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Pseudo-Riemannian Lie groups admitting left-invariant conformal vector fields

Groupes de Lie pseudo-riemanniens admettant des champs vectoriels conformes invariants à gauche

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Abstract. Let *G* be a Lorentzian Lie group or a pseudo-Riemannian Lie group of type (n - 2, 2). If *G* admits a non-Killing left-invariant conformal vector field, then *G* is solvable.

Résumé. Soit *G* un groupe de Lie lorentzien ou un groupe de Lie pseudo-riemannien de type (n - 2, 2). Si *G* admet un champ vectoriel invariant à gauche non-Killing, alors *G* est résoluble.

Keywords. Conformal vector fields, Killing vector fields, Pseudo-Riemannian Lie groups, Lorenztian Lie groups.

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1. Introduction

Let (M, g) be a pseudo-Riemannian manifold. The conformal transformation group of (M, g), denoted by Conf(M, g), is called *essential* if no metric in the conformal class of g is preserved by Conf(M, g). If (M, g) is Riemannian of dimension ≥ 2 , then Conf(M, g) is essential if and only if (M, g) is conformally diffeomorphic to the standard sphere \mathbb{S}^n or \mathbb{R}^n with the canonical flat metric. This is the famous *Lichnérowicz's Conjecture*, which was finally proved by J. Ferrand in [7]. However, in the pseudo-Riemannian case, the situation is quite different as shown in [1,8,11,14].

Thus it is meaningful to study the structure of a pseudo-Riemannian manifold (M, g) with Conf(M, g) essential, for example a homogeneous pseudo-Riemannian manifold with a non-Killing conformal vector field. Here a vector field X on (M, g) is said to be conformal, if

$$L_X g = 2\rho g, \tag{1}$$

where L_X is the Lie derivation and ρ is a smooth function on M called the conformal factor with respect to g. If g is a Riemannian metric, the existence of the function ρ might give some information about the topological structure of the Riemannian manifold (see [5, 12]). As a subclass of conformal vector fields, a *Yamabe soliton* vector field, i.e. a vector field X satisfies $L_X g = 2(\text{scal} - \lambda)g$ where scal is the scalar curvature of the metric g and λ is a constant, plays an important role in exploring Yamabe flow (see [2–4, 6, 9]). If $\rho = 0$, we call X a *Killing* vector field which provides a close link between the geometry of a manifold M and the algebra of I(M), where I(M) denotes the set of all isometries in (M, g) (see [13]).

Here we restrict (M, g) to be a pseudo-Riemannian Lie group which is a Lie group with a leftinvariant pseudo-Riemannian metric. All Lie groups are assumed to be connected. Furthermore the Lie group is called type (p, q) if the signature of the pseudo-Riemannian metric is of (p, q). As we know, there are some studies on Lorentzian Lie groups with non-Killing left-invariant conformal vector fields, i.e. pseudo-Riemannian Lie groups of type (n - 1, 1). For example the results in [2, 4, 15, 17].

The paper is organized as follows. First, we recall some facts on non-Killing left-invariant conformal vector fields on pseudo-Riemannian Lie groups in Section 2, and then prove the following Theorem in Section 3.

Theorem 1. Let G be a Lorentzian Lie group or a pseudo-Riemannian Lie group of type (n - 2, 2), where $n \ge 4$. If G admits a non-Killing left-invariant conformal vector field, then G is solvable.

But it is unknown for general type (p, q) for $p, q \ge 3$, and we conjecture that Theorem 1 holds for any type (p, q) for $p, q \ge 3$.

In Section 4, we construct a class of pseudo-Riemannian solvable Lie groups of type (p,q) which admit non-Killing left-invariant conformal vector fields, and then prove they are conformally flat. That is, they satisfy Lichnérowicz conjecture in the pseudo-Riemannian case.

2. Preliminaries

Let *G* be a Lie group with the Lie algebra \mathfrak{g} consisting of left-invariant vector fileds and let $\langle \cdot, \cdot \rangle$ be a pseudo-Riemannian metric on *G*. Assume that ∇ is the Levi-Civita connection associated with $\langle \cdot, \cdot \rangle$. Then

$$[X,Y] = \nabla_X Y - \nabla_Y X. \tag{2}$$

If $\langle \cdot, \cdot \rangle$ is left-invariant on *G*, we have

$$\langle \nabla_Z X, Y \rangle + \langle X, \nabla_Z Y \rangle = 0, \tag{3}$$

for any $X, Y, Z \in \mathfrak{g}$. By (2) and (3),

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2} (\langle [X, Y], Z \rangle - \langle [Y, Z], X \rangle + \langle [Z, X], Y \rangle), \tag{4}$$

where X, Y, Z are all left-invariant vector fields. Assume that $X \in \mathfrak{g}$ is a conformal vector field, by (1), we have

$$0 = L_X \langle X, X \rangle = 2\rho |X|^2.$$

Furthermore if $\langle \cdot, \cdot \rangle$ is Riemannian, then $\rho = 0$ or X = 0. That is, X is Killing or trivial.

For left-invariant pseudo-Riemannian metrics $\langle \cdot, \cdot \rangle$, we have

Lemma 2 ([2]). Let G be a unimodular pesudo-Riemannian Lie group. Then any left-invariant conformal vector field on G is a Killing vector field.

If *G* is a non-unimodular pseudo-Riemannian Lie group, we have the following result.

Lemma 3 ([2]). Let *G* be an *n*-dimensional non-unimodular pseudo-Riemannian Lie group of type (p,q). If \mathfrak{g} admits a non-Killing conformal vector field, then dim $C(\mathfrak{g}) \leq \min(p,q)$ and dim $[\mathfrak{g},\mathfrak{g}] \geq \dim \mathfrak{g} - \min(p,q)$.

Furthermore, for a Lorentzian Lie group, we have the following Lemma.

Lemma 4 ([15]). *Let G be a Lorentzian Lie group admitting a non-Killing left-invariant conformal vector field. Then* dim $[\mathfrak{g},\mathfrak{g}] = \dim \mathfrak{g} - 1$.

3. The proof of Theorem 1

In order to prove Theorem 1, we first recall two facts.

Lemma 5 ([10]). Let \mathfrak{g} be a Lie algebra over \mathbb{R} . If there is an invertible derivation on \mathfrak{g} , then \mathfrak{g} is nilpotent.

Lemma 6. For any matrix $H \in \mathfrak{so}(n, 1)$, either H has n-1 purely imaginary and two non-zero real eigenvalues $\pm r \in \mathbb{R}$, or H has n+1 purely imaginary eigenvalues. Here, we consider 0 as a purely imaginary number and $\mathfrak{so}(n, 1)$ is defined by

$$\mathfrak{so}(n,1) = \left\{ \begin{pmatrix} A & C \\ C' & 0 \end{pmatrix} : A = -A' \in \mathbb{R}^{n \times n}, C \in \mathbb{R}^{n \times 1} \right\},\$$

where A' denotes the transpose of A.

Let *G* be a pseudo-Riemannian Lie group whose Lie algebra is \mathfrak{g} and let *X* be a non-Killing left-invariant conformal vector field on *G*. Denote by $\langle \cdot, \cdot \rangle$ the pseudo-Riemannian metric of signature (p, q) on *G*. Clearly $\langle X, X \rangle = 0$. By the definition of a conformal vector field (1), we have

$$\langle [X, U], V \rangle + \langle U, [X, V] \rangle = -2\rho \langle U, V \rangle, \tag{5}$$

where $U, V \in \mathfrak{g}$ and $0 \neq \rho$ is a constant.

Lemma 7 ([2]). Let *G* be a Lorentzian Lie group admitting a non-Killing left-invariant conformal vector field. Then the restriction of $\langle \cdot, \cdot \rangle$ on $[\mathfrak{g}, \mathfrak{g}]$ is degenerate.

In fact, this lemma holds for any pseudo-Riemannian Lie group.

Lemma 8. Let G be a pseudo-Riemannian Lie group admitting a non-Killing left-invariant conformal vector field X. Then the restriction $\langle \cdot, \cdot \rangle$ on $[\mathfrak{g}, \mathfrak{g}]$ is degenerate.

Proof. Assume that the restriction $\langle \cdot, \cdot \rangle$ on $[\mathfrak{g}, \mathfrak{g}]$ is non-degenerate. Then the restriction $\langle \cdot, \cdot \rangle$ on $[\mathfrak{g}, \mathfrak{g}]^{\perp}$ is non-degenerate. Thus there is a vector field $U \in [\mathfrak{g}, \mathfrak{g}]^{\perp}$ such that $\langle U, U \rangle \neq 0$. By (5),

$$0 = \langle [X, U], U \rangle + \langle U, [X, U] \rangle = -2\rho \langle U, U \rangle.$$

Thus $\rho = 0$, it is a contradiction. So the restriction $\langle \cdot, \cdot \rangle$ on $[\mathfrak{g}, \mathfrak{g}]$ is degenerate.

Proposition 9. Assume that G is a pseudo-Riemannian Lie group of type (p, q). If G admits a non-Killing left-invariant conformal vector field and dim $[g,g] = \dim g - \min(p,q)$, then G is solvable.

 \Box

Proof. Assume that $X \in \mathfrak{g}$ is a non-Killing conformal vector filed. Let $V_0 = \{y \in \mathfrak{g} \mid adX(y) = 0\}$. By (5), for any $u, v \in V_0$ we have

$$0 = \langle adX(u), v \rangle + \langle u, adX(v) \rangle = -2\rho \langle u, v \rangle.$$

Since $\rho \neq 0$, it follows $\langle u, v \rangle = 0$ for any $u, v \in V_0$. So V_0 is an isotropy subspace of $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$. Let $k = \dim V_0$. Since $adX(\mathfrak{g}) \subset [\mathfrak{g}, \mathfrak{g}]$, we have

$$k \ge \dim \mathfrak{g} - \dim[\mathfrak{g}, \mathfrak{g}] = \min(p, q)$$

Let *m* denote the multiplicity of the eigenvalue 0 of adX. Clearly $m \ge k$. We claim k = m. Otherwise, m > k. By the Jordan canonical form theory for a nilpotent matrix, there are non-zero vectors $w \in \mathfrak{g}, v_0 \in V_0$ satisfying $adX(w) = v_0$. Obviously, $w \notin V_0$. Since V_0 is an isotropy subspace, for any $v \in V_0$, we have

$$0 = \langle adX(w), v \rangle + \langle w, adX(v) \rangle = -2\rho \langle w, v \rangle,$$

which implies $\langle w, v \rangle = 0$ since $\rho \neq 0$. It follows that

$$0 = \langle adX(w), w \rangle + \langle w, adX(w) \rangle = -2\rho \langle w, w \rangle,$$

which implies $\langle w, w \rangle = 0$. Then we have an isotropy subspace of dimension $\geq \min(p, q) + 1$ spanned by V_0 and w, which is impossible. That is, k = m.

If adX isn't invertible on $[\mathfrak{g},\mathfrak{g}]$, then we have

$$k = m \ge \min(p, q) + 1.$$

Namely dim $V_0 \ge \min(p, q) + 1$. It is a contradiction since V_0 is an isotropy subspace of $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$. Hence adX must be invertible on $[\mathfrak{g}, \mathfrak{g}]$. By Lemma 5, we know that $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent which implies the solvability of \mathfrak{g} .

The proof of Theorem 1. For the Lorentzian case, by Lemma 4, $\dim[\mathfrak{g},\mathfrak{g}] = \dim \mathfrak{g} - 1$. Thus *G* is solvable by Proposition 9.

For the pseudo-Riemannian Lie group of type (n - 2, 2), by Lemma 3, dim $[\mathfrak{g}, \mathfrak{g}] \ge \dim \mathfrak{g} - 2$. By Lemma 2, we must have dim $[\mathfrak{g}, \mathfrak{g}] = \dim \mathfrak{g} - 2$ or dim $[\mathfrak{g}, \mathfrak{g}] = \dim \mathfrak{g} - 1$. If dim $[\mathfrak{g}, \mathfrak{g}] = \dim \mathfrak{g} - 2$, then *G* is solvable by Proposition 9. If dim $[\mathfrak{g}, \mathfrak{g}] = n - 1$, by Lemma 8, there is a basis $\{e_1, e_2, \dots, e_{n-1}, e_n\}$ of \mathfrak{g} such that $[\mathfrak{g}, \mathfrak{g}] = \operatorname{span}\{e_1, e_2, \dots, e_{n-1}\}$ and the metric matrix associated with this basis is defined by

$$\begin{pmatrix} I_{n-3} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$
 (6)

Set $adX(e_j) = \sum_{i=1}^n a_{ij}e_i, a_{ij} \in \mathbb{R}, 1 \le i, j \le n$. By (5), we know the matrix of adX associated with this basis is represented by

$$\begin{pmatrix} H-\rho I_{n-3} & \mathbf{0} & \boldsymbol{\alpha} \\ \boldsymbol{\beta} & -2\rho & 0 \\ \mathbf{0}' & 0 & 0 \end{pmatrix}.$$

where $H \in \mathfrak{so}(n-3,1)$, $\boldsymbol{\alpha} \in \mathbb{R}^{(n-2)\times 1}$ and $\boldsymbol{\beta} \in \mathbb{R}^{1\times (n-2)}$. By Lemma 6, we know the eigenvalues of $(adX)|_{[\mathfrak{g},\mathfrak{g}]}$ are of the forms:

$$-\rho, -2\rho, -\rho \pm \lambda, -\rho + \mathbf{i}a \ (0 \neq a \in \mathbb{R}).$$

If $\lambda \neq \pm \rho$, then $(adX)|_{[\mathfrak{g},\mathfrak{g}]}$ is invertible. By Lemma 5, $[\mathfrak{g},\mathfrak{g}]$ is nilpotent which implies the solvability of \mathfrak{g} . If $\lambda = \rho$ or $\lambda = -\rho$, then the eigenvalues of $(adX)|_{[\mathfrak{g},\mathfrak{g}]}$ are of the forms:

 $-\rho$, -2ρ (of multiplicity 2), 0 (of multiplicity 1), $-\rho + ia$ ($0 \neq a \in \mathbb{R}$).

Assume that $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}$ is a Levi decomposition of \mathfrak{g} with $\mathfrak{s} \neq 0$. Let $X = X_{\mathfrak{s}} + X_{\mathfrak{r}}$ be the corresponding decomposition of *X*. Then we have

$$adX = \begin{pmatrix} (adX_{\mathfrak{s}})|_{\mathfrak{s}} & 0\\ * & (adX)|_{\mathfrak{r}} \end{pmatrix} : \quad \mathfrak{s} \ltimes \mathfrak{r} \to \mathfrak{s} \ltimes \mathfrak{r}.$$

In particular, the eigenvalues of $(adX_{\mathfrak{s}})|_{\mathfrak{s}}$ would be also eigenvalues of adX. It contradicts to $\operatorname{tr}(adX_{\mathfrak{s}})|_{\mathfrak{s}}) = 0$. So $\mathfrak{s} = 0$, i.e. \mathfrak{g} is solvable.

That is, we have Theorem 1.

In general, we have the following conjecture.

Conjecture 10. Let G be a pseudo-Riemannian Lie group of type (p, q) where $p, q \ge 3$. If G admits a non-Killing left-invariant conformal vector field, then G is solvable.

4. Conformally flat pseudo-Riemannian Lie groups

The following example generalizes the Lorentzian case in [2] to type (p, q).

Example 11. Consider the Lie algebra g defined by

$$[e_n, e_i] = -\rho e_i, \ 1 \le i \le n-2, \ [e_n, e_{n-1}] = -2\rho e_{n-1}, \tag{7}$$

where $\{e_1, e_2, ..., e_n\}$ is a basis of \mathfrak{g} , and ρ is a non-zero constant. Clearly, \mathfrak{g} is a non-umimodular solvable Lie algebra with abelian derived algebra $[\mathfrak{g}, \mathfrak{g}] = \operatorname{span}\{e_1, e_2, ..., e_{n-1}\}$. Define an inner product $\langle \cdot, \cdot \rangle$ of signature (p, q) $(p, q \ge 1)$ on \mathfrak{g} associated with the basis $\{e_1, e_2, ..., e_n\}$ by

$$\begin{pmatrix} I_{p-1} & & \\ & -I_{q-1} & \\ & & 0 & 1 \\ & & & 1 & 0 \end{pmatrix}.$$
 (8)

Let *G* denote the simply connected Lie group with the Lie algebra \mathfrak{g} , and we also use the symbol $\langle \cdot, \cdot \rangle$ to denote the induced left-invariant pseudo-Riemannian metric on *G*. Then $X \in \mathfrak{g}$ is a non-Killing conformal vector field on $(G, \langle \cdot, \cdot \rangle)$ if and only if

$$\langle [X, e_i], e_j \rangle + \langle e_i, [X, e_j] \rangle = -2c \langle e_i, e_j \rangle, \tag{9}$$

where $1 \le i, j \le n$, and *c* is a non-zero constant. By a straightforward computation, we know $X = e_n$ is a left-invariant non-Killing conformal vector field on $(G, \langle \cdot, \cdot \rangle)$ satisfying

$$L_X\langle\cdot,\cdot\rangle = 2\rho\langle\cdot,\cdot\rangle. \tag{10}$$

The following is to prove that $(G, \langle \cdot, \cdot \rangle)$ is conformally flat. We first recall some definitions and a theorem of Weyl. For a pseudo-Riemannian manifold (M, g), denote by ∇ , R, Ric and scal the Levi-Civita connection, the Riemann curvature tensor, the Ricci tensor and the scalar curvature respectively. For symmetric (0,2)-type tensor fields h, k on (M, g), define the *Kulkarni–Nomizu* product as the (0,4)-type tensor field by

$$h \circ k(v_1, v_2, v_3, v_4) = \frac{1}{2} (h(v_1, v_4)k(v_2, v_3) + h(v_2, v_3)k(v_1, v_4)) - \frac{1}{2} (h(v_1, v_3)k(v_2, v_4) + h(v_2, v_4)k(v_1, v_3)).$$

The *Schouten tensor* for n > 2 is given by

$$P = \frac{2}{n-2} \operatorname{Ric} - \frac{\operatorname{scal}}{(n-1)(n-2)} \cdot g,$$
(11)

and the Weyl conformal curvature tensor W is defined by

$$R = P \circ g + W,\tag{12}$$

where *R* is the (0,4)-type Riemann curvature tensor and \circ is the Kulkarni–Nomizu product. The following result of Weyl is well-known.

Theorem 12 ([16]). A pseudo-Riemannian manifold M of dimension ≥ 4 is conformally flat if and only if the Weyl conformal curvature tensor W vanishes identically.

Let the notations as Example 11 for $n \ge 4$. Denote by *B* the left-invariant pseudo-Riemannian metric $\langle \cdot, \cdot \rangle$ and denote by ∇ the Levi-Civita connection as usual. Let

$$B_{ij} = B(e_i, e_j), [e_i, e_j] = C_{ij}{}^k e_k, \nabla_{e_i} e_j = \Gamma_{ji}{}^k e_k,$$

$$R(e_i, e_j) e_k = R_{ijk}{}^l e_l, R_{ijkl} = R_{ijk}{}^s B_{sl}, H_{ijkl} = (P \circ B)(e_i, e_j, e_k, e_l).$$

Using the Koszul's formula, we have

$$\Gamma_{ij}{}^{k} = \frac{1}{2} (-C_{ij}{}^{s}B_{st} - C_{jt}{}^{s}B_{si} + C_{ti}{}^{s}B_{sj})B^{tk}.$$

where (B^{ij}) denotes the inverse matrix of (B_{ij}) . Furthermore, we have

$$R_{ijk}^{\ \ l} = \Gamma_{kj}^{\ \ s} \Gamma_{si}^{\ \ l} - \Gamma_{ki}^{\ \ s} \Gamma_{sj}^{\ \ l} - C_{ij}^{\ \ s} \Gamma_{ks}^{\ \ l}$$

By the definition of \mathfrak{g} in Example 11, the non-zero C_{ij}^{k} are

$$C_{in}{}^{i} = \rho = -C_{ni}{}^{i}, i \le n-2; C_{n-1,n}{}^{n-1} = 2\rho = -C_{n,n-1}{}^{n-1}$$

Set $\varepsilon_i = B(e_i, e_i) \in \{\pm 1\}$ for $i \le n-2$. By a straightforward computation, the non-zero Γ_{ij}^k are

$$\Gamma_{ii}^{n-1} = -\rho \varepsilon_i, \ \Gamma_{ni}^{i} = \rho, \ i \le n-2; \Gamma_{n-1,n}^{n-1} = -2\rho, \ \Gamma_{nn}^{n} = 2\rho,$$

and by the symmetry of R corresponding to subscript, the fundamental non-zero R_{ijk}^{l} are

$$R_{ini}^{n-1} = -\rho^2 \varepsilon_i, \ R_{inn}^i = \rho^2, \ i \le n-2.$$

and consequently the fundamental non-zero R_{ijkl} are

$$R_{inni} = \rho^2 \varepsilon_i, \ i \le n-2.$$

In particular, the Ricci tensor is

$$(\operatorname{Ric}(e_i, e_j)) = \begin{pmatrix} 0 \cdots 0 & 0 \\ \vdots \ddots \vdots & \vdots \\ 0 \cdots 0 & 0 \\ 0 \cdots 0 & (n-2)\rho^2 \end{pmatrix},$$

and the scalar curvature vanishes. Then by (11), we know that the Schouten tensor $P = \frac{2}{n-2}$ Ric, and the fundamental nonzero $H_{ijkl} = (P \circ B)(e_i, e_j, e_k, e_l)$ are

$$H_{inni} = \rho^2 \varepsilon_i, \ i \le n-2.$$

Since the Weyl conformal curvature $W_{ijkl} = R_{ijkl} - H_{ijkl}$, by Theorem 12, we know that $(G, \langle \cdot, \cdot \rangle)$ is conformally flat.

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