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The K-theory of the conjugation action

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Abstract. In 1999, Brylinski and Zhang computed the complex equivariant K-theory of the conjugation self-action of a compact, connected Lie group with torsion-free fundamental group. In this note we show it is possible to do so in under a page.

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Brylinski and Zhang [2] showed that if G is a compact, connected Lie group with torsion-free fundamental group, then the equivariant K-theory of its conjugation action G^{Ad} is isomorphic to the ring $\Omega^*_{RG/\mathbb{Z}}$ of Grothendieck differentials on the complex representation ring RG of G. Their proof uses results on holomorphic differentials on complex manifolds, a reduction to the case G is a torus, and some algebraic geometry. We show a more concrete and arguably more natural expression for the ring $K^*_G(G^{\mathrm{Ad}})$ can be obtained rapidly using only Hodgkin's Künneth spectral sequence [5], in the same manner they already use it, and elementary algebraic considerations. We then show this purely algebraic isomorphism admits a satisfying geometric interpretation, and remark finally that this geometric version gives back Brylinski and Zhang's description in terms of Grothendieck differentials at no added cost.

Theorem (Brylinski–Zhang [2, Thm. 3.2]). *Let G be a compact, connected Lie group with torsion-free fundamental group. Then $K^*_G(G^{\mathrm{Ad}})$ is isomorphic to $RG \otimes K^*G$ as an RG-algebra. Under this identification, the forgetful map $f\colon K^*_G(G^{\mathrm{Ad}}) \longrightarrow K^*G$ becomes reduction with respect to the augmentation ideal IG of RG.*

Proof. Write G^{bi} for G under the $(G \times G)$-action $(h,k)\cdot g = hgk^{-1}$. The orbit space of $G^{\mathrm{bi}} \times G^{\mathrm{bi}}$ under the restricted, free diagonal action of $1 \times G$ is $(G\times 1)$-equivariantly diffeomorphic to G^{Ad} via $(g', g) \longmapsto g'g^{-1}$, so when $X = Y = G^{\mathrm{bi}}$, Hodgkin's $(\mathbb{Z}\times\mathbb{Z}/2)$-graded Künneth spectral sequence $\mathrm{Tor}_{R(G\times G)}(K^*_{G\times G}X, K^*_{G\times G}Y) \implies K^*_{G\times G}(X \times Y)$ reduces to $\mathrm{Tor}_{RG\otimes RG}(RG, RG) \implies K^*_G(G^{\mathrm{Ad}})$. Here the two structure maps $RG \otimes RG \longrightarrow RG$ are the both the multiplication of RG. Recall [5, Prop. 11.1] that under our hypotheses, RG is the tensor product of a polynomial ring on generators $y_i \in IG$ and a Laurent polynomial ring on generators $t_j \in 1 + IG$. Let P be the free abelian group on generators q_i and w_j and let $\gamma\colon P \longrightarrow IG$ be the linear map taking q_i to y_i and w_j to $t_j - 1$. Then an $(RG)^{\otimes 2}$-module resolution of RG is given by $RG \otimes \Lambda P \otimes RG$, with differential the derivation vanishing on $RG \otimes \mathbb{Z} \otimes RG$ and sending $1 \otimes z \otimes 1$, for $z \in P$, to $1 \otimes 1 \otimes \gamma(z) - \gamma(z) \otimes 1 \otimes 1$. To compute the Tor, apply $- \otimes_{RG\otimes RG} RG$ to this resolution to obtain the CDGA $RG \otimes \Lambda P$ with 0 differential, RG in bidegree $(0,0)$, and P in bidegree $(-1,0)$.

The spectral sequence collapses because the differentials d_r for $r \geq 2$ send all generators into the right half-plane. Since $\pi_1 G$ is torsion-free and $X = G$ is locally contractible of finite covering dimension, the spectral sequence strongly converges to the intended target. Hence $RG \otimes \Lambda P$ is the graded algebra associated to a filtration $(F_p)_{p \leq 0}$ of $K_G^*(G^{\mathrm{Ad}})$ with $F_0 \cong RG$ and $F_{-1}/F_0 \cong RG \otimes P$. Since RG and ΛP are both free abelian, there is no additive extension problem, so $K_G^*(G^{\mathrm{Ad}})$ is also free abelian as a group. Let $\tilde{z}_k$ be elements in F_{-1} lifting $1 \otimes q_i$ and $1 \otimes w_j$ under the isomorphism $F_{-1}/F_0 \cong RG \otimes P$. Then the $\tilde{z}_k$ anticommute with each other because they lie in $K_G^1(G^{\mathrm{Ad}})$ and square to 0 since $K_G^*(G^{\mathrm{Ad}})$ contains no 2-torsion, and by induction, they generate $K_G^*(G^{\mathrm{Ad}})$ as an RG-algebra, so $K_G^*(G^{\mathrm{Ad}}) = RG \otimes \Lambda[\tilde{z}_k]$.

To see the forgetful map f is as claimed, note that forgetting the $(G \times 1)$-action on G^{bi} induces a map to the spectral sequence $\mathrm{Tor}_{RG}(\mathbb{Z}, \mathbb{Z}) \Longrightarrow K^* G$, which again collapses by lacunary considerations. Computing $\mathrm{Tor}_{RG}(\mathbb{Z}, \mathbb{Z}) \cong \Lambda P$ with the resolution $\Lambda P \otimes RG$ of $\mathbb{Z}$ shows the map $E_2(f)\colon RG \otimes \Lambda P \longrightarrow \Lambda P$ is reduction modulo IG and $\Lambda P \cong K^* G$. □

Remark. We can be completely explicit about the exterior generators. As observed by Hodgkin [4, Thm. A], the injection $\mathrm{U}(n) \rightarrowtail \mathrm{U} := \varinjlim \mathrm{U}(n)$ induces an additive map $\beta\colon RG \longrightarrow K^1 G$ descending to a group isomorphism between the module $IG/(IG)^2$ of indecomposables of RG and the module $PK^* G$ of primitives of the exterior Hopf algebra $K^* G \cong \Lambda PK^* G$. In particular, a set of generators is given by $\beta(\lambda_i) = \beta(\lambda_i - \dim \lambda_i)$ for λ_i lifts in G of the fundamental representations of the commutator subgroup G' and $\beta(t_j) = \beta(t_j - 1)$ for $t_j\colon G \to G/G' \twoheadrightarrow \mathrm{U}(1)$ circular coordinate functions of the torus $G^{\mathrm{ab}} = G/G' \cong (S^1)^{\mathrm{rk}\, G - \mathrm{rk}\, G'}$. Let $Q = \{\lambda_i, t_j\}_{i,j}$.

The map β in fact factors as $f \circ \beta^{\mathrm{Ad}}$ for a map $\beta^{\mathrm{Ad}}\colon RG \longrightarrow K_G^*(G^{\mathrm{Ad}})$, already giving surjectivity of f since $PK^* G$ generates $K^* G = \Lambda PK^* G$ as a ring. Atiyah [1, Lem. 2, pf.] described β^{Ad} and hence β geometrically: given a representation $\rho\colon G \longrightarrow \mathrm{U}(n)$, we can build a representative E of $\beta(\rho)$ via the clutching construction, taking with two trivial bundles $CG \times \mathbb{C}^n$ over the cone CG on G and gluing them along $G \times \mathbb{C}^n$ via the relation $(g, v) \sim \big(g, \rho(g)v\big)$ to obtain a bundle over the suspension $CG \cup_G CG$. The action $h \cdot (g, v) = \big(hgh^{-1}, \rho(h)v\big)$ of G on $G^{\mathrm{Ad}} \times \mathbb{C}^n$ preserves this relation and so induces a G-action on E making it a G-equivariant bundle over the suspension of G^{Ad}.

The RG-module structure on $K_G^*(X)$ is always given by $\sigma \cdot [E] = \big[\alpha(\sigma) \otimes E\big]$, where if $\sigma\colon G \longrightarrow \mathrm{Aut}\, V$ is a representation, then $\alpha(\sigma)$ is the trivial bundle $X \times V$ equipped with the diagonal G-action. Thus $\sigma \otimes \prod_k \beta(\rho_k) \longmapsto \alpha(\sigma) \cdot \prod_k \beta^{\mathrm{Ad}}(\rho_k)$ for $\rho_k \in Q$ gives an explicit isomorphism $RG \otimes K^* G \xrightarrow{\sim} K_G^*(G^{\mathrm{Ad}})$.

Remark. The first paragraph of our proof is a variant of Brylinski–Zhang's §4 [2]. Once the ring structure is determined as in our proof's second paragraph, replacing §§5–6, one knows their map $\phi\colon \Omega^*_{RG/\mathbb{Z}} \longrightarrow K_G^*(G^{\mathrm{Ad}})$ from the ring of Grothendieck differentials is an isomorphism as soon as one knows it is a well-defined RG-algebra map [2, Prop 3.1], for the class $\beta^{\mathrm{Ad}}(\rho) \in K_G^1(G^{\mathrm{Ad}})$ from the previous remark is in fact the same as[1] Brylinski–Zhang's $\phi(d\rho)$, so that $f \circ \phi$ takes a basis $\{d\rho : \rho \in Q\}$ of the free RG-module $\Omega^1_{RG/\mathbb{Z}}$ to a $\mathbb{Z}$-basis $\big\{\beta(\rho) : \rho \in Q\big\}$ of $PK^* G$.

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[1] the corrected version of—see Fok [3, Rmk. 2.8.1]