INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus
Mathématique

Snehajit Misra and Nabanita Ray

On Ampleness of vector bundles

Volume 359, issue 6 (2021), p. 763-772

<https://doi.org/10.5802/crmath.222>

Les Comptes Rendus. Mathématique *sont membres du*
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org

Comptes Rendus
Mathématique
2021, 359, nº 6, p. 763-772
https://doi.org/10.5802/crmath.222

Algebraic geometry / *Géométrie algébrique*

On Ampleness of vector bundles

Snehajit Misra*, [a] and Nabanita Ray [a]

[a] Tata Institute of Fundamental Research (TIFR), Homi Bhabha Road, Colaba, Mumbai 400005, India.

E-mails: smisra@math.tifr.res.in (Snehajit Misra), nray@math.tifr.res.in (Nabanita Ray)

Abstract. In this article, we give a necessary and sufficient condition for ampleness of semistable vector bundles with vanishing discriminant on a smooth projective variety X. As an application, we show ampleness of some special vector bundles on certain ruled surfaces. We prove similar results for parabolic ampleness.

Mathematical subject classification (2010). 14J60, 14N05, 14J40, 14C17.

Manuscript received 9th February 2021, revised 16th February 2021 and 2nd May 2021, accepted 2nd May 2021.

1. Introduction

Let X be a complex manifold of dimension n, and E be a holomorphic vector bundle of rank r on X endowed with a hermitian metric h. The hermitian bundle (E,h) determines a unique hermitian connection compatible with the complex structure on X and E, called as Chern connection, and it is denoted by D_E. This connection D_E in turn gives rise to a curvature tensor, called as Chern curvature tensor and denoted by $\Theta(E,h) \in C^\infty\big(X, \wedge^{1,1} T_X^* \otimes \mathrm{End}(E)\big)$ a $\mathrm{End}(E)$-valued $(1,1)$ form on X. If $z_1, z_2, \ldots, z_n$ are local coordinates on X, and if $(e_\lambda)_{1\le\lambda\le r}$ is a local orthonormal frame on E, then one can write

$$i\Theta(E,h) = \sum_{1\le j,k\le n, 1\le \lambda,\mu\le r} c_{jk\lambda\mu} \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k \otimes e_\lambda^* \otimes e_\mu,$$

where $c_j k\lambda\mu = c_k j\mu\lambda$. One looks at the associated quadratic form on $S = T_X \otimes E$ as follows:

$$\widetilde{\Theta}_{E,h}(\xi\otimes v) = \big\langle \Theta_{E,h}(\xi,\bar{\xi})\cdot v, v\big\rangle_h = \sum_{1\le j,k\le n, 1\le\lambda,\mu\le r} c_j k\lambda\mu \xi_j \bar{\xi}_k v_\lambda \bar{v}_\mu.$$

The hermitian bundle (E,h) is said to be Griffiths positive if at any point $z \in X$, we have $\widetilde{\Theta}_{E,h}(\xi\otimes v) > 0$ for all $0 \neq \xi \in T_{X,z}$ and for all $0 \neq v \in E_z$.

A holomorphic vector bundle E on a complex projective manifold is called ample in the sense of Hartshorne if the tautological line bundle $\mathscr{O}_{\mathbb{P}(E)}(1)$ is ample. i.e. there exists a smooth hermitian metric on $\mathscr{O}_{\mathbb{P}(E)}(1)$ with positive curvature.

* Corresponding author.

It is always true that if a hermitian holomorphic vector bundle (E,h) on a complex projective manifold X is Griffiths positive, then E is ample in the sense of Hartshorne. A famous conjecture of Griffiths asks whether ample bundles in the sense of Hartshorne admit Griffiths positively curved metrics. Also it is well known that if E is ample, then $\det(E)$ is ample. However, ampleness of $\det(E)$ does not ensure ampleness of E in general.

For a vector bundle of rank r on a complex manifold X, the characteristic class

$$c_2\big(\mathrm{End}(E)\big) = 2r c_2(E) - (r-1)c_1^2(E) \in H^4(X,\mathbb{Q})$$

is called the discriminant of E, denoted by $\triangle(E)$.

In Section 3, we prove the following.

Theorem 1. *Let X be a projective variety of dimension n and (E,h) be a hermitian holomorphic bundle of rank r on X. Further assume that E is a semistable vector bundle with $\triangle(E) = 0$. Then the following are equivalent:*

(i) *(E,h) is Griffiths positive.*
(ii) *E is ample in the sense of Hartshorne.*
(iii) $\det(E)$ *is ample.*

The Nakai–Moishezon criterion for ampleness says that a line bundle L on a projective variety X is ample if and only if $L^{\dim Y}\cdot Y > 0$ for every positive dimensional subvarieties Y of X. Mumford gave an example of a non-ample line bundle on a ruled surface whose intersection with every curve is positive (see [14, Chapter 1]). Therefore, in general, it is not sufficient to check the condition only for curves in Nakai–Moishezon criterion. However, in some special cases, to check ampleness of a line bundle L on X, it is enough to check that $L\cdot C > 0$ for every irreducible curve $C \subset X$ (e.g., on abelian varieties [21], on flag bundles [7]). One must also note that for a globally generated vector bundle E on X, E is ample if and only if it's restriction to every curve $C \subset X$ is ample. This follows easily from Gieseker's Lemma (see [15, Proposition 6.1.7]). In general, there is no straight forward way to check ampleness of a given vector bundle on a projective variety X. In [12], it is proved that an equivariant vector bundle on a toric variety X is ample if and only if its restriction to finitely many invariant curves in X are ample. Similar result holds for torus equivariant vector bundles on certain homogenous variety (see [6]). In [1], a sufficient condition is given to check ampleness of a vector bundle of rank 2 on some specific smooth surfaces with Picard rank 1.

We recall from [11, Chapter 5] that a vector bundle W of rank 2 on a smooth projective curve C is said to be normalized if $H^0(W) \neq 0$, but $H^0(W\otimes L) = 0$ for all line bundle L on C with $\deg(L) < 0$. We notice that a normalized bundle W is semistable if and only if $\deg(W) \geq 0$. An important consequence of Theorem 1 is the following.

Corollary 2. *Let $\rho : X = \mathbb{P}(W) \longrightarrow C$ be a ruled surface defined by a normalized rank 2 bundle on a smooth curve C such that $\mu_{\min}(W) = \deg(W)$. Let E be a semistable vector bundle of rank r on X with discriminant $\triangle(E) = 0$. Then, E is ample if and only if $E|_\sigma$ and $E|_f$ are ample, where σ is the smooth section of ρ such that $\mathscr{O}_X(\sigma) \cong \mathscr{O}_{\mathbb{P}(W)}(1)$ and f is a fibre of ρ.*

The above Corollary 2 implies the following:

Corollary 3. *Let $\rho : X = \mathbb{P}(W) \longrightarrow C$ be a ruled surface on a smooth curve C defined by a normalized rank 2 bundle W on C with $\mu_{\min}(W) = \deg(W)$, and E be a vector bundle on C. Then the vector bundle $E = \rho^*(V)\otimes\mathscr{O}_{\mathbb{P}(W)}(m)$ is ample on X if and only if $m > 0$ and $\mu_{\min}(E) > -m\deg(W)$.*

We also prove similar result for parabolic ampleness in Section 4.

2. Preliminaries

2.1. *Harder–Narasimhan Filtration*

A non-zero torsion-free coherent sheaf $\mathscr{G}$ on X is said to be H-semistable if

$$\mu_H(\mathscr{F}) = \frac{c_1(\mathscr{F})\cdot H^{n-1}}{\operatorname{rank}(\mathscr{F})} \le \mu_H(\mathscr{G}) = \frac{c_1(\mathscr{G})\cdot H^{n-1}}{\operatorname{rank}(\mathscr{G})}$$

for all subsheaves $\mathscr{F}$ of $\mathscr{G}$. For every vector bundle E on X, there is a unique filtration

$$0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_{k-1} \subsetneq E_k = E$$

of subbundles of E, called the Harder–Narasimhan filtration of E, such that E_i/E_{i-1} is H-semistable torsion free sheaf for each $i \in \{1,2,\dots,k\}$ and $\mu_H\big(E_i/E_{i-1}\big) > \mu_H\big(E_{i+1}/E_i\big)$ for each $i \in \{1,2,\dots,k-1\}$. We define $Q_k := E_k/E_{k-1}$ and $\mu_{\min}(E) := \mu_H(Q_k) = \mu_H\big(E_k/E_{k-1}\big)$.

Let $N_1(X)_{\mathbb{R}}$ be the set of all numerical equivalence classes of real one cycles on X. Inside $N_1(X)_{\mathbb{R}}$, the closure of the convex cone generated by effective one cycles is called the closed cone of curves and it is denoted by $\overline{\mathrm{NE}}(X)$. By Theorem 1.4.29 of [14], a divisor D is ample if and only if $D\cdot\gamma > 0$ for all $\gamma \in \overline{\mathrm{NE}}(X) - \{0\}$.

3. Main result and applications

We begin this section by proving our main result.

Proof of Theorem 1. (i) $\implies$ (ii)**.** See Theorem 6.1.25 in [15] for a proof.

(ii) $\implies$ (iii)**.** See Corollary 5.3 in [10] for a proof.

(iii) $\implies$ (i)**.** There exists a filtration

$$0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_{l-1} \subsetneq E_l = E$$

such that on each $G_j = E_j/E_{j-1}$, there exists a hermitian metric h_j on G_j for which the curvature tensor is equal to $\frac{1}{r}\gamma\otimes \mathrm{Id}_{G_j}$ where γ is (1,1)-form representing the first Chern class $c_1(E)$ (see [19]). Since $\det(E)$ is ample, each (G_j,h_j) is Griffiths positive. As extension of two Griffiths positive bundles is again Griffiths positive, we have inductively each E_i is Griffiths positive and thus E is also Griffiths positive. □

Remark 4. Theorem 1 can be thought of as a generalization of Gieseker's ampleness criterion for semistable vector bundles on smooth curves (see [13, Theorem 3.2.7]). However, the condition about vanishing discriminant is not essential for both V and $\det(V)$ to be ample. For example, consider the tangent bundle $T_{\mathbb{P}^2}$. Then $T_{\mathbb{P}^2}$ sits in the following exact sequence:

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^2} \longrightarrow \mathscr{O}_{\mathbb{P}^2}(1)^{\oplus 3} \longrightarrow T_{\mathbb{P}^2} \longrightarrow 0.$$

Hence, $T_{\mathbb{P}^2}$ being quotient of an ample bundle is ample and $\det(T_{\mathbb{P}^2}) \cong \mathscr{O}_{\mathbb{P}^2}(3)$ is also ample. But $T_{\mathbb{P}^2}$ is semistable with $\triangle(T_{\mathbb{P}^2}) \neq 0$.

Remark 5. Note that for a vector bundle E on a smooth projective curve C, we have $\triangle(E) = 0$. Hence our result Theorem 1 is analogus to the result in [22]. Also one can compare our result with the results in [16] and [20].

A vector bundle V on an abelian variety X is called weakly-translation invariant (semi-homogeneous in the sense of Mukai) if for every closed point $x \in X$, there is a line bundle L_x on X depending on x such that $T_x^*(V) \simeq V \otimes L_x$ for all $x \in X$, where T_x is the translation morphism given by $x \in X$.

Corollary 6. *A semi-homogeneous vector bundle E of rank r on an abelian variety X is ample if and only if* $\det(E)$ *is ample if and only if* $\det(E)\cdot C > 0$ *for all irreducible curve C in X.*

Proof. Mukai proved that E is Gieseker semistable (see [13, Chapter 1] for definition) with respect to some polarization and it has projective Chern classes zero, i.e., if $c(E)$ is the total Chern class, then $c(E) = \{1+c_1(E)/r\}^r$ (see [18, Theorem 5.8, p. 260], [18, Proposition 6.13, p. 266]; also see [17, p. 2]). Gieseker semistablity implies slope semistability (see [13]). So, in particular, we have E is slope semistable with $\triangle(E) = 2rc_2(E)-(r-1)c_1^2(E) = 0$. Hence, the result follows from Theorem 19 and Proposition 1.4 in [21]. □

Corollary 7. *Let W be a vector bundle of rank m over a smooth complex projective curve C and $\rho : \mathbb{P}(W) \longrightarrow C$ be the projectivisation map. Let E be a semistable vector bundle on $\mathbb{P}(W)$ of rank r with discriminant $\triangle(E) = 0$, and $c_1(E) \equiv x\xi + yf$, where ξ and f are the numerical classes of $\mathcal{O}_{\mathbb{P}(W)}(1)$ and a fibre of ρ respectively. Then, E is ample if and only if $x > 0$ and $\big(x\mu_{\min}(W)+y\big) > 0$.*

Proof. We note that by Lemma 2.1 of [9], the nef cone of divisors in $\mathbb{P}(W)$ is given by

$$\operatorname{Nef}\big(\mathbb{P}(W)\big) = \big\{a\big(\xi - \mu_{\min}(W)f\big) + bf \,\big|\, a,b \in \mathbb{R}_{\geq 0}\big\}.$$

Applying duality (see [14, Proposition 1.4.28]), we get

$$\overline{\operatorname{NE}}\big(\mathbb{P}(W)\big) = \Big\{a\big(\xi^{m-1} - \big(\deg(W) - \mu_{\min}(W)\big)\xi^{m-2}f\big) + b\xi^{m-2}f \,\Big|\, a,b \in \mathbb{R}_{\geq 0}\Big\}.$$

Hence, $\det(E)$ is ample if and only if

- $c_1(E)\cdot\big\{\xi^{m-1} - \big(\deg(W) - \mu_{\min}(W)\big)\xi^{m-2}f\big\} = \big(x\mu_{\min}(W)+y\big) > 0$ and
- $c_1(E)\cdot\xi^{m-2}f = x > 0$.

Therefore, the result follows from the previous Theorem. □

Corollary 8. *Let $\rho : X = \mathbb{P}(W) \longrightarrow C$ be a ruled surface defined by a normalized rank 2 bundle on a smooth curve C such that $\mu_{\min}(W) = \deg(W)$. Let E be a semistable vector bundle of rank r on X with discriminant $\triangle(E) = 0$. Then, E is ample if and only if $E|_\sigma$ and $E|_f$ are ample, where σ is the smooth section of ρ such that $\mathcal{O}_X(\sigma) \simeq \mathcal{O}_{\mathbb{P}(W)}(1)$ and f is a fibre of ρ.*

Proof. Let $c_1(E) \equiv x\zeta + yf$, where $\zeta = [\sigma] \in N^1(X)$. Note that, by the given hypothesis, both $E|_\sigma$ and $E|_f$ are semistable, and hence both are ample if and only if

- $\deg\big(E|_\sigma\big) = (x\zeta + yf)\cdot\zeta = (x\deg(W) + y) > 0$, and
- $\deg\big(E|_f\big) = (x\zeta + yf)\cdot f = x > 0$.

But, in that case, $(x\mu_{\min}(W) + y) = (x\deg(W) + y) > 0$. Therefore, the result follows from the previous corollary. □

Remark 9. Let $\rho : X = \mathbb{P}(W) \longrightarrow C$ be a ruled surface on a smooth curve C as in Corollary 8. Then, for any semistable vector bundle R on C and any integer m, $E := \rho^*(R)\otimes\mathcal{O}_{\mathbb{P}(W)}(m)$ is a semistable vector bundle with vanishing discriminant. Hence by Corollary 8, any semistable vector bundle V on X of this form $\rho^*(R)\otimes\mathcal{O}_{\mathbb{P}(W)}(m)$ is ample if and only if $E|_\sigma$ and $E|_f$ are ample if and only if $m > 0$ and $\deg(R) > -m\deg(W)$.

For example, we consider the ruled surface $\rho : X = \mathbb{P}(W) \longrightarrow C$ over the elliptic curve C defined by the nonsplit extension $0 \longrightarrow \mathcal{O}_C \longrightarrow W \longrightarrow \mathcal{O}_C \longrightarrow 0$. Then for any semistable bundle R on C of positive degree, $E := \rho^*(R)\otimes\mathcal{O}_X(m)$ is ample for every positive integer m.

Corollary 10. *Let $\rho : X = \mathbb{P}(W) \longrightarrow C$ be a ruled surface on a smooth curve C defined by a normalized rank 2 bundle W on C with $\mu_{\min}(W) = \deg(W)$, and V be a vector bundle on C. Then the vector bundle $E = \rho^*(V)\otimes\mathcal{O}_{\mathbb{P}(W)}(m)$ is ample on X if and only if $m > 0$ and $\mu_{\min}(V) > -m\deg(W)$.*

Proof. Let

$$0 = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_{k-1} \subsetneq V_k = V$$

be the Harder–Narasimhan filtration of V, and $Q_i = V_i/V_{i-1}$ for each i. Since ρ is a smooth map, in particular it is flat and hence ρ^* is an exact functor. We also observe that for any ample line bundle H on X, we have $\mu_H\big(\rho^* Q_i\big) = \mu(Q_i)\big(f \cdot H\big)$ and $f \cdot H > 0$, where f denotes a fiber of ρ. Fix $E_i := \rho^*(V_i) \otimes \mathscr{O}_{\mathbb{P}(E)}(m)$. Then above observation and the uniqueness of Harder Narasimhan filtration imply that

$$0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_{k-1} \subsetneq E_k = E$$

is the Harder Narasimhan filtration of E with respect to any polarization H.

Now, suppose m satisfies $m > 0$ and $\mu_{\min}(V) > -m\deg(W)$. Then by the previous remark, we conclude that each $R_i := \rho^*(Q_i) \otimes \mathscr{O}_{\mathbb{P}(W)}(m)$ is ample. Inductively, each E_i is ample. In particular E is also ample.

Conversely, if V is ample for some m, then $R_k = \rho^*(Q_k) \otimes \mathscr{O}_{\mathbb{P}(W)}(m)$ is ample for each k. Thus m must satisfy $m > 0$ and $\mu_{\min}(E) > -m\deg(W)$. □

Example 11. Let us consider the ruled surface $\rho : X = \mathbb{P}(W) \longrightarrow C$ over a curve C where $W = \mathscr{O}_C \oplus \mathscr{L}$ for some line bundle $\mathscr{L}$ on C with $\deg(\mathscr{L}) < 0$. Then for any vector bundle E on C with $\mu_{\min}(E) > -m\deg(\mathscr{L})$ for some positive integer m, the bundle $V = \rho^*(E) \otimes \mathscr{O}_X(m)$ is ample.

Let $\rho : \mathbb{F}_e = \mathbb{P}\big(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(-e)\big) \longrightarrow \mathbb{P}^1$ be a Hirzebruch surface for some $e \geq 0$. Let C_0 be its normalized section such that $\mathscr{O}_{\mathbb{F}_e}(C_0)$ is the tautological bundle on $\mathbb{F}_e$, and f be a fibre of ρ. We recall the following results from [11, Chapter 5].

Theorem 12. *Let $D \sim aC_0 + bf$ be a divisor on $\mathbb{F}_e$. Then*

(a) *If D is an irreducible curve $\neq C_0, f$, then $a > 0$ and $b \geq ae$.*
(b) *The linear system $|D|$ contains a section of ρ if and only if $a = 1$ and either $b = 0$ or $b \geq e$.*
(c) *The linear system $|D|$ contains an irreducible non-singular curve if and only if it contains an irreducible curve if and only if $a = 0$, $b = 1$ (namely f) ; or $a = 1, b = 0$ (namely C_0) ; or $a > 0$, $b > ae$; or $e > 0$, $a > 0$, $b = ae$.*
(d) *D is very ample if and only if D is ample if and only if $a > 0$ and $b > ae$.*

Lemma 13. *Any irreducible curve of $\mathbb{F}_e$ other than the fibers of ρ is linearly equivalent to an effective curve which is a union of sections of the map ρ.*

Proof. Let C be an irreducible curve in $\mathbb{F}_e$ other than a fibre and the section C_0. Then $C \sim xC_0 + yf$ for some $x > 0$ and $y \geq xe$. Let $y = mxe + r$ for some $m > 0$ and $0 \leq r < xe$. Now, $C \sim xC_0 + yf \sim (x-1)\big(C_0 + ef\big) + \big(C_0 + (e(xm - x + 1) + r)f\big)$. This proves the result. □

Proposition 14. *Let C be an irreducible curve in $\mathbb{F}_e$ and $C \sim C_1 + \cdots + C_r$ where C_i's are sections of the map ρ. Let E be a vector bundle on $\mathbb{F}_e$ such that for any two curves B and B' in $\mathbb{F}_e$ with $B \sim B'$, we have $E|_B \cong E|_{B'}$. Then $\rho_*\big(E|_C\big) \cong \bigoplus_i \rho_*\big(E|_{C_i}\big)$ as vector bundles on $\mathbb{P}^1$.*

Proof. We first observe that for any two curves B and B' in $\mathbb{F}_e$ which are linearly equivalent to each other,

$$\rho_*\big(E \otimes \mathscr{O}_B\big) \cong \rho_*\big(E \otimes \mathscr{O}_{B'}\big) \quad \text{on } \mathbb{P}^1.$$

In other words, $\rho_*\big(E|_B\big) \cong \rho_*\big(E|_{B'}\big)$ on $\mathbb{P}^1$. So, without loss of generality we assume that $C = C_1 + \cdots + C_r$ and $C_i \hookrightarrow C$ be an irreducible component of it. Then, $\mathscr{O}_C \twoheadrightarrow \mathscr{O}_{C_i}$, which induces a sheaf map $\rho_*\big(E \otimes \mathscr{O}_C\big) \longrightarrow \rho_*\big(E \otimes \mathscr{O}_{C_i}\big)$ on $\mathbb{P}^1$ for all i, and hence induces a map $\rho_*\big(E \otimes \mathscr{O}_C\big) \longrightarrow \bigoplus_i \rho_*\big(E \otimes \mathscr{O}_{C_i}\big)$ as well.

We claim that

$$\rho_*\big(E \otimes \mathscr{O}_C\big) \longrightarrow \bigoplus_i \rho_*\big(E \otimes \mathscr{O}_{C_i}\big)$$

is an isomorphism on $\mathbb{P}^1$. Indeed, for any $y \in \mathbb{P}^1$,

$$\big(\rho_*(E\otimes\mathcal{O}_C)\big)_y \cong \bigoplus_{x\in\{C\cap\rho^{-1}(y)\}} E_x.$$

On the other hand,

$$\left(\bigoplus_i \rho_*(E\otimes\mathcal{O}_{C_i})\right)_y \cong \bigoplus_i \big(\rho_*(E\otimes\mathcal{O}_{C_i})\big)_y \cong \bigoplus_{x\in C_i,\rho(x)=y} E_x.$$

Hence, the map is isomorphic at the stalk level. This proves our claim and the result. □

Any rank two vector bundle E on $\mathbb{F}_e$ has two numerical invariants describing it as an extension in a cannonical manner. The first invariant d_E is defined by the splitting type of E on a general fiber f, i.e., if $E|_f = \mathcal{O}_{\mathbb{P}^1}(d)\oplus\mathcal{O}_{\mathbb{P}^1}(d')$ and $d \geq d'$, then $d_E = d$. The second invariant $r_E = r = \deg\big(\rho_*(E(-dC_0))\big)$. See [8] for more information about these numerical invariants d and r. Note that, if E is globally generated, then for a generic fibre f, $E|_f = \mathcal{O}_{\mathbb{P}^1}(d)\oplus\mathcal{O}_{\mathbb{P}^1}(d')$, where $d \geq d' \geq 0$.

Theorem 15. *Let E be a globally generated rank two bundle on $\mathbb{F}_e$ with numerical invariants d and r, and E sits in the exact sequence*

$$0 \to \mathcal{O}(dC_0 + rf) \to E \to \mathcal{O}(d'C_0 + r'f) \to 0. \tag{1}$$

Further assume that for any two curves B and B' in $\mathbb{F}_e$ with $B \sim B'$, we have $E|_B \cong E|_{B'}$. Then, E is ample if and only if $E|_f$, $E|_{C_0}$ and $E|_{C_0+nf}$ are ample on a generic fibre f, on C_0 and sections of ρ of the forms $C_0 + nf$ with $d(n-e)+r \leq 0$ respectively.

Proof. Restriction of ample bundle being ample, $E|_C$ is ample for any curve C in $\mathbb{F}_e$ whenever E is ample.

Conversely, let $E|_f$, $E|_{C_0}$ and $E|_{C_0+nf}$ are ample on a generic fibre f, on C_0 and sections of ρ of the forms $C_0 + nf$ with $d(n-e)+r \leq 0$ respectively. Now, if

$$E|_f = \mathcal{O}_{\mathbb{P}^1}(d)\oplus\mathcal{O}_{\mathbb{P}^1}(d'),$$

for a generic fibre f of ρ with $d \geq d' \geq 0$, then the ampleness of $E|_f$ implies that $d, d' > 0$.

Let f' be a fibre among those finitely many fibre which has different splitting type of E than that of a generic fibre. Restricting the exact sequence (1) to f', we get

$$0 \to \mathcal{O}_{\mathbb{P}^1}(d) \to E|_{f'} \to \mathcal{O}_{\mathbb{P}^1}(d') \to 0$$

Hence, $E|_{f'}$ being an extension of two ample line bundle, is also ample.

Let $C \sim C_0 + nf$ be any section of ρ, where either $n = 0$ or $n \geq e$. Now, restricting the exact sequence (1) to C, we get

$$0 \to \mathcal{O}_{\mathbb{P}^1}\big(d(n-e)+r\big) \to E|_C \to \mathcal{O}_{\mathbb{P}^1}\big(d'(n-e)+r'\big) \to 0.$$

As $E|_{C_0}$ is ample on C_0, and $\mathcal{O}_{\mathbb{P}^1}(-d'e+r')$ being the quotient is also ample. Hence, $(-d'e+r') > 0$, which implies $d'(n-e)+r' > 0$ for any $n \geq 1$. Note that, if $d(n-e)+r > 0$ then $E|_{C_0+nf}$ is ample, as it is then an extension of two ample bundles. If $d(n-e)+r \leq 0$ then $E|_{C_0+nf}$ is also ample by the given hypothesis. Therefore, we conclude that restriction of E onto each fibre and each section is ample.

Let C be any curve of $\mathbb{F}_e$ other than a fibre of ρ, and $C \sim C_1 + \cdots + C_r$ where C_i's are sections. Now, using Proposition 14, we get that $\rho_*(E|_C)$ is an ample vector bundle on $\mathbb{P}^1$.

If E is not ample, then by Gieseker's lemma [15, Proposition 6.1.7], there exists an irreducible curve C in $\mathbb{F}_e$ other than the fibres and a surjective homomorphism $u : E|_C \twoheadrightarrow \mathcal{O}_C$. This induces the sujection $\rho_*(E|_C) \twoheadrightarrow \rho_*(\mathcal{O}_C) \cong \mathcal{O}_{\mathbb{P}^1}$, as well as the injection $\mathcal{O}_{\mathbb{P}^1} \hookrightarrow \big(\rho_*(E|_C)\big)^*$ which contradicts the fact that $\rho_*(E|_C)$ is an ample bundle on $\mathbb{P}^1$. Therefore, E is ample. This completes the proof. □

4. Remark about Parabolic Ampleness

Let X be a connected smooth complex projective variety of dimension d and $D \subset X$ be an effective divisor on X.

Definition 16. *A quasi parabolic structure on a coherent sheaf E with respect to D is a filtration by $\mathscr{O}_X$-coherent subsheaves*

$$E = \mathscr{F}_1(E) \supset \mathscr{F}_2(E) \supset \cdots \supset \mathscr{F}_l(E) \supset \mathscr{F}_{l+1}(E) = E(-D)$$

where $E(-D) = E \otimes_{\mathscr{O}_X} \mathscr{O}_X(-D)$. The integer l is called the length of the filtration.

A parabolic structure is a quasi-parabolic structure, as above, together with a system of parabolic weights $\{\alpha_1, \alpha_2, \ldots, \alpha_l\}$ such that $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_{l-1} < \alpha_l < 1$, where each α_i is attached to $\mathscr{F}_i(E)$.

We shall denote the parabolic sheaf by $(E, \mathscr{F}_*, \alpha_*)$ or simply by E_* when there is no confusion. For any parabolic sheaf E_* defined as above, for any $t \in \mathbb{R}$, we define the following filtration $\{E_t\}_{t\in\mathbb{R}}$ of coherent sheaves parametrized by $\mathbb{R}$:

$$E_t = \mathscr{F}_i(E)(-[t]D)$$

where $[t]$ is the integral part of t and $\alpha_{i-1} < t - [t] \leq \alpha_i$ with $\alpha_0 = \alpha_l - 1$ and $\alpha_{l+1} = 1$. Note that, any coherent subsheaf M of E has an induced parabolic structure such that if $\{M_t\}_{t\in\mathbb{R}}$ is the corresponding filtration then $M_t = E_t \cap M$ for any $t \geq 0$.

The *parabolic degree* of E_* with respect to a fixed ample bundle L on X, denoted by par_deg(E_*) is defined as follows :

$$\text{par\_deg}(E_*) := \int_{-1}^{0} \deg(E_t)\,\mathrm{d}t$$

The *parabolic slope* of E_*, denoted by par_$\mu(E_*)$ is the quotient par_deg$(E_*)/\operatorname{rank}(E)$.

Definition 17. *The parabolic sheaf E_* is called parabolic semistable (resp. parabolic stable) if for every subsheaf M of E with $0 < \operatorname{rank}(M) < \operatorname{rank}(E)$, and E/M being torsion-free sheaf, the inequality* par_$\mu(M_*) \leq$ par_$\mu(E_*)$ *(resp.* par_$\mu(M_*) < \mu(E_*)$*) is satisfied.*

Consider the decomposition

$$D = \sum_{i=1}^{n} n_i D_i \tag{2}$$

where any D_i is a reduced irreducible divisor and $n_i \geq 1$. Let

$$f_i : n_i D_i \longrightarrow X$$

denote the inclusion of the subscheme $n_i D_i$. For $1 \leq i \leq n$, let

$$0 = F^i_{l_i+1} \subset F^i_{l_i} \subset F^i_{l_i-1} \subset \cdots \subset F^i_1 = f_i^* E \tag{3}$$

Let α^i_j, $1 \leq j \leq l_i + 1$ be real numbers satisfying

$$1 = \alpha^i_{l_i+1} > \alpha^i_{l_i} > \alpha^i_{l_i-1} > \cdots > \alpha^i_2 > \alpha^i_1 \geq 0.$$

From now on we will always impose the following three conditions on the parabolic bundles E_* that we will consider:

(a) the parabolic divisor $D = \sum_{i=1}^{n} n_i D_i$ is a normal crossing divisor, i.e., all $n_i = 1$ and D_i are smooth divisors and they intersect transversally.
(b) all F^i_j on D_i in sequence (3) are subbundles of $f_i^* E$ for every i.
(c) all the weights α^i_j are rational numbers; so $\alpha^i_j = m^i_j/N$, where N is a fixed integer and $m^i_j \in \{0, 1, \ldots, N-1\}$.

In [2], parabolic tensor product has been defined. The parabolic m-fold symmetric product $S^m(E_*)$, is the invariant subsheaf of the m-fold parabolic tensor product of E_* for the natural action of the permutation group for the factors of the tensor product. The underlying sheaf of the parabolic sheaf $S^m(E_*)$ will be denoted by $S^m(E_*)_0$. We recall the definition of parabolic ampleness from [3].

Definition 18. *The parabolic sheaf E_* is called parabolic ample if for any coherent sheaf F on X there is an integer m_0 such that for any $m \geq m_0$, the tensor product $F \otimes S^m(E_*)_0$ is globally generated.*

Parabolic Chern classes $c_i(E_*) \in H^{2i}(X,\mathbb{Q})$ has been introduced in [3]. For a parabolic vector bundle E_* of rank r we define the parabolic discriminant, denoted by $\triangle_{\text{par}}(E_*)$ as follows:

$$\triangle_{\text{par}}(E_*) := 2rc_2(E_*) - (r-1)c_1^2(E_*).$$

Theorem 19. *Let E_* be a semistable parabolic vector bundle of rank r on a smooth complex projective variety X such that $\triangle_{\text{par}}(E_*) = 0$. Then, E_* is parabolic ample if and only if its parabolic first Chern class $c_1(E_*)$ is in the ample cone of X.*

Proof. Let $p : Y \longrightarrow X$ be the Kawamata cover, V be the corresponding orbifold bundle on Y with $c_1(V) = p^* c_1(E_*)$ (see [2] and [4]). So if E_* is ample, then V is also ample (see [3]) and thus $c_1(V)$ is also ample. Using the finiteness of the surjective map p, we conclude that $c_1(E_*)$ is in the ample cone of X.

Conversely, if $c_1(E_*)$ is in the ample cone of X, then $\det(V)$ is also ample. Also, by the given hypothesis, V is orbifold semistable and hence semistable (in the usual sense) with $\triangle(V) = p^*\triangle_{\text{par}}(E_*) = 0$. Hence V is ample and thus E_* is parabolic ample. □

Proposition 20. *Let $\pi : X \longrightarrow Y$ be a smooth surjective morphism between two smooth connected complex projective varieties X and Y. Let E_* be a parabolic semistable bundle on Y with parabolic divisor $D \subset Y$ and $\triangle_{\text{par}}(E_*) = 0$. Then, the pullback bundle $\pi^*(E_*)$ under the map π is also parabolic semistable on X with parabolic divisor $\pi^*(D) \subset X$ and $\triangle_{\text{par}}\big(\pi^*(E_*)\big) = 0$.*

Conversely, if E_* be a parabolic semistable bundle on projective bundle $\pi : X = \mathbb{P}(\mathscr{E}) \longrightarrow Y$ with parabolic divisor $D' = \pi^{-1}(D)$, with $\triangle_{\text{par}}(E_*) = 0$ and the parabolic first Chern class $c_1(E_*) = \pi^*(\mathscr{L})$ for some line bundle $\mathscr{L}$ on Y, then there exists a semistable parabolic bundle E'_* on Y with parabolic divisor D and $\triangle_{\text{par}}(E'_*) = 0$ such that $E_* = \pi^*(E'_*)$.

Proof. Let $D = \sum_{i=1}^n D_i$ be the normal crossing divisor on Y and $D' = \pi^*(D)$. Since π is smooth, the pullback divisor D' on X is also a normal crossing divisor satisfying condition (a).

Let $p : Y' \longrightarrow Y$ be a Kawamata cover with Galois group G such that

$$p^*D_i = k_i N(p^*D_i)_{\text{red}}$$

for some positive integers k_i and N. Consider the following fibre product diagram

$$\begin{array}{ccc} X' = X \times_Y Y' & \xrightarrow{\ \tilde{p}\ } & X \\ \big\downarrow{\scriptstyle \tilde{\pi}} & & \big\downarrow{\scriptstyle \pi} \\ Y' & \xrightarrow{\ p\ } & Y \end{array}$$

Then $\widetilde{p} : X' \longrightarrow X$ is a Galois cover with the same Galois group G. Let V be the orbifold bundle on Y' associated to the parabolic bundle E_* on Y (see [4]). The pullback orbifold bundle $V' := \widetilde{\pi}^*(V)$ then corresponds to the parabolic pullback bundle $\pi^*(E_*)$ on X with parabolic divisor D'.

We note that $\triangle(V) = \triangle_{\text{par}}(E_*) = 0$. Since E_* is parabolic semistable, by using the correspondence in [4], V is also orbifold semistable, and hence semistable (in the usual sense). Therefore the pullback bundle V' is also orbifold semistable with $\triangle(V')=0$, proving that $\pi^*(E_*)$ is parabolic semistable with $\triangle_{\text{par}}\big(\pi^*(E_*)\big)=0$.

Conversely, let V be the orbifold bundle on X' associated to the parabolic bundle E_* on X. Then,

$$c_1(V)=\widetilde{p}^*c_1(E_*)=\widetilde{p}^*\pi^*(\mathscr{L})=\widetilde{\pi}^*p^*(\mathscr{L})$$

Now by the given hypothesis, V is orbifold semistable and hence semistable (in the usual sense). Since $\triangle(V)=0$, by Theorem 1.2 in [5] $V|_f$ is semistable on $f\simeq\mathbb{P}^m$ for every fibre f of the map $\widetilde{\pi}$ $\big($Here $\operatorname{rank}(\mathscr{E})=m+1\big)$ and $\deg(V|_f)=0$. This implies $V\simeq\widetilde{\pi}^*(W)$ for some orbifold bundle W on Y' which must be semistable. Let E'_* be the associated semistable parabolic bundle on Y. Note that $\triangle(V)=\widetilde{\pi}^*(\triangle(W))=0$ and $\widetilde{\pi}^*$ is injective. Hence $\triangle(W)=0$. By a similar argument we have $\triangle_{\text{par}}(E'_*)=0$. Then by the construction of E'_*, the result follows. □

Corollary 21. *Let W be a vector bundle of rank m over a smooth complex projective curve C and $\rho: X=\mathbb{P}(W)\longrightarrow C$ be the projectivisation map. Let E_* be a semistable vector bundle on X of rank r with parabolic discriminant $\triangle_{\text{par}}(E_*)=0$, and parabolic 1st Chern class $c_1(E_*)\equiv x\xi+yf$, where ξ and f are the numerical classes of $\mathscr{O}_{\mathbb{P}(W)}(1)$ and a fibre of ρ respectively. Then, E_* is ample if and only if $x>0$ and $\big(x\mu_{\min}(W)+y\big)>0$.*

Proof. We note that

$$\overline{\text{NE}}\big(\mathbb{P}(W)\big)=\Big\{a\big(\xi^{m-1}-(\deg(W)-\mu_{\min}(W))\xi^{m-2}f\big)+b\xi^{m-2}f\,\Big|\,a,b\in\mathbb{R}_{\geq 0}\Big\}$$

Hence, $c_1(E_*)$ is in the ample cone if and only if

- $c_1(E_*)\cdot\{\xi^{m-1}-(\deg(W)-\mu_{\min}(W))\xi^{m-2}f\}=\big(x\mu_{\min}(W)+y\big)>0$ and
- $c_1(E_*)\cdot\xi^{m-2}f=x>0$.

Therefore, the result follows from the previous theorem. □

Example 22. Let $\rho: X=\mathbb{P}(W)\longrightarrow Y$ be a projective bundle on a smooth projective variety Y. Let $D\subset Y$ be a normal crossing divisor in Y and F_* be a semistable parabolic bundle of rank r on Y with parabolic divisor D. Then $\rho^*(F_*)$ is a parabolic semistable bundle on X with parabolic divisor $D'=\rho^*(D)$. Let $D'=\sum_{i=1}^n D'_i$ be the decomposition into irreducible components of D'. A parabolic line bundle with parabolic divisor D' is a data of the form $L_*=\big(L,\{\alpha_1,\dots,\alpha_n\}\big)$, where L is a line bundle on X and each $0\leq\alpha_i<1$ corresponds to the divisor D'_i. Assume $\alpha_i\in\mathbb{Q}$ for all i. Then $E_*=\rho^*(F_*)\otimes L_*$ is parabolic semistable with $\triangle_{\text{par}}(E_*)=0$. Note that $c_1(L_*):=c_1(L)+\sum_{i=1}^n\alpha_i[D_i]$. One can choose L_* in such a way that $c_1(E_*)=c_1(\rho^*F_*)+rc_1(L_*)$ is in the ample cone of X. This way one can produce parabolic ample bundles on X.

Acknowledgement

The authors would like to thank Prof. Indranil Biswas for many helpful discussions.

References

[1] A. Beauville, "An ampleness criterion for rank 2 vector bundles on surfaces", *Vietnam J. Math.* **48** (2020), no. 1, p. 125-129.

[2] I. Biswas, "Chern classes for parabolic bundles", *J. Math. Kyoto Univ.* **37** (1997), no. 4, p. 597-613.

[3] ———, "Parabolic ample bundles", *Math. Ann.* **307** (1997), no. 3, p. 511-529.

[4] ———, "Parabolic bundles as orbifold bundles", *Duke Math. J.* **88** (1997), no. 2, p. 305-325.

[5] I. Biswas, U. Bruzzo, "On semistable principal bundles over a complex projective manifold", *Int. Math. Res. Not.* **2008** (2008), article no. rnn035 (28 pages).

[6] I. Biswas, K. Hanumanthu, D. S. Nagaraj, "Positivity of vector bundles on homogeneous varieties", *Int. J. Math.* **31** (2020), no. 12, article no. 2050097 (11 pages).

[7] I. Biswas, S. Mehrotra, A. J. Parameswaran, "Nef line bundles on flag bundles on a curve over $\overline{\mathbb{F}_p}$", *Arch. Math.* **101** (2013), no. 2, p. 105-110.
[8] V. Brînzănescu, "Algebraic 2-vector bundles on ruled surfaces", *Ann. Univ. Ferrara, Nuova Ser., Sez. VII* **37** (1991), p. 55-64.
[9] M. Fulger, "The cones of effective cycles on projective bundles over curves", *Math. Z.* **269** (2011), no. 1-2, p. 449-459.
[10] R. Hartshorne, "Ample Vector bundles", *Publ. Math., Inst. Hautes Étud. Sci.* **29** (1966), p. 63-94.
[11] ———, *Algebraic Geometry*, Graduate Texts in Mathematics, vol. 52, Springer, 1977.
[12] M. Hering, M. Mustaţă, S. Payne, "Positivity properties of toric vector bundles", *Ann. Inst. Fourier* **60** (2010), no. 2, p. 607-640.
[13] D. Huybrechts, M. Lehn, *The Geometry of Moduli Spaces of Sheaves*, Cambridge University Press, 2010.
[14] R. Lazarsfeld, *Positivity in algebraic geometry. I. Classical setting: line bundles and linear series*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 48, Springer, 2004.
[15] ———, *Positivity in algebraic geometry. II. Positivity for vector bundles, and multiplier ideals*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 49, Springer, 204.
[16] M. Lübke, "Stability of Einstein–Hermitian vector bundles", *Manuscr. Math.* **42** (1983), p. 245-257.
[17] V. B. Mehta, M. V. Nori, "Semistable sheaves on homogeneous spaces and abelian varieties", *Proc. Indian Acad. Sci., Math. Sci.* **93** (1984), no. 1, p. 1-12.
[18] S. Mukai, "Semi-homogeneous vector bundles on an abelian variety", *J. Math. Kyoto Univ.* **18** (1978), no. 2, p. 239-272.
[19] N. Nakayama, "Normalized Tautological divisors of semi-stable vector bundles", in *Free resolutions of coordinate rings of projective varieties and related topics*, Sūrikaisekikenkyūsho Kōkyūroku, vol. 1075, RIMS, Kyoto University, 1999, p. 167-173.
[20] M. Schneider, A. Tancredi, "Positive vector bundles on complex surfaces", *Manuscr. Math.* **50** (1985), p. 133-144.
[21] F. Serrano, "Strictly nef divisors and Fano threefolds", *J. Reine Angew. Math.* **464** (1995), p. 187-206.
[22] H. Umemura, "Some results in the theory of vector bundles", *Nagoya Math. J.* **52** (1973), p. 97-128.