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Sergey A. Denisov

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Partial differential equations / *Équations aux dérivées partielles*

Subharmonic functions in scattering theory

Sergey A. Denisov [a]

[a] University of Wisconsin–Madison Mathematics Department 480 Lincoln Dr., Madison, WI, 53706, USA
E-mail: denissov@wisc.edu

Abstract. We present a method that uses the properties of subharmonic functions to control spatial asymptotics of Green's kernel of multidimensional Schrödinger operator with rough potential.

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1. Introduction

In this note, we look at the multidimensional Schrödinger operator H: $H = -\Delta + V$, $x \in \mathbb{R}^3$ in the case when the real-valued potential $V \in L^\infty(\mathbb{R}^3)$ does not satisfy the classical assumptions of the scattering theory [14]. Our motivation is to present a method based on the elementary properties of subharmonic functions which provides control on spatial asymptotics of Green's function for H in the situation when the standard tools of perturbation theory do not work. In the classical situation, when V is short-range, i.e., $|V(x)| < C(|x|+1)^{-\gamma}, \gamma > 1$ the results of Kato [9] and Agmon [1] provide existence of wave operators in Schrödinger dynamics and the positive spectrum of H is purely absolutely continuous. The case when V is long-range, e.g., $|D^j V| < C(|x|+1)^{-\gamma-j}$, $\gamma > 0.5$, $0 \le j \le j_0$ was addressed in the works of Hörmander [7, 8] and, again, the positive spectrum was shown to be purely a.c. (see also Saito [12] for results with even weaker conditions on V). Following the significant progress in one-dimensional scattering theory (see [13] for the survey), the focus shifted to understanding the spectrum of multidimensional operator with rough potentials, e.g., V satisfying only an upper bound $|V| < C(|x|+1)^{-\gamma}$ with $\gamma \in (0.5, 1]$. For that case, the existence of a.c. spectrum is not known, but some results were obtained [4, 5, 11] if V is assumed to oscillate additionally. We notice that for potentials so rough the singular spectrum can coexist with the absolutely continuous one and the standard tools of scattering theory, such as absorption principle, has no chance to work. The alternative method rooted in complex analysis rather than harmonic analysis is based on the study of spatial asymptotics of Green's function. Here is its outline. Given $z \in \mathbb{C}^+$ and $F \in L^2(\mathbb{R}^3)$, we define $R_z F = (H-z)^{-1} F$ and Green's function $G(x,y,z)$ as $(R_z F)(x) = \int_{\mathbb{R}^3} G(x,y,z) F(y) \mathrm{d}y$. For the free Schrödinger equation when $V = 0$, the formula for Green's function is known:

$$G_0(x,y,k^2) = e^{ik|x-y|}(4\pi|x-y|)^{-1},\ k = \sqrt{z},\ k \in \mathbb{C}^+\,.$$

Let F have compact support and $F \not\equiv 0$. If one defines $U_F(x,k) = R_{k^2}F$, then the central quantity of interest is an *amplitude* $A_F(k,\theta)$ defined by

$$A_F(k,\theta) = \lim_{|x|\to\infty, x/|x|=\theta\in\mathbb{S}^2} |x|e^{-ik|x|}U_F(x,k)\,. \tag{1}$$

If one formally takes $F = \delta_0$, then A_F tells how the perturbed Green's function deviates from the unperturbed one. Suppose $\sigma_F(E)$ denotes the spectral measure of F relative to H. The importance of the amplitude in the study of the spectral type is explained by the following Lemma (see [3, formula (4.2)]).

Lemma 1. *Suppose V and F are compactly supported, then*

$$\sigma_F'(k^2) = \pi^{-1}k\|A_F(k,\theta)\|^2_{L^2(\mathbb{S}^2)}, \quad k \in \mathbb{R}^+\,. \tag{2}$$

This identity is remarkable for two reasons. First, by the Spectral Theorem, we always have equality

$$\int_{\mathbb{R}} \mathrm{d}\sigma_F(E) = \|F\|^2\,. \tag{3}$$

Second, the function $h(k) := \log\|A_F(k,\theta)\|_{L^2(\mathbb{S}^2)}$ is subharmonic in $k \in \{\Im k > 0, \Re k > 0\}$. Thus, if we can obtain

(A) *rough upper bounds for h*, e.g., $h(k) \le c(\Im k)^{-\alpha}$ for some $\alpha > 0$,
(B) *some lower bound away from the real line*, e.g., $h(k_0) > \delta, \Im k_0 = \epsilon$ with some positive ϵ and δ

then the mean-value inequality for subharmonic functions written for a suitable isosceles trapezoid $\mathscr{T} \subset \mathbb{C}^+$ with the base $I \subset \mathbb{R}^+$ gives

$$\int_{\partial\mathscr{T}} h(\xi)\mathrm{d}\omega_{k_0} \ge h(k_0)\,. \tag{4}$$

The symbol ω_{k_0} denotes the harmonic measure with the reference point $k_0 \in \mathscr{T}$. In particular, (A) and (B) provide the lower estimate for h on interval I. Combined with (3) and Chebyshev's inequality, this gives the following bound

$$\left|\left\{k \in I : \|A_F(k,\theta)\|^2_{L^2(\mathbb{S}^2)} \le e^{-\Lambda}\right\}\right| + \left|\left\{k \in I : \|A_F(k,\theta)\|^2_{L^2(\mathbb{S}^2)} \ge e^{\Lambda}\right\}\right| < C(I,c,\alpha,\epsilon,\delta)\Lambda^{-1} \tag{5}$$

for every $\Lambda > 1$. Thanks to (2), we also have control on the spectral measure σ_F:

$$\left|\left\{k \in I : \sigma_F'(k^2) \le e^{-\Lambda}\right\}\right| + \left|\left\{k \in I : \sigma_F'(k^2) \ge e^{\Lambda}\right\}\right| < C(I,c,\alpha,\epsilon,\delta)\Lambda^{-1}$$

These bounds indicate that "scattering happens" for most positive energies $E = k^2$. At the same time, they are also consistent with the possibility that some energies carry the singular part of σ_F.

In the next section, we use that strategy to study potentials V supported on large annuli. In [2], this technique was applied to show the presence of a.c. spectral type for 1d Schrödinger with operator-valued rough potential. We expect this approach to be useful for other problems, e.g., spectral theory of difference operators on graphs.

Notation.

- Given an interval $I \subset \mathbb{R}^+$ and positive α, the symbol $R_{I,\alpha}$ denotes the rectangle $I \times (0,\alpha)$ in complex plane; c_I is the center of I.
- For the function $g \in L^2(\mathbb{S}^2)$, the symbol $\|g\|$ stands for its $L^2(\mathbb{S}^2)$ norm. $B_r(x)$ is the ball with radius r centered at x.
- Symbol B is Laplace-Beltrami operator on the unit sphere. It is nonpositive in $L^2(\mathbb{S}^2)$. Given a set $E \subset \mathbb{R}^3$, the symbol χ_E denotes the characteristic function of E.
- We define the angular component of the gradient (in spherical coordinates) as

$$\nabla_\tau V = \nabla V - \frac{x}{|x|}\left\langle \frac{x}{|x|}, \nabla V\right\rangle.$$

2. An example: V supported on the large annulus

We are looking for conditions on V that provide estimates (A) and (B) discussed in Introduction. Suppose $\operatorname{supp} V \subset \{x : T/2 < |x| < T\}$, $F = \chi_{B_1(0)}$, A_F is defined as in (1), and T is a large parameter.

Theorem 2. *Consider $H = -\Delta + V$, $x \in \mathbb{R}^3$ where non-negative real-valued potential V satisfies*

$$\operatorname{supp} V \subset \{x : T/2 < |x| < T\}, \quad |V| \le CT^{-\gamma}, \quad |\nabla_\tau V| \le CT^{-\gamma-\delta}, \quad \delta, \gamma > 0. \tag{6}$$

If $\gamma \in \left(\frac{2}{3}, 1\right]$ and $\gamma + \frac{\delta}{2} > 1$, then

$$\sup_{T>1} \int_I |\log \|A_F(k,\theta)\|| \mathrm{d}k < \infty$$

for every segment $I \subset \mathbb{R}^+, |I| \sim 1$.

Operator H with potential V that satisfies stronger assumptions on angular gradient has been studied in [10].

Remark 3. As a consequence, we get

$$\left|\left\{k \in I : \|A_F(k,\theta)\|^2_{L^2(\mathbb{S}^2)} \le e^{-\Lambda}\right\}\right| + \left|\left\{k \in I : \|A_F(k,\theta)\|^2_{L^2(\mathbb{S}^2)} \ge e^{\Lambda}\right\}\right| < C(\gamma,\delta,I)\Lambda^{-1} \tag{7}$$

for every $\Lambda > 1$ and $T > 1$.

To get bounds (A) and (B), we will use Schrödinger equation itself. Recall that the operator H can be written as

$$H = -\partial^2_{rr} - \frac{B}{r^2} + V(r,\theta)$$

in the spherical coordinates $(r,\theta) \in \mathbb{R}^+ \times \mathbb{S}^2$ and the corresponding map $F(x) \to f(r,\theta) = rF(r,\theta)$ satisfies $\|F\|_{L^2(\mathbb{R}^3)} = \|f\|_{L^2(\mathbb{R}^+)}$ where f is considered as $L^2(\mathbb{S}^2)$-valued function in r. If f represents F and u represents $U_F = R_{k^2}F$, we have an equation for u:

$$-u'' - \frac{Bu}{r^2} + Vu = k^2 u + f, \quad r > 0.$$

Similarly, if a represents $e^{-ik|x|}U_F$, then $a = ue^{-ikr}$ and

$$2ika' = -a'' - \frac{Ba}{r^2} + Va - fe^{-ikr} \tag{8}$$

or

$$a' = (2ik)^{-1}\left(-a'' - \frac{Ba}{r^2} + Va - fe^{-ikr}\right). \tag{9}$$

Recall that $\operatorname{supp} f \subset [0,1]$. Then, the last equation gives the following identity if one takes the inner product and integrates the real part of resulting equation.

Lemma 4. *Let $1 < r_1 < r_2$ and $k : \Re k > 0, \Im k > 0$, then*

$$\|a(r_2,k)\|^2 + \frac{\Im k}{|k|^2}\int_{r_1}^{r_2}\left(\|a'\|^2 - \frac{\langle Ba,a\rangle}{\rho^2}\right)\mathrm{d}\rho + \frac{\Im k}{|k|^2}\int_{r_1}^{r_2}\langle Va,a\rangle \mathrm{d}\rho = \|a(r_1,k)\|^2 + Q(r_2) - Q(r_1) \tag{10}$$

where

$$Q(r) := \frac{i}{2k}\langle a'(r,k), a(r,k)\rangle - \frac{i}{2\bar{k}}\langle a(r,k), a'(r,k)\rangle.$$

If V satisfies

$$V \ge 0, \quad \operatorname{supp} V \in [T/2, T], \quad \|V\|_{L^\infty(\mathbb{S}^2)} < \infty$$

all terms in the left-hand side of (10) are non-negative and we get the following apriori estimate after setting $r_2 = \infty$ and applying Cauchy-Schwartz to $Q(r_1)$:

$$\|a(\infty,k)\|^2 < C_I\left(\|a(r_1,k)\|^2 + \|a'(r_1,k)\|^2\right) \tag{11}$$

provided that $k \in R_{I,1}$ and $I \subset \mathbb{R}^+$. The Spectral Theorem implies that $\|U\|_{L^2(\mathbb{R}^3)} \le C_I(\Im k)^{-1}$. Thus, from equation $HU = k^2U + F$, we get $\|\Delta U\|_{L^2(\mathbb{R}^3)} < C_I(\Im k)^{-1}$. So, the theorem about traces of functions in Sobolev class implies $\|a(2,k)\|^2+\|a'(2,k)\|^2<C_I(\Im k)^{-1}$ and (11) yields $\|a(\infty,k)\|^2<C_I(\Im k)^{-1}$ if we set $r_1=2$. Thus,

$$h=\log\|a(\infty,k)\|<C_1+C_2|\log\Im k|,\quad k\in R_{I,1} \tag{12}$$

with some positive T-independent C_1 and C_2. This is a rough bound (A) from the Introduction that we are looking for and it does not require any smallness assumption on V at all. Our next immediate goal is to obtain better control on h away from the real line, i.e., an estimate (B). The idea is to use (10) with $r_2=\infty$, $r_1=T/4$, and $k\in R_{I,1}$ to write

$$\begin{aligned}&|\|a(\infty,k)\|^2-\|a(T/4,k)\|^2|\\ &\qquad\le C(I)\left(|Q(T/4)|+(\Im k)T^{-\gamma}\int_{T/2}^{T}\|a\|^2\mathrm{d}\rho+(\Im k)\int_{T/4}^{\infty}\left(\|a'\|^2-\frac{\langle Ba,a\rangle}{\rho^2}\right)\mathrm{d}\rho\right)\end{aligned} \tag{13}$$

Here, we choose $r_1=T/4$ to guarantee enough separation from the support of V. Our goal is to show that the right-hand side is small if k is separated from the real axis. If

$$\|V\|_{L^\infty(\mathbb{S}^2)}\le CT^{-\gamma},\quad k\in R_{I,1},\quad \Im k>cT^{-\gamma} \tag{14}$$

with suitable T-independent constant c, the standard perturbation theory based on Combes-Thomas estimates (see, e.g., [6]) yields

$$\|a(T/4,k)\|^2\sim 1,\quad \|a'(T/4,k)\|^2\le C_Ie^{-T^\kappa},\quad \langle Ba(T/4,k),a(T/4,k)\rangle/T^2\le C_Ie^{-T^\kappa},\quad \kappa>0. \tag{15}$$

Hence, $|Q(T/4)|<C_Ie^{-T^\kappa}$. In what follows, we will assume that (14) holds. We need to control the other three terms in the right-hand side of (13). Taking the inner product with a' in (8) and integrating the real part from r to ∞ gives the following lemma.

Lemma 5. *Let $\Re k>0,\Im k>0$, and $r>1$, then*

$$\begin{aligned}(4\Im k)\int_r^\infty\langle a',a'\rangle\mathrm{d}\rho+\|a'(r,k)\|^2-2\int_r^\infty\frac{\langle Ba(\rho,k),a(\rho,k)\rangle}{\rho^3}\mathrm{d}\rho\\ =-\frac{\langle Ba(r,k),a(r,k)\rangle}{r^2}-2\Re\left(\int_r^\infty\langle Va,a'\rangle\mathrm{d}\rho\right).\end{aligned}$$

Since the terms in the left-hand side are all non-negative, we obtain another apriori estimate for $k\in R_{I,1},\Im k>CT^{-\gamma}$:

$$(\Im k)\int_{T/4}^\infty\|a'(\rho,k)\|^2\mathrm{d}\rho\le C_I\left(e^{-T^\kappa}+T^{-\gamma}\left(\int_{T/4}^T\|a\|^2\mathrm{d}r\right)^{1/2}\left(\int_{T/4}^\infty\|a'\|^2\mathrm{d}r\right)^{1/2}\right). \tag{16}$$

Taking $r_1=T/4$ and integrating (10) in r_2 from $T/4$ to T, we apply Cauchy-Schwartz and (15) to get

$$\int_{T/4}^T\|a\|^2\mathrm{d}r\le C_I\left(T+\left(\int_{T/4}^T\|a\|^2\mathrm{d}r\right)^{1/2}\left(\int_{T/4}^T\|a'\|^2\mathrm{d}r\right)^{1/2}\right)$$

for $k\in R_{I,1},\Im k>CT^{-\gamma}$. This implies

$$\int_{T/4}^T\|a\|^2\mathrm{d}r\le C_I\left(T+\int_{T/4}^T\|a'\|^2\mathrm{d}r\right)\le C_I\left(T+\int_{T/4}^\infty\|a'\|^2\mathrm{d}r\right) \tag{17}$$

which, being substituted into (16), yields

$$\int_{T/4}^\infty\|a'\|^2\mathrm{d}r<C_I((\Im k)^{-1}e^{-T^\kappa}+(\Im k)^{-2}T^{1-2\gamma})$$

and, from (17),

$$T^{-1}\int_{T/4}^T\|a\|^2\mathrm{d}r\le C_I. \tag{18}$$

These two last bounds control two additional terms in the right-hand side of (13) in the regime when $k \in R_{I,1}$ and $\Im k > CT^{-\gamma}$. Thus, we are only left with estimating

$$\int_{T/4}^{\infty} -\frac{\langle Ba, a\rangle}{\rho^2} \mathrm{d}\rho\,.$$

in (13). To this end, we use an extra smoothness of V, i.e., the last condition in (6). Taking the gradient of (9) in the angular variables, we rewrite (6) as

$$(\nabla_\theta a)' = (2ik)^{-1}\left(-(\nabla_\theta a)'' - \frac{B\nabla_\theta a}{r^2} + V\nabla_\theta a + a\nabla_\theta V\right). \tag{19}$$

This equation has the same structure as equation for a except for the term $a\nabla_\theta V$. The last condition on V in (6) can be written as $\|\nabla_\theta V\|_{L^\infty(\mathbb{S}^2)} \le CT^{1-\gamma-\delta}$. Repeating the previous arguments and using the bound (18), we obtain

$$\int_{T/4}^{T} -\frac{\langle Ba, a\rangle}{\rho^2} \mathrm{d}\rho \sim T^{-2}\int_{T/4}^{T} \|\nabla_\theta a\|^2 \mathrm{d}\rho \le C_I\left((\Im k)^{-2} T^{1-2\gamma-2\delta} + T^{3-2\gamma-2\delta}\right).$$

For $r > T$, one has $V = 0$ and the bound

$$\int_{T/4}^{\infty} -\frac{\langle Ba, a\rangle}{\rho^2} \mathrm{d}\rho = \int_{T/4}^{\infty} \frac{\|\nabla_\theta a\|^2}{\rho^2} \mathrm{d}\rho \le C_I\left((\Im k)^{-2} T^{1-2\gamma-2\delta} + T^{3-2\gamma-2\delta}\right)$$

follows by the similar reasoning. In (13), we apply the obtained estimates to control the right-hand side. This gives

$$\left|\|a(\infty,k)\|^2 - \|a(a,k)\|^2\right| \le C_I\left(T^{-1} + (\Im k)(T^{1-\gamma} + T^{3-2\gamma-2\delta}) + (\Im k)^{-1} T^{1-2\gamma}\right).$$

Given (6), we can always assume $T^{3-2\gamma-2\delta} > T^{1-\gamma}$ and find a small positive ϵ such that the set $P_{I,T}$ defined by $P_{I,T} = \left\{k \in \mathbb{C}^+ : \Re k \in I,\ T^{\epsilon}(T^{1-2\gamma}) < \Im k < T^{-\epsilon}(T^{2\gamma+2\delta-3})\right\}$ is nonempty. Hence, if $k \in P_{I,T}$, then

$$\|a(\infty,k)\|^2 = \|a(T/4,k)\|^2 + O(T^{-\epsilon_1}) \tag{20}$$

with some $\epsilon_1 > 0$ and thus $\|a(\infty,k)\|^2 \sim 1$ there. This establishes the required bound (B) and we are ready to prove Theorem 2.

Proof of Theorem 2. For every $I \subset \mathbb{R}^+$, consider R_{I,δ_T} with $\delta_T = T^{-\epsilon}(T^{2\gamma+2\delta-3})$ and write mean-value inequality for subharmonic function $h := \log\|a(\infty,k)\|$:

$$\int_{\partial R_{I,\delta_T}} h \mathrm{d}\omega_{k_0} \ge h(k_0)$$

where $k_0 = x + i\delta_T/2, x \in I' \subset I$ and the subinterval I' is chosen such that $c_{I'} = c_I, |I'| > c|I|$ with fixed T-independent c.

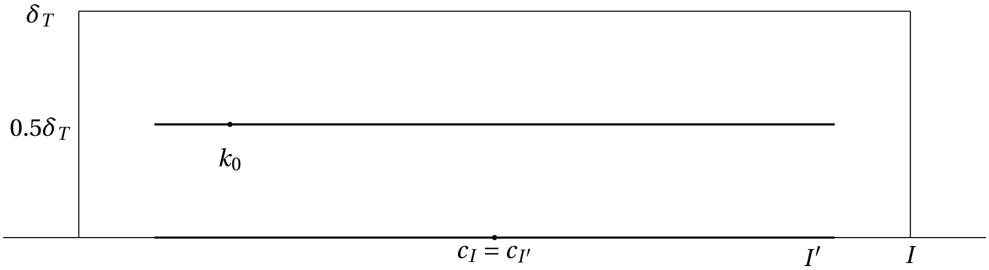

Figure 1. Rectangle R_{I,δ_T}

If $\Im k > \delta_T/2$ and $k \in \partial R_{I,\delta_T}$, we have $\|a(\infty,k)\| \sim 1$ by (20). On the other hand, (12) guarantees that $\int_\Gamma |h| \mathrm{d}\omega_{k_0} < C$ uniformly in T where Γ are the legs of R_{I,δ_T}, i.e., $\Gamma = \{k \in \partial R_{I,\delta_T}, 0 < \Im k < \delta_T\}$.

In the end, if we use the standard estimates on the harmonic measure of the rectangle and integrate in $x \in I'$, we obtain the bound $\int_{I'} h(k)\mathrm{d}k > C$ uniformly in T. The formula (3) yields $\int_I \|a(\infty,k)\|^2 \mathrm{d}k < C$ uniformly in T and the trivial estimate $\max\{0, \log t\} \le t,\ t > 0$ gives

$$\sup_{T>1} \int_{I'} |\log \|a(\infty,k)\|| \mathrm{d}k < \infty.$$

For an arbitrary I', we can find the corresponding $I \subset \mathbb{R}^+$ so the proof is finished. □

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