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Versions L^p des inégalités généralisées de Korn pour les champs de tenseurs incompatibles de dimension quelconque avec dérivée extérieure p-intégrable

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Abstract. For $n \geq 2$ and $1 < p < \infty$ we prove an L^p-version of the generalized Korn-type inequality for incompatible, p-integrable tensor fields $P : \Omega \to \mathbb{R}^{n\times n}$ having p-integrable generalized $\underline{\text{Curl}}$ and generalized vanishing tangential trace $P\,\tau_l = 0$ on $\partial\Omega$, denoting by $\{\tau_l\}_{l=1,\dots,n-1}$ a moving tangent frame on $\partial\Omega$, more precisely we have:

$$\|P\|_{L^p(\Omega,\mathbb{R}^{n\times n})} \leq c\left(\left\|\operatorname{sym} P\right\|_{L^p(\Omega,\mathbb{R}^{n\times n})} + \left\|\underline{\operatorname{Curl}}\, P\right\|_{L^p(\Omega,(\mathfrak{so}(n))^n)}\right),$$

where the generalized $\underline{\text{Curl}}$ is given by $(\underline{\operatorname{Curl}}\, P)_{ijk} := \partial_i P_{kj} - \partial_j P_{ki}$ and $c = c(n,p,\Omega) > 0$

Résumé. On montre pour $n \geq 2$ et $1 < p < \infty$ une version L^p de l'inégalité généralisée de Korn pour tous les champs de tenseurs incompatibles et p-intégrables $P : \Omega \to \mathbb{R}^{n\times n}$, avec rotationnel généralisé p-intégrable et avec zéro trace tangentielle $P\,\tau_l = 0$ sur $\partial\Omega$, où $\{\tau_l\}_{l=1,\dots,n-1}$ est un repère tangent sur $\partial\Omega$. Plus précisément on a :

$$\|P\|_{L^p(\Omega,\mathbb{R}^{n\times n})} \leq c\left(\left\|\operatorname{sym} P\right\|_{L^p(\Omega,\mathbb{R}^{n\times n})} + \left\|\underline{\operatorname{Curl}}\, P\right\|_{L^p(\Omega,(\mathfrak{so}(n))^n)}\right),$$

où les composantes du rotationnel généralisé s'écrivent $(\underline{\operatorname{Curl}}\, P)_{ijk} := \partial_i P_{kj} - \partial_j P_{ki}$ et $c = c(n,p,\Omega) > 0$.

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1. Introduction

In [6] we have shown that there exists a constant $c = c(p,\Omega) > 0$ such that

$$\|P\|_{L^p(\Omega,\mathbb{R}^{3\times3})} \le c\left(\|\operatorname{sym} P\|_{L^p(\Omega,\mathbb{R}^{3\times3})} + \|\operatorname{Curl} P\|_{L^p(\Omega,\mathbb{R}^{3\times3})}\right)$$

holds for all tensor fields $P \in W_0^{1,p}(\operatorname{Curl};\Omega,\mathbb{R}^{3\times3})$, i.e., for all $P \in W^{1,p}(\operatorname{Curl};\Omega,\mathbb{R}^{3\times3})$ with vanishing tangential trace $P\times\nu = 0$ ($\Leftrightarrow\ P\tau_l = 0$) on $\partial\Omega$ where ν denotes the outward unit normal vector field and $\{\tau_l\}_{l=1,2,3}$ a moving tangent frame on $\partial\Omega$ and $\Omega\subset\mathbb{R}^3$ is a bounded Lipschitz domain. The crucial ingredients for our proof were the Lions lemma and Nečas estimate, the compactness of $W_0^{1,p}(\Omega)\subset\subset L^p(\Omega)$ and an algebraic identity in terms of components of the cross product of a skew-symmetric matrix with a vector. Recall, that for a bounded Lipschitz domain (i.e. bounded open connected with Lipschitz boundary) $\Omega\subset\mathbb{R}^n$, the Lions lemma states that $f\in L^p(\Omega)$ if and only if $f\in W^{-1,p}(\Omega)$ and $\nabla f\in W^{-1,p}(\Omega,\mathbb{R}^n)$, which is equivalently expressed by the Nečas estimate

$$\|f\|_{L^p(\Omega)} \le c\left(\|f\|_{W^{-1,p}(\Omega)} + \|\nabla f\|_{W^{-1,p}(\Omega,\mathbb{R}^n)}\right) \tag{1}$$

with a positive constant $c = c(p,n,\Omega)$. In fact, such an argumentation scheme is also used to prove the classical Korn inequalities, cf. e.g. [1–6] and the discussions contained therein. However, [1–5] focus on the compatible case, i.e. $P = \mathrm{D}u$, where we deal with general square matrices $P\in\mathbb{R}^{n\times n}$, thus, the incompatible case.

Here, we extend our results from [6] to the n-dimensional case, hence generalizing the main result from [8] to the L^p-setting. This is, we prove

$$\|P\|_{L^p(\Omega,\mathbb{R}^{n\times n})} \le c\left(\|\operatorname{sym} P\|_{L^p(\Omega,\mathbb{R}^{n\times n})} + \|\underline{\operatorname{Curl}} P\|_{L^p(\Omega,(\mathfrak{so}(n))^n)}\right) \quad \forall\ P\in W_0^{1,p}\left(\underline{\operatorname{Curl}};\Omega,\mathbb{R}^{n\times n}\right), \tag{2}$$

where the generalized $\underline{\operatorname{Curl}}$ is given by $(\underline{\operatorname{Curl}} P)_{ijk} := \partial_i P_{kj} - \partial_j P_{ki}$ and the vanishing tangential trace condition reads $P\tau_l = 0$ on $\partial\Omega$ denoting by $\{\tau_l\}_{l=1,\dots,n-1}$ a moving tangent frame on $\partial\Omega$.

For a detailed motivation and definitions we refer to [6] and the references contained therein. Indeed, we follow the argumentation scheme presented in [6] closely, emphasizing only the necessary modifications coming from the generalization of the vector product. The latter then provides an adequate generalization of the Curl-operator to the n-dimensional setting. Especially, the generalized curl of vector fields can be seen as their exterior derivative, see also the discussion in [8].

2. Notations

Let $n\ge 2$. For vectors $a,b\in\mathbb{R}^n$, we consider the scalar product $\langle a,b\rangle := \sum_{i=1}^n a_i\, b_i \in\mathbb{R}$, the (squared) norm $\|a\|^2 := \langle a,a\rangle$ and the dyadic product $a\otimes b := (a_i\, b_j)_{i,j=1,\dots,n}\in\mathbb{R}^{n\times n}$. Similarly, for matrices $P,Q\in\mathbb{R}^{n\times n}$ we define the scalar product $\langle P,Q\rangle := \sum_{i,j=1}^n P_{ij}\, Q_{ij}\in\mathbb{R}$ and the (squared) Frobenius-norm $\|P\|^2 := \langle P,P\rangle$. Moreover, $P^T := (P_{ji})_{i,j=1,\dots,n}$ denotes the transposition of the matrix $P = (P_{ij})_{i,j=1,\dots,n}$, which decomposes orthogonally into the symmetric part $\operatorname{sym} P := \frac{1}{2}(P+P^T)$ and the skew-symmetric part $\operatorname{skew} P := \frac{1}{2}(P-P^T)$. The Lie-Algebra of skew-symmetric matrices is denoted by $\mathfrak{so}(n) := \{A\in\mathbb{R}^{n\times n} \mid A^T = -A\}$. The identity matrix is denoted by $\mathbb{1}$, so that the trace of a matrix P is given by $\operatorname{tr} P := \langle P,\mathbb{1}\rangle$.

The cross product for vectors $a, b \in \mathbb{R}^n$ generalizes to

$$a \underline{\times} b := \left(a_i\, b_j - a_j\, b_i\right)_{i,j=1,\dots,n} = a\otimes b - b\otimes a = 2\cdot \operatorname{skew}(a\otimes b) \in \mathfrak{so}(n) \cong \mathbb{R}^{\frac{n(n-1)}{2}}. \tag{3}$$

Using the bijection $\operatorname{axl}: \mathfrak{so}(3)\to\mathbb{R}^3$ we obtain back the standard cross product for $a, b\in\mathbb{R}^3$:

$$a\times b = -\operatorname{axl}(a\underline{\times} b) \tag{4}$$

where $\operatorname{axl}: \mathfrak{so}(3)\to\mathbb{R}^3$ is given in such a way that

$$A\, b = \operatorname{axl}(A)\times b \quad \forall\ A\in\mathfrak{so}(3), \quad b\in\mathbb{R}^3. \tag{5}$$

Like in 3-dimensions it holds:

Observation 1. *Let $n\geq 2$. For non-zero vectors $a, b\in\mathbb{R}^n$ we have $a\underline{\times} b = 0$ if and only if a and b are parallel.*

Proof. Since the "if" part is obvious we show the "only if" direction:

$$\begin{aligned} a\underline{\times} b = 0 \quad &\Leftrightarrow\quad \operatorname{skew}(a\otimes b) = 0 \quad\Leftrightarrow\quad a\otimes b = b\otimes a \quad\Rightarrow\quad (a\otimes b)b = (b\otimes a)b \\ &\Leftrightarrow\quad a\,\|b\|^2 = b\,\langle a, b\rangle. \end{aligned}$$

□

As in the 3-dimensional case, we understand the vector product of a square-matrix $P\in\mathbb{R}^{n\times n}$ and a vector $b\in\mathbb{R}^n$ row-wise, i.e.

$$P\underline{\times} b := \left((P^T e_k)\underline{\times} b\right)_{k=1,\dots,n} = \left(P_{ki}\, b_j - P_{kj}\, b_i\right)_{i,j,k=1,\dots,n} \in (\mathfrak{so}(n))^n. \tag{6}$$

For index notations we set: $\quad (P\underline{\times} b)_{ijk} := P_{ki}\, b_j - P_{kj}\, b_i$.

Especially, for skew-symmetric matrices $A\in\mathfrak{so}(n)$ we note the following crucial relation for our considerations:

$$\begin{aligned} \left(A\underline{\times} b\right)_{kij} - \left(A\underline{\times} b\right)_{kji} + \left(A\underline{\times} b\right)_{jik} &= A_{jk}\, b_i - A_{ji}\, b_k - \left(A_{ik}\, b_j - A_{ij}\, b_k\right) + A_{kj}\, b_i - A_{ki}\, b_j \\ &\overset{(A_{ij}=-A_{ji})}{=} 2A_{ij}\, b_k \qquad \forall\ i,j,k = 1,\dots n \end{aligned} \tag{7}$$

with the direct consequence

Observation 2. *Let $n\geq 2$. For $A\in\mathfrak{so}(n)$ and a non-zero vector $b\in\mathbb{R}^n$ we have $A\underline{\times} b = 0$ if and only if $A = 0$.*

Let $\Omega\subset\mathbb{R}^n$, $n\geq 2$, be a domain. As in $\mathbb{R}^3$ we formally introduce the generalized $\underline{\operatorname{curl}}$ of a vector field $v\in\mathscr{D}'(\Omega,\mathbb{R}^n)$ via

$$\underline{\operatorname{curl}}\ v := v\underline{\times}(-\nabla) = \nabla\underline{\times} v = -2\cdot\operatorname{skew}(v\otimes\nabla) = -2\cdot\operatorname{skew}(\mathrm{D}v)\in\mathfrak{so}(n). \tag{8}$$

Furthermore, for $(n\times n)$-square matrix fields we understand this operation row-wise:

$$\underline{\operatorname{Curl}}\, P := P\underline{\times}(-\nabla) = \left(\underline{\operatorname{curl}}\left(P^T e_k\right)\right)_{k=1,\dots,n} = \left(\partial_i P_{kj} - \partial_j P_{ki}\right)_{i,j,k=1,\dots,n} \in (\mathfrak{so}(n))^n. \tag{9}$$

For index notations we define: $\quad (\underline{\operatorname{Curl}}\, P)_{ijk} := \partial_i P_{kj} - \partial_j P_{ki}$. Of course, $\quad \underline{\operatorname{Curl}}\,\mathrm{D}v\equiv 0$.

Moreover, we make use of the generalized divergence Div for matrix fields $P\in\mathscr{D}'(\Omega,\mathbb{R}^{n\times n})$ row-wise, via

$$\operatorname{Div} P := \left(\operatorname{div}\left(P^T e_k\right)\right)_{k=1,\dots,n}. \tag{10}$$

In fact, the crucial relation (7) implies that the full gradient of a skew-symmetric matrix is already determined by its generalized $\underline{\operatorname{Curl}}$, cf. also [7, p. 155]:

Corollary 3. *Let $n\geq 2$. For $A\in\mathscr{D}'(\Omega,\mathfrak{so}(n))$ the entries of the gradient $\mathrm{D}A$ are linear combinations of the entries from $\underline{\operatorname{Curl}}\, A$.*

Proof. Replacing b by $-\nabla$ in (7) we see that

$$\left(\underline{\operatorname{Curl}}\, A\right)_{kij} - \left(\underline{\operatorname{Curl}}\, A\right)_{kji} + \left(\underline{\operatorname{Curl}}\, A\right)_{jik} = -2\,\partial_k A_{ij}.$$

□

This control of all first partial derivatives of a skew-symmetric matrix field in terms of the generalized $\underline{\mathrm{Curl}}$ then immediately yields in all dimensions

Corollary 4. *Let $n \geq 2$. For $A \in L^p(\Omega, \mathfrak{so}(n))$ we have $\underline{\mathrm{Curl}}\, A \equiv 0$ in the distributional sense if and only if $A = \mathrm{const}$ almost everywhere in Ω.*

2.1. *Function spaces*

Having above relations at hand we can now catch up the arguments from [6]. For that purpose let us define for $n \geq 2$ and $1 < p < \infty$ the space

$$W^{1,p}\left(\underline{\mathrm{Curl}};\Omega,\mathbb{R}^{n\times n}\right) := \left\{P \in L^p\left(\Omega,\mathbb{R}^{n\times n}\right) \mid \underline{\mathrm{Curl}}\, P \in L^p\left(\Omega,(\mathfrak{so}(n))^n\right)\right\} \tag{11a}$$

equipped with the norm

$$\|P\|_{W^{1,p}(\underline{\mathrm{Curl}};\Omega,\mathbb{R}^{n\times n})} := \left(\|P\|^p_{L^p(\Omega,\mathbb{R}^{n\times n})} + \left\|\underline{\mathrm{Curl}}\, P\right\|^p_{L^p(\Omega,(\mathfrak{so}(n))^n)}\right)^{\frac{1}{p}}. \tag{11b}$$

By definition of the norm in the dual space, we have

$$P \in L^p\left(\Omega,\mathbb{R}^{n\times n}\right) \quad\Rightarrow\quad \underline{\mathrm{Curl}}\, P \in W^{-1,p}\left(\Omega,(\mathfrak{so}(n))^n\right) \quad \text{with } \left\|\underline{\mathrm{Curl}}\, P\right\|_{W^{-1,p}(\Omega,(\mathfrak{so}(n))^n)} \leq c\,\|P\|_{L^p(\Omega,\mathbb{R}^{n\times n})}. \tag{12}$$

Furthermore, we consider the subspace

$$\begin{aligned} W_0^{1,p}\left(\underline{\mathrm{Curl}};\Omega,\mathbb{R}^{n\times n}\right) &:= \left\{P \in W^{1,p}\left(\underline{\mathrm{Curl}};\Omega,\mathbb{R}^{n\times n}\right) \mid P\,\underline{\times}\,\nu = 0 \text{ on } \partial\Omega\right\} \\ &= \left\{P \in W^{1,p}\left(\underline{\mathrm{Curl}};\Omega,\mathbb{R}^{n\times n}\right) \mid P\,\tau_l = 0 \text{ on } \partial\Omega \text{ for all } l = 1,\dots,n-1\right\}, \end{aligned} \tag{13}$$

where ν stands for the outward unit normal vector field and $\{\tau_l\}_{l=1,\dots,n-1}$ denotes a moving tangent frame on $\partial\Omega$. Here, the generalized tangential trace $P\,\underline{\times}\,\nu$ is understood in the sense of $W^{-\frac{1}{p},p}(\partial\Omega,\mathbb{R}^{n\times n})$ which is justified by partial integration, so that its trace is defined by

$$\begin{aligned} &\forall\ k = 1,\dots n,\ \forall\ Q \in W^{1-\frac{1}{p'},p'}\left(\partial\Omega,\mathbb{R}^{n\times n}\right): \\ &\qquad \left\langle\left(P^T e_k\right)\underline{\times}\,\nu, Q\right\rangle_{\partial\Omega} = \int_\Omega \left\langle \underline{\mathrm{curl}}\left(P^T e_k\right), \widetilde{Q}\right\rangle_{\mathbb{R}^{n\times n}} + 2\left\langle P^T e_k, \mathrm{Div}\left(\mathrm{skew}\,\widetilde{Q}\right)\right\rangle_{\mathbb{R}^n} \mathrm{d}x \end{aligned}$$

having denoted by $\widetilde{Q} \in W^{1,p'}(\Omega,\mathbb{R}^{n\times n})$ any extension of Q in Ω, where, $\langle\cdot,\cdot\rangle_{\partial\Omega}$ indicates the duality pairing between $W^{-\frac{1}{p},p}(\partial\Omega,\mathbb{R}^{n\times n})$ and $W^{1-\frac{1}{p'},p'}(\partial\Omega,\mathbb{R}^{n\times n})$. Indeed, for $P, Q \in C^1(\Omega,\mathbb{R}^{n\times n}) \cap C^0(\overline{\Omega},\mathbb{R}^{n\times n})$ we have

$$\begin{aligned} \frac{1}{2}\left\langle\left(P^T e_k\right)\underline{\times}\,\nu, Q\right\rangle_{\mathbb{R}^{n\times n}} &= \left\langle \mathrm{skew}\left(\left(P^T e_k\right)\otimes\nu\right), Q\right\rangle_{\mathbb{R}^{n\times n}} = \left\langle\left(P^T e_k\right)\otimes\nu, \mathrm{skew}\,Q\right\rangle_{\mathbb{R}^{n\times n}} \\ &= \sum_{i,j=1}^n P_{ki}\,\nu_j\,(\mathrm{skew}\,Q)_{ij} = -\sum_{i,j=1}^n \nu_j\,(\mathrm{skew}\,Q)_{ji}\,P_{ki} \\ &= -\left\langle \nu, (\mathrm{skew}\,Q)\left(P^T e_k\right)\right\rangle_{\mathbb{R}^n}, \end{aligned} \tag{14}$$

so that using the divergence-theorem, for $k = 1, \dots, n$ we have[1]

$$\begin{aligned}
&\int_{\partial\Omega} \langle (P^T e_k) \underline{\times} \nu, Q \rangle_{\mathbb{R}^{n\times n}} \, \mathrm{d}S \\
&\qquad \overset{(14)}{=} -2 \int_{\partial\Omega} \langle \nu, (\operatorname{skew} Q)\, (P^T e_k) \rangle_{\mathbb{R}^n} \, \mathrm{d}S \\
&\qquad = -2 \int_{\Omega} \operatorname{div}\big((\operatorname{skew} Q)\, (P^T e_k) \big) \, \mathrm{d}x \\
&\qquad = -2 \int_{\Omega} \langle \operatorname{Div}\left[(\operatorname{skew} Q)^T \right], P^T e_k \rangle_{\mathbb{R}^n} + \langle (\operatorname{skew} Q), \mathrm{D}(P^T e_k) \rangle_{\mathbb{R}^{n\times n}} \, \mathrm{d}x \\
&\qquad = \int_{\Omega} \langle \underline{\operatorname{curl}}\, (P^T e_k), Q \rangle_{\mathbb{R}^{n\times n}} + 2 \langle P^T e_k, \operatorname{Div}(\operatorname{skew} Q) \rangle_{\mathbb{R}^n} \, \mathrm{d}x.
\end{aligned} \tag{15}$$

Further, following [6] we introduce also the space $W^{1,p}_{\Gamma,0}(\operatorname{Curl};\Omega,\mathbb{R}^{n\times n})$ of functions with vanishing tangential trace only on a relatively open (non-empty) subset $\Gamma \subseteq \partial\Omega$ of the boundary by completion of $C^{\infty}_{\Gamma,0}(\Omega,\mathbb{R}^{n\times n})$ with respect to the $W^{1,p}(\operatorname{Curl};\Omega,\mathbb{R}^{n\times n})$-norm.

Remark 5 (Tangential trace condition). Note, that the vanishing of the tangential trace $P\underline{\times}\nu$ at some point is equivalent to $P\tau_l = 0$ for all $l = 1, \dots, n-1$, denoting by $\{\tau_l\}_{l=1,\dots,n-1}$ a frame of the corresponding tangent space. Indeed, by Observation 1 we have

$$\begin{aligned}
&P\underline{\times}\nu = 0 \\
&\quad \Leftrightarrow \quad \operatorname{skew}\big((P^T e_k) \otimes \nu \big) = 0,\ k = 1, \dots, n, \quad \Leftrightarrow \quad (P^T e_k) \text{ parallel to } \nu \text{ for all } k = 1, \dots, n \\
&\quad \Leftrightarrow \quad \langle P^T e_k, \tau_l \rangle = 0 \quad \forall\ l = 1, \dots, n-1,\ \forall\ k = 1, \dots, n \quad \Leftrightarrow \quad P\tau_l = 0 \quad \forall \quad l = 1, \dots, n-1.
\end{aligned}$$

3. Main results

We will now state the results from [6] in the n-dimensional case, for details of the proofs we refer to the corresponding results therein:

Lemma 6. *Let $n \geq 2$, $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $1 < p < \infty$. Then $P \in \mathscr{D}'(\Omega,\mathbb{R}^{n\times n})$, $\operatorname{sym} P \in L^p(\Omega,\mathbb{R}^{n\times n})$ and $\underline{\operatorname{Curl}}\, P \in W^{-1,p}(\Omega,(\mathfrak{so}(n))^n)$ imply $P \in L^p(\Omega,\mathbb{R}^{n\times n})$. Moreover, we have the estimate*

$$\|P\|_{L^p(\Omega,\mathbb{R}^{n\times n})} \leq c \left(\|\operatorname{skew} P\|_{W^{-1,p}(\Omega,\mathbb{R}^{n\times n})} + \|\operatorname{sym} P\|_{L^p(\Omega,\mathbb{R}^{n\times n})} + \|\underline{\operatorname{Curl}}\, P\|_{W^{-1,p}(\Omega,(\mathfrak{so}(n))^n)} \right), \tag{16}$$

with a constant $c = c(n,p,\Omega) > 0$.

Proof. Use Corollary 3 and apply the Lions lemma and Nečas estimate, [6, Theorem 2.6] to $\operatorname{skew} P$, cf. [6, proof of Lemma 3.1]. □

The general Korn-type inequalities then follow by eliminating the first term on the right-hand side of (16):

Theorem 7. *Let $n \geq 2$, $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $1 < p < \infty$. There exists a constant $c = c(n,p,\Omega) > 0$, such that for all $P \in L^p(\Omega,\mathbb{R}^{n\times n})$ we have*

$$\inf_{A \in \mathfrak{so}(n)} \|P - A\|_{L^p(\Omega,\mathbb{R}^{n\times n})} \leq c \left(\|\operatorname{sym} P\|_{L^p(\Omega,\mathbb{R}^{n\times n})} + \|\underline{\operatorname{Curl}}\, P\|_{W^{-1,p}(\Omega,(\mathfrak{so}(n))^n)} \right). \tag{17}$$

[1]This partial integration formula slightly differs from the situation in $\mathbb{R}^3$ since the generalized $\underline{\operatorname{Curl}}$ has image in $(\mathfrak{so}(n))^n$ which corresponds to $\mathbb{R}^{n\times n}$ only for $n = 3$.

Proof. By Corollary 4 the kernel of the right-hand side consists only of constant skew-symmetric matrices:

$$\begin{aligned} K &:= \left\{P \in L^p\left(\Omega, \mathbb{R}^{n\times n}\right) \middle| \operatorname{sym} P = 0 \text{ a.e. and } \underline{\operatorname{Curl}}\, P = 0 \text{ in the distributional sense}\right\} \\ &= \{P = A \text{ a.e.} \,|\, A \in \mathfrak{so}(n)\}. \end{aligned} \tag{18}$$

Then there exist $M := \dim K = \frac{n(n-1)}{2}$ linear forms ℓ_α on $L^p(\Omega, \mathbb{R}^{n\times n})$ such that $P \in K$ is equal to 0 if and only if $\ell_\alpha(P) = 0$ for all $\alpha = 1, \ldots, M$. Exploiting the compactness $L^p(\Omega, \mathbb{R}^{n\times n}) \subset\subset W^{-1,p}(\Omega, \mathbb{R}^{n\times n})$ allows us to eliminate the first term on the right-hand side of (16) so that we arrive at

$$\|P\|_{L^p(\Omega,\mathbb{R}^{n\times n})} \le c\left(\left\|\operatorname{sym} P\right\|_{L^p(\Omega,\mathbb{R}^{n\times n})} + \left\|\underline{\operatorname{Curl}}\, P\right\|_{W^{-1,p}(\Omega,(\mathfrak{so}(n))^n)} + \sum_{\alpha=1}^{M} |\ell_\alpha(P)|\right). \tag{19}$$

Considering $P - A_P$ in (19), where the skew-symmetric matrix $A_P \in K$ is chosen in such a way that $\ell_\alpha(P - A_P) = 0$ for all $\alpha = 1, \ldots, M$, then yields the conclusion, cf. [6, proof of Theorem 3.4]. □

Moreover, the kernel is killed by the tangential trace condition $P \underline{\times} \nu \equiv 0$ (or $P\tau_l \equiv 0$ for all $l = 1, \ldots, n-1$), cf. (18) together with Observation 2 (and also Remark 5), so that we arrive at

Theorem 8. *Let $n \ge 2$, $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $1 < p < \infty$. There exists a constant $c = c(n, p, \Omega) > 0$, such that for all $P \in W_0^{1,p}(\underline{\operatorname{Curl}}; \Omega, \mathbb{R}^{n\times n})$ we have*

$$\|P\|_{L^p(\Omega,\mathbb{R}^{n\times n})} \le c\left(\left\|\operatorname{sym} P\right\|_{L^p(\Omega,\mathbb{R}^{n\times n})} + \left\|\underline{\operatorname{Curl}}\, P\right\|_{L^p(\Omega,(\mathfrak{so}(n))^n)}\right). \tag{20}$$

Proof. Having Observation 2 we can closely follow the proof of [6, Theorem 3.5]. □

Similar argumentations show that (20) also holds true for functions with vanishing tangential trace only on a relatively open (non-empty) subset $\Gamma \subseteq \partial\Omega$ of the boundary, namely

Theorem 9. *Let $n \ge 2$, $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $1 < p < \infty$. There exists a constant $c = c(n, p, \Omega) > 0$, such that for all $P \in W_{\Gamma,0}^{1,p}(\underline{\operatorname{Curl}}; \Omega, \mathbb{R}^{n\times n})$ we have*

$$\|P\|_{L^p(\Omega,\mathbb{R}^{n\times n})} \le c\left(\left\|\operatorname{sym} P\right\|_{L^p(\Omega,\mathbb{R}^{n\times n})} + \left\|\underline{\operatorname{Curl}}\, P\right\|_{L^p(\Omega,(\mathfrak{so}(n))^n)}\right). \tag{21}$$

Furthermore, Theorem 9 reduces for compatible $P = \mathrm{D}u$ to a tangential Korn inequality (Corollary 10) and for skew-symmetric $P = A$ to a Poincaré inequality in arbitrary dimensions (Corollary 12):

Corollary 10. *Let $n \ge 2$, $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $1 < p < \infty$. There exists a constant $c = c(n, p, \Omega) > 0$, such that for all $u \in W_{\Gamma,0}^{1,p}(\Omega, \mathbb{R}^n)$ we have*

$$\|\mathrm{D}u\|_{L^p(\Omega,\mathbb{R}^{n\times n})} \le c\left\|\operatorname{sym} \mathrm{D}u\right\|_{L^p(\Omega,\mathbb{R}^n)} \quad \text{with } \mathrm{D}u \underline{\times} \nu = 0 \quad \text{on } \Gamma. \tag{22}$$

Remark 11. On Γ the boundary condition $\mathrm{D}u \underline{\times} \nu = 0$ is equivalent to $\mathrm{D}u\,\tau_l = 0$ for all $l = 1, \ldots, n-1$ and is, e.g., fulfilled if $u_{|\Gamma} \equiv \text{const.}$, see Remark 5.

Corollary 12. *Let $n \ge 2$, $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $1 < p < \infty$. There exists a constant $c = c(n, p, \Omega) > 0$, such that for all $A \in W_{\Gamma,0}^{1,p}(\underline{\operatorname{Curl}}; \Omega, \mathfrak{so}(n)) = W_{\Gamma,0}^{1,p}(\Omega, \mathfrak{so}(n))$ we have*

$$\|A\|_{L^p(\Omega,\mathfrak{so}(n))} \le c\left\|\underline{\operatorname{Curl}}\, A\right\|_{L^p(\Omega,(\mathfrak{so}(n))^n)} \quad \text{with } A \underline{\times} \nu = 0 \overset{*}{\Leftrightarrow} A = 0 \quad \text{on } \Gamma. \tag{23}$$

Remark 13. The equivalence of condition $*$ is seen in the following way: In any dimension the rank of the skew-symmetric matrix A is an even number, cf. [9, p. 30], and by Remark 5 the rows $A^T e_k$ are all parallel ($\Leftrightarrow$ $A\tau_l = 0$ for all $l = 1, \ldots, n-1$) such that the rank of A is not greater then 1.

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