INSTITUT DE FRANCE
Académie des sciences

# Comptes Rendus
# Mathématique

José Carlos Sierra

**On morphisms from $\mathbb{P}^3$ to $\mathbb{G}(1,3)$**

Volume 359, issue 7 (2021), p. 853-860

<https://doi.org/10.5802/crmath.219>

*Comptes Rendus*
*Mathématique*
**2021**, 359, nº 7, p. 853-860
https://doi.org/10.5802/crmath.219

Algebraic geometry / *Géométrie algébrique*

# On morphisms from $\mathbb{P}^3$ to $\mathbb{G}(1,3)$

**José Carlos Sierra**[*, a]

[a] Departamento de Matemáticas Fundamentales, Facultad de Ciencias, UNED,
C/Juan del Rosal 10, 28040 Madrid, Spain.
*E-mail:* jcsierra@mat.uned.es

**Abstract.** Every morphism from $\mathbb{P}^n$ to $\mathbb{G}(k,m)$ is constant if $m<n$, and nonconstant morphisms from $\mathbb{P}^n$ to $\mathbb{G}(k,n)$ rarely appear when $0<k<n-1$. In this setting, Tango proved that a morphism from $\mathbb{P}^n$ to $\mathbb{G}(1,n)$ is constant if $n \notin \{3,5\}$. Here we focus on the case $n=3$ and show that if $\phi : \mathscr{O}_{\mathbb{P}^3}^{\oplus 4} \to E$ is the surjection onto a rank 2 vector bundle $E$ inducing a morphism $\varphi : \mathbb{P}^3 \to \mathbb{G}(1,3)$, then $h^1(E^*) \le 1$. Furthermore, a complete classification is given if equality holds.

**2020 Mathematics Subject Classification.** 14J60, 14M15, 14N05.

*Manuscript received 12th February 2021, accepted 29th April 2021.*

## 1. Introduction

Morphisms from $\mathbb{P}^n$ to $\mathbb{G}(k,m)$ were studied by Tango in the series of papers [12–14] over an algebraically closed field of arbitrary characteristic. He showed in [12, Corollary 3.2] that a morphism from $\mathbb{P}^n$ to $\mathbb{G}(k,m)$ is constant if $m<n$ (cf. [8, Theorem 3.1]). Then he considered the case $m=n$, proving that a morphism from $\mathbb{P}^n$ to $\mathbb{G}(k,n)$ is constant when (cf. Remark 4):

(a) $k \in \{1,2\}$ and $n \ge 6$ [12, Proposition 3.4 and Proposition 3.5];
(b) $0<k<n-1$ and $kn$ is even, $(k,n) \ne (2,5)$ [13, Theorem];
(c) $0<k<n-1$ and $kn$ is odd, $3 \le k \le 9$ [14].

In [13, Section 3] he also gave the example of a nonconstant morphism from $\mathbb{P}^3$ to $\mathbb{G}(1,3)$ that we reproduce in Example 9. Furthermore, in the case when the defining field is of characteristic 2, he gave an example of a nonconstant morphism from $\mathbb{P}^5$ to $\mathbb{G}(2,5)$. Using this, he gave a remarkable example of indecomposable vector bundle of rank 2 on $\mathbb{P}^5$ (cf. [6, Section 6]). However, no example of nonconstant morphism from $\mathbb{P}^5$ to $\mathbb{G}(1,5)$ is known.

In this paper we work in characteristic zero and focus on the case $k=1$, and hence $n \in \{3,5\}$ according to (a) and (b). Our main result, concerning the case $n=3$, is the bound $h^1(E^*) \le 1$ and the characterization of the rank 2 vector bundles $E$ with $h^1(E^*)=1$ giving a morphism from $\mathbb{P}^3$ to $\mathbb{G}(1,3)$. More precisely:

[*] Corresponding author.

**Theorem 1.** *Let $\varphi : \mathbb{P}^3 \to \mathbb{G}(1,3)$ be the morphism induced by a surjection $\phi : \mathscr{O}_{\mathbb{P}^3}^{\oplus 4} \to E$ onto a rank 2 vector bundle. Then $h^1(E^*) \le 1$, and equality holds if and only if there exists a nontrivial extension $0 \to \mathscr{O}_{\mathbb{P}^3} \to E' \to E \to 0$ in which $E'$ is the kernel of a surjection $\mathscr{O}_{\mathbb{P}^3}^{\oplus 4}(e) \to \mathscr{O}_{\mathbb{P}^3}(2e)$ for some integer $e \ge 1$.*

By contrast, the case $h^1(E^*) = 0$ remains very much open. Actually, in this situation the only known example of surjection $\mathscr{O}_{\mathbb{P}^3}^{\oplus 4} \to E$ was given in a completely different context [2] (see also [1, Remark 2.11]). It would be nice to complete the picture in this case, or at least to find some other examples (see Question 17 and the discussion at the end of Section 3).

On the other hand, concerning the case $n = 5$ and closely related to the question whether there are vector bundles of small rank on projective spaces [6, Section 6], we pose the following (see Remark 20):

**Conjecture 2.** *Every morphism from $\mathbb{P}^5$ to $\mathbb{G}(1,5)$ is constant.*

The paper is organized as follows. In Section 2 we recall the correspondence between morphisms to Grassmannians and vector bundles generated by their global sections (see for example [4, Theorem 3.4]). Following [7], a brief account on the correspondence between rank 2 vector bundles and subcanonical curves, due to the Serre construction [10], is also provided. In Section 3 we first analyze the Tango example in Remark 10 and Remark 11. This suggests the generalization given in Proposition 12, which is the first step towards the main result. We then proceed with the proof of Theorem 1. A key point in the argument is Proposition 13, which is based on an elementary property that can be traced back at least to [11, Lemma, p. 44]. A few comments on the case $h^1(E^*) = 0$ follow in 15, 16, and 17. Finally, Section 4 is devoted to Conjecture 2.

## 2. Preliminaries

### 2.1. *Morphisms to Grassmannians and globally generated vector bundles*

Let $V$ a vector space of dimension $m+1$ over an algebraically closed field of characteristic zero, and let

$$G := G(k+1, m+1) = \mathbb{G}(k, m)$$

denote the Grassmannian of $k$-planes in $\mathbb{P}(V) = \mathbb{P}^m$, whose points correspond to one-dimensional quotients of $V$ according to the Grothendieck convention (note that this is opposite to the usage in [4], which is the main reference for this subsection).

Let $Q$ denote the universal quotient bundle of rank $k+1$ on $G$, and let $S$ denote the universal subbundle of rank $m-k$ on $G$ giving the exact sequence

$$0 \to S \to V \otimes \mathscr{O}_G \to Q \to 0. \tag{1}$$

Thus if $X$ is a scheme and $\varphi : X \to \mathbb{G}(k,m)$ is a morphism (i.e., a regular map), then the pull-back of (1) shows that $\varphi^*(Q)$ is a vector bundle of rank $k+1$ on $X$ generated by $m+1$ global sections.

Conversely, if $X$ is a scheme, $E$ is a vector bundle of rank $k+1$ generated by its global sections, and the evaluation morphism $\phi = \phi_V : V \otimes \mathscr{O}_X \to E$ is surjective for some vector subspace $V \subseteq H^0(E)$ of dimension $m+1$, then we get a morphism $\varphi = \varphi_V : X \to \mathbb{G}(k,m)$ such that $\varphi^*(Q) = E$. Moreover, if $p : \mathbb{G}(k,m) \to \mathbb{P}(\Lambda^{k+1} V)$ denotes the Plücker embedding of $G$ then the composition

$$f = p \circ \varphi : X \to \mathbb{G}(k,m) \to \mathbb{P}(\Lambda^{k+1} V) \tag{2}$$

is the morphism induced by the surjection $\phi_W : W \otimes \mathscr{O}_X \to \Lambda^{k+1} E$ onto the determinant line bundle, corresponding to the vector subspace

$$W := \Lambda^{k+1} V \subseteq \Lambda^{k+1} H^0(E) \subseteq H^0(\Lambda^{k+1} E). \tag{3}$$

**Remark 3.** How many global sections are needed to generate $E$? If $\dim(X) = n$ then $\dim\mathbb{P}(E) = n+k$, so $\min\{n+k+1, h^0(E)\}$ will be enough. Let us see how to improve this when $X$ is a smooth projective variety. Let $E$ be a globally generated vector bundle of rank $k+1$ on $X$. Then there exists a vector subspace $V \subseteq H^0(E)$ of dimension $n+k+1-\delta$ and a surjection $\phi_V : V \otimes \mathcal{O}_X \to E$ if and only if $s_{n+1-\delta}(E) = c_{n+1-\delta}(F) = 0$, where $F$ denotes the vector bundle on $X$ obtained as the kernel of the evaluation morphism

$$0 \to F \to H^0(E) \otimes \mathcal{O}_X \to E \to 0, \tag{4}$$

and $s_{n+1-\delta}$ and $c_{n+1-\delta}$ denote the Segre and Chern classes, respectively (see for example [4, Propositions 10.2 and 10.3]). This shows the connection between morphisms from $X$ to Grassmannians and vector bundles of rank $< n$ on $X$, i.e., if $\delta > 0$ then the kernel of the surjection $\phi_V : V \otimes \mathcal{O}_X \to E$ is a vector bundle of rank $n-\delta$ on $X$. In particular $\delta \le n$, with equality holding if and only if $E = \mathcal{O}_X^{\oplus k+1}$ and $V = H^0(E)$. This happens if and only if the morphism $\varphi$ induced by $\phi_V : V \otimes \mathcal{O}_X \to E$ is constant.

**Remark 4.** Let us explain the Tango assumption $0 < k < n-1$. If $k = 0$, that is $E = \mathcal{O}_{\mathbb{P}^n}(a)$, then every surjection $\phi : \mathcal{O}_{\mathbb{P}^n}^{\oplus n+1} \to E$ is given by a vector subspace $V \subseteq H^0(E)$ generated by $n+1$ homogeneous polynomials of degree $a$ without common zeros. And if $k = n-1$ then every surjection $\phi : \mathcal{O}_{\mathbb{P}^n}^{\oplus n+1} \to E$ is given by the cokernel of an injection $\mathcal{O}_{\mathbb{P}^n}(-a) \to \mathcal{O}_{\mathbb{P}^n}^{\oplus n+1}$, where $a = c_1(E)$. Therefore, these two dual cases are not interesting.

### 2.2. *The correspondence between vector bundles and curves*

Here we follow [7, Theorem 1.1]. If $E$ is a globally generated rank 2 vector bundle on $\mathbb{P}^3$ then the scheme of zeros of a general section $s \in H^0(E)$ is either empty, or a smooth —but not necessarily connected— curve $Y \subset \mathbb{P}^3$ with canonical sheaf $\omega_Y \cong \mathcal{O}_{\mathbb{P}^3}(c_1(E)-4)|_Y$ [7, Proposition 1.4]. Furthermore, the Koszul complex gives the exact sequence of sheaves

$$0 \to \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\cdot s} E \to \mathcal{I}_Y(c_1(E)) \to 0. \tag{5}$$

Conversely, if $Y \subset \mathbb{P}^3$ is a smooth, not necessarily connected, curve with $\omega_Y \cong \mathcal{O}_{\mathbb{P}^3}(a)|_Y$ then the Serre construction [10] gives a (unique up to isomorphism) rank 2 vector bundle $E$, a (unique up to a scalar multiple) section $s \in H^0(E)$ whose scheme of zeros is $Y \subset \mathbb{P}^3$, and the exact sequence of sheaves (5). Of course, in this case $E$ need not be globally generated.

**Remark 5.** If $c_1 := c_1(E)$ and $c_2 := c_2(E)$ denote the Chern classes of $E$ then the numerical invariants of $Y \subset \mathbb{P}^3$ can be expressed in terms of $c_1$ and $c_2$ [7, Proposition 2.1]. More precisely, the degree and arithmetic genus of $Y$ are:

(i) $\deg(Y) = c_2$.
(ii) $2p_a(Y) - 2 = c_2(c_1 - 4)$.

Moreover, since $E^* \cong E(-c_1)$, the number of connected components of $Y$ is:

(iii) $h^0(\mathcal{O}_Y) = 1 + h^1(\mathcal{I}_Y) = 1 + h^1(E^*)$.

**Remark 6.** Going back to our context, if the morphism $\varphi : \mathbb{P}^3 \to \mathbb{G}(1,m)$ is induced by a surjection $\phi_V : V \otimes \mathcal{O}_{\mathbb{P}^3} \to E$ and $s \in V$, then $\varphi(Y)$ is the set of lines contained in the hyperplane of $\mathbb{P}^m$ corresponding to $[s] \in \mathbb{P}(V^*)$.

## 3. Proof of the main result

**Lemma 7.** *Let $E$ be a globally generated rank 2 vector bundle on $\mathbb{P}^3$. Then there exists a surjection $\phi : \mathcal{O}_{\mathbb{P}^3}^{\oplus 4} \to E$ if and only if $c_1 = 2e$ and $c_2 = 2e^2$.*

**Proof.** Consider the exact sequence (4). Let $c_t(E) = 1 + c_1 t + c_2 t^2$ and $c_t(F)$ denote the Chern polynomials of $E$ and $F$. Since $c_t(E) \cdot c_t(F) = 1$ we deduce that

$$c_t(F) = 1 - c_1 t + \left({c_1}^2 - c_2\right) t^2 - \left({c_1}^3 - 2c_1 c_2\right) t^3.$$

By Remark 3, there exists a surjection $\phi : \mathscr{O}_{\mathbb{P}^3}^{\oplus 4} \to E$ if and only if $s_3(E) = c_3(F) = 0$, that is, if and only if $c_1^2 = 2c_2$. But this happens if and only if $c_1$ is even, say $c_1 = 2e$, and $c_2 = 2e^2$. $\square$

Combining the preliminary material with Lemma 7 and Remark 5, we get:

**Corollary 8.** *Let $\phi : \mathscr{O}_{\mathbb{P}^3}^{\oplus 4} \to E$ be a surjection onto a nontrivial rank 2 vector bundle. Then the scheme of zeros of a general section $s \in H^0(E)$ is a smooth curve $Y \subset \mathbb{P}^3$ with $\omega_Y \cong \mathscr{O}_{\mathbb{P}^3}(2e-4)|_Y$, and its ideal sheaf is given by the exact sequence*

$$0 \to \mathscr{O}_{\mathbb{P}^3} \xrightarrow{\cdot s} E \to \mathscr{I}_Y(2e) \to 0.$$

*In particular, $Y$ has degree $2e^2$, arithmetic genus $2e^2(e-2)+1$ and $h^1(E^*)+1$ connected components.*

The starting point of this paper was [13, Example 1], that is worth quoting in full:

**Example 9.** Let $S_4$ be the quadric hypersurface of $\mathbb{P}^5$ defined by the homogeneous equation

$$X_0 X_1 + X_2 X_3 + X_4 X_5 = 0.$$

Then, it is well-known that $Gr(3,1)$ is isomorphic to $S_4$. Let $f$ be a morphism from $\mathbb{P}^3$ to $S_4$ defined by

$$f(x_0, x_1, x_2, x_3) = \left(x_0^2, -x_1^2, x_2^2, x_3^2, x_0 x_1 + x_2 x_3, x_0 x_1 - x_2 x_3\right).$$

It is easy to see that $f$ is not a constant morphism.

**Remark 10.** Let us analyze the Tango example in detail: what is the rank 2 vector bundle $E$ on $\mathbb{P}^3$ and the surjection $\phi_V : V \otimes \mathscr{O}_{\mathbb{P}^3} \to E$ defining $\varphi : \mathbb{P}^3 \to \mathbb{G}(1,3)$? In this case, $f$ is given by a linear system of quadrics. Thus $c_1(E) = c_1(\Lambda^2 E) = 2$ by (2), and Corollary 8 yields $\deg(Y) = 2$ and $p_a(Y) = -1$. We conclude that $Y = Y_1 \cup Y_2$ is the disjoint union of two lines in $\mathbb{P}^3$, whence $h^1(E^*) = 1$. Therefore $E \cong N(1)$, where $N(1)$ denotes the twisted null-correlation bundle on $\mathbb{P}^3$ defined by the exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^3} \to \Omega_{\mathbb{P}^3}(2) \to N(1) \to 0.$$

In particular, $H^0(E) = 5$ and $V \subset H^0(E)$ is a general hyperplane.

**Remark 11.** Let us now show a more geometric interpretation of Example 9 and Remark 10. The surjection $\bar{\phi} = \phi_{H^0(E)} : H^0(E) \otimes \mathscr{O}_{\mathbb{P}^3} \to E$ induces a morphism $\bar{\varphi} = \varphi_{H^0(E)} : \mathbb{P}^3 \to \mathbb{G}(1,4)$, that is an isomorphism onto its image since equality holds in (3). More precisely, $\bar{\varphi}(\mathbb{P}^3)$ is the set of lines of a smooth quadric hypersurface $Q^3$ in $\mathbb{P}^4$ (cf. [12, Section 6]; in particular, $E \cong N(1)$ is the missing vector bundle denoted by $(*)$ in [12, p. 417]). Therefore, we get a commutative diagram

$$\begin{array}{ccc} & & \mathbb{G}(1,4) \\ & \overset{\bar{\varphi}}{\nearrow} & \Big\downarrow \pi \\ \mathbb{P}^3 & \xrightarrow{\varphi} & \mathbb{G}(1,3) \end{array} \tag{6}$$

where $\pi : \mathbb{G}(1,4) \dashrightarrow \mathbb{G}(1,3)$ is induced by the linear projection from $\mathbb{P}(H^0(E)) = \mathbb{P}^4$ onto $\mathbb{P}(V) = \mathbb{P}^3$ given by a point $p \notin Q^3$, hence defined everywhere in $\bar{\varphi}(\mathbb{P}^3)$.

According to the previous discussion, the Tango example can be generalized as follows:

**Proposition 12.** *Let $Y = Y_1 \cup Y_2$ be the disjoint union of two smooth curves in $\mathbb{P}^3$, each of them a complete intersection of two surfaces of degree $e$. Let $E$ be the rank 2 vector bundle corresponding to $Y$. Then there exist a rank 3 vector bundle $E'$ and a nontrivial extension $0 \to \mathscr{O}_{\mathbb{P}^3} \to E' \to E \to 0$ such that:*

(i) *$E'$ is the kernel of a surjection $\mathscr{O}_{\mathbb{P}^3}^{\oplus 4}(e) \to \mathscr{O}_{\mathbb{P}^3}(2e)$. In particular, $E'$ and $E$ are globally generated, $H^0(E') = 6$, and $H^0(E) = 5$.*

(ii) *The surjection $\phi_V : V \otimes \mathscr{O}_X \to E$ given by a general hyperplane $V \subset H^0(E)$ induces a morphism $\varphi : \mathbb{P}^3 \to \mathbb{G}(1,3)$.*

**Proof.** Since each $Y_i \subset \mathbb{P}^3$ is a smooth complete intersection of degree $e^2$ and genus $e^2(e-2)+1$ we get $\omega_Y \cong \mathscr{O}_{\mathbb{P}^3}(2e-4)|_Y$, and the Serre construction gives a rank 2 vector bundle $E$ with Chern classes $c_1 = 2e$ and $c_2 = 2e^2$ such that $h^1(E^*) = 1$. To make the construction explicit, consider the two resolutions

$$0 \to \mathscr{O}_{\mathbb{P}^3} \to \mathscr{O}_{\mathbb{P}^3}^{\oplus 2}(e) \to \mathscr{I}_{Y_i}(2e) \to 0$$

that, together with the exact sequence of sheaves

$$0 \to \mathscr{I}_Y(2e) \to \mathscr{I}_{Y_1}(2e) \oplus \mathscr{I}_{Y_2}(2e) \to \mathscr{O}_{\mathbb{P}^3}(2e) \to 0,$$

give a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathscr{O}_{\mathbb{P}^3}^{\oplus 2} & = & \mathscr{O}_{\mathbb{P}^3}^{\oplus 2} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & E' & \longrightarrow & \mathscr{O}_{\mathbb{P}^3}^{\oplus 4}(e) & \longrightarrow & \mathscr{O}_{\mathbb{P}^3}(2e) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathscr{I}_Y(2e) & \longrightarrow & \mathscr{I}_{Y_1}(2e) \oplus \mathscr{I}_{Y_2}(2e) & \longrightarrow & \mathscr{O}_{\mathbb{P}^3}(2e) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $E'$ denotes the kernel of the surjection $\mathscr{O}_{\mathbb{P}^3}^{\oplus 4}(e) \to \mathscr{O}_{\mathbb{P}^3}(2e)$. In particular, $c_3(E') = 0$ and $h^1(E'^*) = 0$. The surjection determines the exact Koszul complex

$$0 \to \mathscr{O}_{\mathbb{P}^3}(-2e) \to \mathscr{O}_{\mathbb{P}^3}^{\oplus 4}(-e) \to \mathscr{O}_{\mathbb{P}^3}^{\oplus 6} \to \mathscr{O}_{\mathbb{P}^3}^{\oplus 4}(e) \to \mathscr{O}_{\mathbb{P}^3}(2e) \to 0,$$

whence an exact sequence

$$0 \to F' \to \mathscr{O}_{\mathbb{P}^3}^{\oplus 6} \to E' \to 0,$$

where $F'$ denotes the cokernel of the injection $\mathscr{O}_{\mathbb{P}^3}(-2e) \to \mathscr{O}_{\mathbb{P}^3}^{\oplus 4}(-e)$. Since $c_3(E') = 0$ and $h^1(E'^*) = 0$, a general global section of $E'$ yields a nontrivial extension $0 \to \mathscr{O}_{\mathbb{P}^3} \to E' \to E \to 0$, proving (i). Lemma 7 gives (ii). □

A key point in the proof of Theorem 1 is the following result, that actually works for many other three-dimensional varieties $X$ (for example, those with Picard number $\rho(X) = 1$). From now on $\bar{\varphi}$ denotes the morphism induced by the surjection $\bar{\phi} : H^0(E) \otimes \mathscr{O}_{\mathbb{P}^3} \to E$, as in Remark 11.

**Proposition 13.** *Let $\phi : \mathscr{O}_{\mathbb{P}^3}^{\oplus 4} \to E$ be a surjection onto a rank 2 vector bundle. Then $h^0(E) \le 5$, and equality holds if and only if $\varphi : \mathbb{P}^3 \to \mathbb{G}(1,3)$ factors as in (6) and $\bar{\varphi}(\mathbb{P}^3)$ is the set of lines of a smooth quadric hypersurface $Q^3$ in $\mathbb{P}^4$.*

**Proof.** Let $P(E)$ denote the subvariety of $\mathbb{P}(H^0(E))$ swept out by the lines corresponding to points of $\bar{\varphi}(\mathbb{P}^3)$, that is, $P(E)$ is the image of the map

$$\varphi_\xi : \mathbb{P}(E) \to \mathbb{P}\left(H^0(E)\right)$$

given by $H^0(\xi) \cong H^0(E)$, where $\xi := \mathscr{O}_{\mathbb{P}(E)}(1)$ denotes the tautological line bundle of $\mathbb{P}(E)$. Since there exists a surjection $\phi : \mathscr{O}_{\mathbb{P}^3}^{\oplus 4} \to E$, Remark 3 gives $s_3(E) = 0$ or, equivalently, $\dim P(E) \le 3$. On the other hand, we may assume that $\dim \bar{\varphi}(\mathbb{P}^3) = 3$ (if not, $\bar{\varphi}$ is constant and $E = \mathscr{O}_{\mathbb{P}^3}^{\oplus 2}$), so $P(E)$ is swept out by a three-dimensional family of lines. Thus $P(E)$ is either $\mathbb{P}^3$, or a smooth quadric hypersurface $Q^3$ in $\mathbb{P}^4$, or it is swept out by a one-dimensional family $B$ of planes (see for example [11, Lemma, p. 44]). But the latter cannot occur, since otherwise $\mathbb{P}^3$ would dominate $B$. Then $h^0(E) \le 5$, with equality if and only if $\varphi : \mathbb{P}^3 \to \mathbb{G}(1,3)$ factors as in (6) and $\bar{\varphi}(\mathbb{P}^3)$ is the set of lines of $Q^3$ in $\mathbb{P}^4$. □

**Corollary 14.** *Let $\phi : \mathscr{O}_{\mathbb{P}^3}^{\oplus 4} \to E$ be a surjection onto a rank 2 vector bundle. Then $h^1(E^*) \le 1$, and $h^1(E^*) = 1$ if and only if $h^0(E) = 5$.*

**Proof.** Let $F$ denote the kernel of $\phi$, and consider the exact sequence

$$0 \to F \to \mathscr{O}_{\mathbb{P}^3}^{\oplus 4} \to E \to 0. \tag{7}$$

We may assume $h^0(E^*) = 0$; otherwise the Splitting Lemma gives $E \cong \mathscr{O}_{\mathbb{P}^3} \oplus \mathscr{O}_{\mathbb{P}^3}(c_1)$ and hence $E = \mathscr{O}_{\mathbb{P}^3}^{\oplus 2}$ by Lemma 7. Similarly, we can assume $h^0(F) = 0$. Thus Proposition 13 applied to (7) and its dual sequence yields

$$h^0(E) = 4 + h^1(F) \le 5, \quad h^0\left(F^*\right) = 4 + h^1\left(E^*\right) \le 5. \tag{8}$$

This gives $h^1(E^*) \le 1$. Suppose now $h^0(E) = 5$. The scheme of zeros of a general section $s \in H^0(E)$ is a smooth curve $Y$, and $\bar{\varphi}(Y)$ is not connected by Proposition 13 and Remark 6, since the set of lines of $Q^3$ contained in a general hyperplane of $\mathbb{P}^4$ consists of the two rulings of a smooth quadric surface. Therefore $Y$ is not connected, so $h^1(E^*) > 0$, and hence $h^1(E^*) = 1$ by (8). Conversely, if $h^1(E^*) = 1$ then $h^0(F^*) = 5$ and the same argument yields $h^1(F) = 1$, whence $h^0(E) = 5$. □

We can now prove the main result of the paper:

**Proof of Theorem 1.** The inequality $h^1(E^*) \le 1$ has been proved in Corollary 14. If $h^1(E^*) = 1$ then the scheme of zeros of a general section $s \in H^0(E)$ is a smooth curve $Y = Y_1 \cup Y_2$ with two connected components by Corollary 8. Moreover $h^0(E) = 5$ and $\bar{\varphi}(Y)$ consists of the two rulings of a smooth quadric surface, as shown in Corollary 14, so we have $\deg(Y_1) = \deg(Y_2)$. Then $\deg(Y_i) = e^2$ and $g(Y_i) = e^2(e-2)+1$ by Corollary 8. If $h^0(\mathscr{I}_{Y_i}(e)) = 0$ for some $i \in \{1,2\}$, then equality holds for $s = e$ in the genus bound [5] and we get a contradiction. Thus

$$h^0\left(\mathscr{I}_{Y_1}(e)\right) \cdot h^0\left(\mathscr{I}_{Y_2}(e)\right) \ne 0. \tag{9}$$

On the other hand, since $\mathscr{I}_Y(2e)$ is globally generated and $h^0(\mathscr{I}_Y(2e)) = h^0(E) - 1 = 4$, we conclude that $h^0(\mathscr{I}_Y(2e-1)) = 0$. In particular, we deduce from (9) that

$$h^0\left(\mathscr{I}_{Y_1}(e-1)\right) = h^0\left(\mathscr{I}_{Y_2}(e-1)\right) = 0. \tag{10}$$

It follows from (10) and [5] that each $Y_i$ is a complete intersection of two surfaces of degree $e$, and Proposition 12 applies. The converse is immediate. □

**Remark 15.** While Theorem 1 gives the bound $h^1(E^*) \le 1$ and a complete classification when $h^1(E^*) = 1$, the case $h^1(E^*) = 0$ remains open. To the best of the author's knowledge, the only known example of surjection $\phi : \mathscr{O}_{\mathbb{P}^3}^{\oplus 4} \to E$ onto a rank 2 vector bundle with $h^1(E^*) = 0$ corresponds to an elliptic curve $Y \subset \mathbb{P}^3$ of degree 8 with no 5-secant lines [2, Théorème 1]. A more detailed proof of the global generation of $\mathscr{I}_Y(4)$ was given in [1, Lemma 2.10] thanks to [9, Remark 4 and Proposition 6], that allows to find a smooth quartic surface $S \subset \mathbb{P}^3$ containing $Y$.

In particular, there exist two pencils $S \to \mathbb{P}^1$ corresponding to $|Y|$ and $|4H - Y|$. This is a remarkable property, so we may also state the following.

**Corollary 16.** *Let $S \in H^0(\mathscr{I}_Y(2e))$ be general in Proposition 12. Then $S \subset \mathbb{P}^3$ is a smooth surface with two pencils $S \to \mathbb{P}^1$ given by $|Y|$ and $|2eH - Y|$.*

**Proof.** If $s, s' \in H^0(E)$ are general sections, then the scheme of zeros of $s \wedge s' \in H^0(\Lambda^2 E)$ is a smooth surface $S \subset \mathbb{P}^3$ of degree $2e$ containing the scheme of zeros $Y$ of $s$. A concrete homogeneous equation of $S$ can be derived from the equality

$$I(Y) = I(Y_1) \cap I(Y_2) = (F_1, G_1) \cdot (F_2, G_2),$$

where $(F_i, G_i)$ denote the homogeneous polynomials of degree $e$ defining $Y_i$ in $\mathbb{P}^3$. Adjunction formula $\omega_Y \cong \omega_S(Y)|_Y$ gives $Y^2 = 0$, whence $(2eH - Y)^2 = 0$. □

Therefore, we pose the following question concerning the case $h^1(E^*) = 0$.

**Question 17.** What are the smooth connected curves $Y \subset \mathbb{P}^3$ of degree $2e^2$ with $\omega_Y \cong \mathscr{O}_{\mathbb{P}^3}(2e - 4)|_Y$ lying on a smooth surface $S \subset \mathbb{P}^3$ of degree $2e$?

## 4. A remark on the conjecture on morphisms from $\mathbb{P}^5$ to $\mathbb{G}(1,5)$

One can modify the statements of Lemma 7 and Corollary 8 to obtain the following two results.

**Lemma 18.** *Let $E$ be a globally generated rank 2 vector bundle on $\mathbb{P}^5$. Then there exists a surjection $\phi : \mathscr{O}_{\mathbb{P}^5}^{\oplus 6} \to E$ if and only if $c_1 = 6e$ and $c_2 = 12e^2$.*

**Corollary 19.** *Let $\phi : \mathscr{O}_{\mathbb{P}^5}^{\oplus 6} \to E$ be a surjection onto a nontrivial rank 2 vector bundle. Then the scheme of zeros of a general section $s \in H^0(E)$ is a smooth irreducible threefold $Y \subset \mathbb{P}^5$ with $\omega_Y \cong \mathscr{O}_{\mathbb{P}^5}(6e - 6)|_Y$, and its ideal sheaf is given by the exact sequence*

$$0 \to \mathscr{O}_{\mathbb{P}^5} \xrightarrow{\cdot s} E \to \mathscr{I}_Y(6e) \to 0.$$

*In particular, $d_i := (K_Y + H_Y)^i \cdot H_Y^{3-i} = 12e^2(6e - 5)^i$ for every integer $0 \le i \le 3$.*

In view of them, we may provide some evidence for Conjecture 2.

**Remark 20.** On the one hand, no example of indecomposable vector bundle of rank 2 on $\mathbb{P}^5$ is known in characteristic zero (see [6, Section 6]). On the other hand, if $E$ is decomposable then $(c_1, c_2) \neq (6e, 12e^2)$ (cf. Lemma 18). Moreover, in the case $e = 1$ there is no smooth threefold $Y \subset \mathbb{P}^5$ with the numerical invariants given in Corollary 19 (see [3, Proposition 4.2]).

## References


[1] L. Chiodera, P. Ellia, "Rank two globally generated vector bundles with $c_1 \le 5$", *Rend. Ist. Mat. Univ. Trieste* **44** (2012), p. 413-422.

[2] J. D'Almeida, "Une involution sur un espace de modules de fibrés instantons", *Bull. Soc. Math. Fr.* **128** (2000), no. 4, p. 577-584.

[3] G. Edelmann, "3-folds in $\mathbb{P}^5$ of degree 12", *Manuscr. Math.* **82** (1994), no. 3-4, p. 393-406.

[4] D. Eisenbud, J. Harris, *3264 and all that. A second course in algebraic geometry*, Cambridge University Press, 2016.

[5] L. Gruson, C. Peskine, "Genre des courbes de l'espace projectif", in *Algebraic geometry (Proc. Sympos., Univ. Tromsø, Tromsø, 1977)*, Lecture Notes in Mathematics, vol. 687, Springer, 1978, p. 31-59.

[6] R. Hartshorne, "Varieties of small codimension in projective space", *Bull. Am. Math. Soc.* **80** (1974), p. 1017-1032.

[7] ———, "Stable vector bundles of rank 2 on $\mathbf{P}^3$", *Math. Ann.* **238** (1978), p. 229-280.

[8] Y. Kachi, E. Sato, *Segre's reflexivity and an inductive characterization of hyperquadrics*, Memoirs of the American Mathematical Society, vol. 763, American Mathematical Society, 2002.

[9] S. Mori, "On degrees and genera of curves on smooth quartic surfaces in $\mathbb{P}^3$", *Nagoya Math. J.* **96** (1984), p. 127-132.

[10] J.-P. Serre, “Sur les modules projectifs”, *Séminaire Dubreil. Algèbre et théorie des nombres* **14** (1960-1961), no. 1, article no. 2.

[11] F. Severi, “Intorno ai punti doppi impropri di una superficie generale dello spazio a quattro dimensioni, e a’ suoi punti tripli apparenti”, *Rend. Circ. Mat. Palermo* **15** (1901), p. 33-51.

[12] H. Tango, “On $(n-1)$-dimensional projective spaces contained in the Grassmann variety $\mathrm{Gr}(n,1)$”, *J. Math. Kyoto Univ.* **14** (1974), p. 415-460.

[13] ———, “On morphisms from projective space $\mathbf{P}^n$ to the Grassmann variety $Gr(n,d)$”, *J. Math. Kyoto Univ.* **16** (1976), p. 201-207.

[14] ———, “On morphisms from projective space $\mathbf{P}^n$ to the Grassmann variety $Gr(n,d)$. II”, *Bull. Kyoto Univ. Educ.* **64** (1984), p. 1-20.