x', y'\right]\right) \mod p \in Z\left(R_p\right).$$

We use the following result [5, Corollary 8].

Lemma 2. *Let* $\mathbf{k}$ *be a perfect field of characteristic* p. *Let* A *be a* p*-adically complete topologically free* $W(\mathbf{k})$*-algebra, such that* $A_1 = A/pA$ *is an Azumaya algebra over its center* Z_1. *Assume that* $\operatorname{Spec}(Z_1)$ *is a smooth symplectic* $\mathbf{k}$*-variety under the reduction modulo p Poisson bracket. Then* $A[p^{-1}]$ *is topologically simple.*

Next we need to recall some results and notations associated with quantum Hamiltonian reduction of the ring of crystalline differential operators in characteristic p from [1].

Let X be a smooth affine variety over an algebraically closed field $\mathbf{k}$ of characteristic p, and G be a reductive algebraic group over $\mathbf{k}$ with the Lie algebra $\mathfrak{g}$. Denote by $D(X)$ the ring of crystalline differential operators on X. As before, we have the moment map $\mu : T^*(X) \to \mathfrak{g}^*$ and the algebra homomorphism $U(\mathfrak{g}) \to D(X)$. Now recall that the p-center of $U(\mathfrak{g})$, denoted by $Z_p(\mathfrak{g})$, is generated by $g^p - g^{[p]}, g \in \mathfrak{g}$. We get an isomorphism

$$i : \operatorname{Sym}(\mathfrak{g})^{(1)} \to Z_p(\mathfrak{g}).$$

On the other hand, the center of $D(X)$ is generated by $\mathcal{O}(X)^p$ and $\xi^p - \xi^{[p]}, \xi \in T_X$ and this leads to an isomorphism

$$\mathcal{O}\left(T^*(X)\right)^{(1)} \to Z(D(X)).$$

We have $\eta' : Z_p(\mathfrak{g}) \to Z(D(X))$ and the corresponding homomorphism

$$\eta : \operatorname{Sym}(\mathfrak{g})^{(1)} \to \mathcal{O}\left(T^*(X)\right)^{(1)}.$$

Given $\chi \in \mathfrak{g}^*$, then $\chi^{[1]} \in \mathfrak{g}^*$ is defined as follows:

$$\chi^{[1]}(g) = \chi(g)^p - \chi\left(g^{[p]}\right), \quad g \in \mathfrak{g}.$$

Using the above homomorphisms it follows that the center of $U_\chi(G, X)$ contains $\mathcal{O}(\mu^{-1}(\chi^{[1]})//G)$. In this setting the following holds.

Lemma 3 ([1]). *Let* $\chi \in (\mathfrak{g}^*)^G$ *be a character. Then* $U_\chi(G, X)$ *is a finite algebra over* $\mu^{-1}(\chi^{[1]})//G$. *If* G *acts freely of* $\mu^{-1}(\chi^{[1]})$, *then* $U_\chi(G, X)$ *is an Azumaya algebra over* $\mu^{-1}(\chi^{[1]})//G$.

We need the following criterion of simplicity of certain filtered quantizations.

Lemma 4. *Let* $S \subset \mathbb{C}$ *be a finitely generated ring, and let* R *be a filtered* S*-algebra, such that* $\operatorname{gr}(R)$ *is a finitely generated commutative ring over* S. *Assume that for all large enough primes p the algebra* $R_p = R/pR$ *is an Azumaya algebra over its center* Z_p, *moreover* $\operatorname{Spec}(Z_p)$ *is a smooth symplectic variety over* S_p *under the reduction modulo p Poisson bracket. Let* F *be the field of fractions of* S. *Then* $R_F = R \otimes_S F$ *is a simple ring.*

Proof. Let I be a nonzero two sided ideal of R such that $(R/I)_F \neq 0$. After localizing S further, we may assume using the generic flatness theorem that $\operatorname{gr}(R/I)$ and $R/I, R$ are free S-modules. Hence for $p \gg 0$, $\bar{I}_p$ (the p-adic completion of I) is a topologically free nontrivial two-sided ideal of $\bar{R}_p$ (the p-adic completion of R). Now Lemma 2 yields a contradiction. □

Next we state a result implying that taking quantum Hamiltonian reduction and reducing modulo a large prime commute. The statement and its proof were kindly provided by W. van der Kallen (via mathoverflow.org.) Possible mistakes in the proof below are solely due to the author.

Theorem 5 (van der Kallen). *Let* S *be a commutative Noetherian ring of finite homological dimension, let* R *be a commutative* S*-algebra flat over* S. *Let* G *be a split reductive group over* S *acting on* R. *Then for all* $p \gg 0$ *and a base change to a characteristic p field* $S \to \mathbf{k}$, *the map* $R^G \otimes_S \mathbf{k} \to R_{\mathbf{k}}^{G_{\mathbf{k}}}$ *is surjective.*

Proof. At first, recall that there exists an integer $n \geq 1$ so that $H^i(G, S[\frac{1}{n}]) = 0$ for all i [3, Theorem 33]. This implies $H^i(G, S[\frac{1}{n}] \otimes_S N) = 0$ for any S-module N with the trivial G-action (since S has a finite global dimension). Let D, N be respectively the image and kernel of the map $S[\frac{1}{n}] \to \mathbf{k}$. As $H^1(G, S[\frac{1}{n}] \otimes_S N) = 0$, we get that $(R \otimes_S S[\frac{1}{n}])^G \to (R \otimes_S D)^G$ is surjective. Now flatness of $\mathbf{k}$ over D yields that

$$(R \otimes_S D)^G \otimes_D \mathbf{k} = R_{\mathbf{k}}^{G_{\mathbf{k}}}.$$

Therefore, we obtain the desired surjectivity $R^G \otimes_S \mathbf{k} \to R_{\mathbf{k}}^{G_{\mathbf{k}}}$ □

Proof of Theorem 1. Recall that some $0 \neq f \in \mathcal{O}((\mathfrak{g}^*)^G)$ has the property that for any $\chi \in (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*$ such that $f(\chi) \neq 0$, the action of G on $\mu^{-1}(\chi)$ is free. Let $S \subset \mathbb{C}$ be a large enough finitely generated subring over which X, f and the action of G on X are defined. Let $U \subset (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])_S^*$ denote the complement of the zero locus of f. Thus, G_S acts freely on $\mu^{-1}(U)$. Localizing S further and using the generic flatness theorem, we may assume that $\mathcal{O}(\mu^{-1}(0)//G)$ and $\mathcal{O}(\mu^{-1}(0))/\mathcal{O}(\mu^{-1}(0)//G)$ is a flat S-module.

Let $e_1, \cdots, e_l$ be a basis of $\mathfrak{g}_{\mathbb{Z}}/[\mathfrak{g}_{\mathbb{Z}}, \mathfrak{g}_{\mathbb{Z}}]$ over $\mathbb{Z}$. Let $S \to \mathbf{k}$ be a base change to a characteristic p field $\mathbf{k}$, let $\bar{\chi}$ denote the image of χ in $\mathfrak{g}_{\mathbf{k}}^*$. Then

$$\bar{\chi}^{[1]}(\bar{e}_i) = \left(\bar{\chi}(\bar{e}_i)\right)^p - \bar{\chi}(\bar{e}_i).$$

Let $W \subset (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*$ be the set of all χ so that $\chi(e_i)$ are algebraically independent over S. Clearly W is a very generic subset. We will show that for any $\chi \in W$ algebra $U_\chi(G, X)$ is simple.

Put $R = U_\chi(G, X)$. We verify that R satisfies assumptions in Lemma 4. Indeed, let $S \to \mathbf{k}$ be a base change to an algebraically closed field $\mathbf{k}$ of characteristic $\gg 0$, let $\bar{\chi}$ denote the base change of χ. Recall that R is equipped with the filtration so that $\mathrm{gr}(R) = \mathcal{O}(\mu^{-1}(0)//G)$. In particular, R is a free S-module. Similarly, $U_{\bar{\chi}}(G_{\mathbf{k}}, X_{\mathbf{k}})$ is equipped with the filtration such that $\mathrm{gr}(U_{\bar{\chi}}(G_{\mathbf{k}}, X_{\mathbf{k}}))$ is a subring of $\mathcal{O}(\mu_{\mathbf{k}}^{-1}(0)//G_{\mathbf{k}})$. Now applying Theorem 5 to the action of G on $\mathcal{O}(\mu^{-1}(0))$, we conclude that $\mathcal{O}(\mu^{-1}(0)//G) \otimes_S \mathbf{k}$ surjects onto $\mathcal{O}(\mu_{\mathbf{k}}^{-1}(0)//G_{\mathbf{k}})$. On the other hand, since $\mathcal{O}(\mu^{-1}(0))/\mathcal{O}(\mu^{-1}(0)//G)$ is flat over S, we get that

$$\mathcal{O}\left(\mu^{-1}(0)//G\right) \otimes_S \mathbf{k} \to \mathcal{O}\left(\mu^{-1}(0)\right) \otimes_S \mathbf{k}$$

is injective. So, the restriction map

$$\mathcal{O}\left(\mu^{-1}(0)//G\right) \otimes_S \mathbf{k} \to \mathcal{O}\left(\mu_{\mathbf{k}}^{-1}(0)//G_{\mathbf{k}}\right)$$

is an isomorphism. Therefore, $\mathrm{gr}(R) \otimes_S \mathbf{k} \cong \mathrm{gr}(U_{\bar{\chi}}(G_{\mathbf{k}}, X_{\mathbf{k}}))$. Now flatnest of $\mathrm{gr}(R)$ over S implies that $\mathrm{gr}(R_{\mathbf{k}}) = \mathrm{gr}(R) \otimes_S \mathbf{k}$. Hence we conclude that $R_{\mathbf{k}} \cong U_{\bar{\chi}}(G_{\mathbf{k}}, X_{\mathbf{k}})$.

Since $\chi(e_1), \cdots, \chi(e_n)$ are algebraically independent over S, we get that $\bar{f}(\chi^{[1]}) \neq 0$ for all $p \gg 0$ and an appropriate base change $S \to \mathbf{k}$. Hence $\bar{\chi} \in U_{\mathbf{k}}$. As G_S acts freely on $\mu^{-1}(U)$, we conclude that $G_{\mathbf{k}}$ acts freely on $\mu^{-1}(\bar{\chi})$. So Lemma 3 implies that $U_{\bar{\chi}}(G_{\mathbf{k}}, X_{\mathbf{k}})$ is an Azumaya algebra over a symplectic variety under the reduction modulo p Poisson bracket. So, conditions of Lemma 4 are met. Hence we have shown that algebra $U_\chi(G, X)$ is simple for very generic values of χ.

Now suppose there exists a nonzero $f \in \mathfrak{g}_{\mathbb{Z}}/[\mathfrak{g}_{\mathbb{Z}}, \mathfrak{g}_{\mathbb{Z}}]$ such that G acts freely on $\mu^{-1}(\chi)$ when $\chi(f) \neq 0$. Let S be a finitely generated subring containing $\chi(e_i)$ satisfying conditions as above. Write $f = \sum_i f_i e_i, f_i \in \mathbb{Z}$. Then given a base change $S \to \mathbf{k}$, we have

$$\bar{\chi}^{[1]}(\bar{f}) = \sum_i \bar{f}_i \left(\left(\bar{\chi}(\bar{e}_i)\right)^p - \bar{\chi}(\bar{e}_i)\right) = \bar{\chi}(\bar{f})^p - \bar{\chi}(\bar{f}).$$

Let χ be so that $\chi(f)$ is irrational. Then it follows from the Chebotarev density theorem that there are arbitrarily large primes p and a base change $S \to \mathbf{k}$ to an algebraically closed field $\mathbf{k}$ of characteristic p, such that $\bar{\chi}(\bar{f}) \notin \mathbb{F}_p$. Hence $\chi^{[1]}(\bar{f})$ is nonzero in $\mathbf{k}$. So, $G_{\mathbf{k}}$ acts freely on $\mu^{-1}(\bar{\chi})$, and arguing just as above we may conclude that the algebra $U_\chi(G, X)$ is simple. □

We may apply the above result to certain filtered quantizations of quiver varieties as follows. Let Q be a quiver with n vertices, let α be a its positive root. Then $G = \prod GL_{\alpha_i}/\mathbb{C}^*$ acts on the space of α-dimensional representations $Rep(Q,\alpha)$ giving rise to the moment map $m_\alpha : T^*(Rep(Q,\alpha)) \to \mathfrak{g}^*$. We will identify $(\mathfrak{g}^*)^G$ with $\lambda \in \mathbb{C}^n$ such that $\lambda \cdot \alpha = 0$. From now on we assume that the moment map m_α is flat. The set of such dimension vectors α was fully described by Crawly-Boevey in [2, Theorem 1.1]. Denote by $A_\lambda(Q,\alpha)$ the corresponding quantum Hamiltonian reduction of the ring of differential operators $D(Rep(Q,\alpha))$ with respect to the character λ.

We have the following direct corollary of Theorem 1. Remark that stronger results on generic simplicity follows from the works of Losev on quantizations of quiver varieties (see for example [4, Theorem 1.4.2].)

Theorem 6. *Let α be a positive root as above. Let $\lambda \cdot \alpha = 0$ be such that $\lambda \cdot \beta \notin \mathbb{Q}$ for any positive root $\beta < \alpha$. Then $A_\lambda(Q,\alpha)$ is simple.*

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