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 $L \exp(\mu_0 \sqrt{2 \log(1+L)})$ -integrable terminal values: an  $L^1$ -solution  
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Probability theory / *Probabilités*

# Uniqueness of solution to scalar BSDEs with $L \exp(\mu_0 \sqrt{2 \log(1 + L)})$ -integrable terminal values: an $L^1$ -solution approach

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**Abstract.** This paper deals with a class of scalar backward stochastic differential equations (BSDEs) with  $L \exp(\mu_0 \sqrt{2 \log(1 + L)})$ -integrable terminal values for a critical parameter  $\mu_0 > 0$ . We show that the solution of these BSDEs is closely connected to the  $L^1$ -solution of the BSDEs with integrable parameters. The key tool is the Girsanov theorem. This idea leads to a new approach to the uniqueness of solution and we obtain a new existence and uniqueness result under general assumptions.

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## 1. Introduction

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a complete probability space,  $T > 0$  a finite time and  $W$  a standard  $d$ -dimensional Brownian motion. Let  $\mathbb{F} := \{\mathcal{F}_t\}_{0 \leq t \leq T}$  be a completion of the augmented filtration generated by the Brownian motion  $W$ . We consider the following backward stochastic differential equation (BSDE for short):

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad t \in [0, T], \quad (1)$$

where the generator  $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^{1 \times d} \rightarrow \mathbb{R}$  is a predictable function and the terminal value  $\xi$  is an  $\mathcal{F}_T$ -measurable random variable. The solution of (1) is denoted by a pair  $\{(Y_t, Z_t), t \in [0, T]\}$  of predictable processes with values in  $\mathbb{R} \times \mathbb{R}^{1 \times d}$  such that  $\mathbb{P}$ -a.s.,  $t \mapsto Y_t$  is continuous,  $t \mapsto Z_t$  belongs to  $L^2(0, T)$  and  $t \mapsto f(t, Y_t, Z_t)$  is integrable, and  $\mathbb{P}$ -a.s.,  $(Y, Z)$  verifies (1).

Pardoux and Peng [12] first introduced the notion of the nonlinear BSDEs and proved the existence and uniqueness of  $L^2$ -solutions. Afterwards, Briand et al. [1] generalized this result in  $L^p$ -integrability setting. Hence, they proved the existence and uniqueness of  $L^p$ -solutions of

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BSDEs. However, in the case of  $p = 1$ , one needs to restrict the generator  $f$  to satisfy the sub-linear growth assumption in  $z$ , i.e., with some  $q \in [0, 1)$ ,

$$|f(t, y, z) - f(t, 0, 0)| \leq a|y| + b|z|^q, \quad (t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^{1 \times d}. \tag{2}$$

for BSDE (1) to have a unique  $L^1$ -solution.

Recently, Hu and Tang [10] studied the scalar BSDEs in  $L\exp(\mu\sqrt{2\log(1+L)})$ -integrability setting with  $\mu > \mu_0$  for a critical value  $\mu_0 = b\sqrt{T}$ . Hence, the terminal value is assumed to be  $L\exp(\mu\sqrt{2\log(1+L)})$ -integrable. This integrability is stronger than  $L\log L$ -integrability and weaker than  $L^p$ -integrability for any  $p > 1$ . They showed that these BSDEs admit a solution without the sub-linear growth condition (2). Furthermore they provided counterpart examples which show that  $L\log L$ -integrability is not sufficient to guarantee the existence of the solution without (2). Afterwards, Buckdahn, Hu and Tang [3] improved the existence result in [10], and gave the uniqueness result for these BSDEs under the Lipschitz condition on generator. A remarkable result for these BSDEs was established by Fan and Hu [8]. They proved the existence and uniqueness of solution to scalar BSDEs with  $L\exp(\mu\sqrt{2\log(1+L)})$ -integrable terminal values in the critical case:  $\mu = \mu_0$ . However, the generator is still assumed to be Lipschitz. Note that if  $\mu < \mu_0$ , then the BSDE does not admit a solution in general (see [10]).

In [3, 8], the authors uses the classical linearization technique to prove the uniqueness result. Hence, the Lipschitz condition (or more generally, monotonicity condition) plays a crucial role in their argument.

In this paper, we prove a new existence and uniqueness result for scalar BSDEs with  $L\exp(\mu_0\sqrt{2\log(1+L)})$ -integrable terminal values under general assumptions. More precisely, we assume that the generator satisfies weak monotonicity and general growth conditions. The existence result under our assumptions can be just proved by following exactly the same method as in [8]. In fact, this observation was already mentioned in [8, Remark 3.7]. For the reader's convenience, we provide a detailed proof of the existence result. The main focus in this paper is on to prove the uniqueness result. Since our generator is neither Lipschitz nor monotonic, the linearization technique in [3, 8] does not work anymore. To overcome this difficulty, we propose an  $L^1$ -solution approach to the BSDEs with  $L\exp(\mu_0\sqrt{2\log(1+L)})$ -integrable terminal values. More precisely, we associate the solution of these BSDEs with the  $L^1$ -solutions of certain BSDEs with integrable parameters. The key tool is the Girsanov theorem. Then, we can easily obtain the uniqueness result by using the known result on  $L^1$ -solutions. Our approach does not require the linearization argument so that the Lipschitz condition is not needed. With the help of  $L^1$ -solution approach, we also show that the second component of solution belongs to  $\bigcap_{\beta \in (0, 1)} M^\beta([0, T])$  (see Theorem 10). As far as we know, this fact was not observed in the literature.

## 2. Preliminaries

### 2.1. Notations and assumptions

For  $p \geq 1$ , we define the following spaces.

- $S^p([0, T]; \mathbb{Q})$  denotes the space of all real-valued, càdlàg, adapted processes  $Y$  such that

$$\mathbb{E}^{\mathbb{Q}} \left[ \sup_{t \in [0, T]} |Y_t|^p \right] < +\infty,$$

and  $S^p([0, T]) := S^p([0, T]; \mathbb{P})$ .

- $M^p([0, T]; \mathbb{Q})$  denotes the space of all predictable processes  $Z$  valued in  $\mathbb{R}^{1 \times d}$  such that

$$\mathbb{E}^{\mathbb{Q}} \left[ \left( \int_0^T |Z_s|^2 ds \right)^{p/2} \right]^{1 \wedge 1/p} < +\infty,$$

and  $M^P([0, T]) := M^P([0, T]; \mathbb{P})$ .

We also use the following notations.

- For  $A \in \mathcal{F}$ ,  $\mathcal{F}$ -measurable random variable  $\eta$  and probability measure  $\mathbb{Q}$ , we define  $\mathbb{E}^{\mathbb{Q}}[\eta; A] := \int_A \eta d\mathbb{Q}$ . In particular,  $\mathbb{E}^{\mathbb{Q}}[\eta] := \mathbb{E}^{\mathbb{Q}}[\eta; \Omega]$ .
- $\mathcal{T}[0, T]$  denotes the set of all stopping times  $\tau$  such that  $0 \leq \tau \leq T$ .
- $\mathcal{E}(\cdot)$  stands for Doléans–Dade exponential. Hence, for any  $\mathbb{R}^{1 \times d}$ -valued predictable process  $\phi$ ,  $\mathcal{E}(\phi \bullet W) := \exp(\int_0^\cdot \phi_r dW_r - \frac{1}{2} \int_0^\cdot |\phi_r|^2 dr)$ , where  $\phi \bullet W := \int_0^\cdot \phi_r dW_r$ .
- We say that the process  $Y = \{Y_t\}_{0 \leq t \leq T}$  belongs to class (D) if the family  $\{Y_\tau, \tau \in \mathcal{T}[0, T]\}$  is uniformly integrable.
- Following [1], we say that a pair  $(Y, Z)$  is an  $L^1$ -solution to the BSDE (1) if the equation (1) holds for any  $t \in [0, T]$ ,  $Y$  belongs to class (D) and for each  $\beta \in (0, 1)$ ,  $(Y, Z)$  belongs to the space  $S^\beta([0, T]) \times M^\beta([0, T])$ .
- We denote by  $I_A(\cdot)$  the indicator of set  $A$ , and  $\text{sgn}(x) := \mathbf{1}_{x>0} - \mathbf{1}_{x \leq 0}$ .
- $x_1 \wedge x_2 := \min\{x_1, x_2\}$ ,  $x_1 \vee x_2 := \max\{x_1, x_2\}$ ,  $x^- := -(x \wedge 0)$ , and  $x^+ := (-x)^-$ .

We work under the following assumptions on generator  $f$ .

- (A1)  $f$  is weakly monotonic in  $y$ , that is, there exists a non-decreasing and concave function  $\rho : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  with  $\rho(0) = 0$ ,  $\rho(t) > 0$  for  $t > 0$  and  $\int_0^+ \frac{dt}{\rho(t)} = +\infty$  such that for any  $y, y' \in \mathbb{R}$  and  $z \in \mathbb{R}^{1 \times d}$ ,

$$\text{sgn}(y - y') (f(t, y, z) - f(t, y', z)) \leq \rho(|y - y'|). \tag{3}$$

- (A2)  $f$  has a general growth with respect to  $y$ , i.e.,  $d\mathbb{P} \times dt$ -a.e.,

$$\forall y \in \mathbb{R}, \quad |f(t, y, 0)| \leq |f(t, 0, 0)| + \Psi(|y|),$$

where  $\Psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is an increasing, continuous function. Furthermore, the map  $y \mapsto f(t, y, z)$  is  $d\mathbb{P} \times dt$ -a.e., continuous.

- (A3)  $f$  is uniformly Lipschitz in  $z$ , i.e., there exists a constant  $b > 0$  such that for any  $y \in \mathbb{R}$  and  $z, z' \in \mathbb{R}^{1 \times d}$ ,

$$|f(t, y, z) - f(t, y, z')| \leq b|z - z'|.$$

**Remark 1.** Since  $\rho(\cdot)$  is a non-decreasing and concave function with  $\rho(0) = 0$ , it increases at most linearly, i.e., there exists a constant  $A > 0$  such that  $\rho(x) \leq A(1 + x)$  for each  $x \geq 0$ . Therefore, (A1) and (A3) imply the following one-sided linear growth condition:

$$\text{sgn}(y) f(t, y, z) \leq h(t) + A|y| + b|z|, \tag{4}$$

with  $h(t) := A + |f(t, 0, 0)|$ . If  $\rho(x) = \lambda x$  for some constant  $\lambda > 0$ , then (A1) implies the usual monotonicity condition. On the other hand, our weak monotonicity condition (3) is stronger than the standard weak monotonicity condition:

$$(y - y') (f(t, y, z) - f(t, y', z)) \leq \rho(|y - y'|^2),$$

(see [9, Remark 2.2] or [5, Proposition 1]).

## 2.2. Some auxiliary results

**Lemma 2.** *Let us consider the following BSDE with generator that depends only on  $y$ , i.e.,*

$$Y_t = \xi + \int_t^T f(s, Y_s) ds - \int_t^T Z_s dW_s, \quad t \in [0, T]. \tag{5}$$

*Let  $(Y, Z)$  be a solution of the BSDE (5) such that  $Y$  belongs to class (D). Suppose that*

$$\text{sgn}(y) f(t, y) \leq h(t) + A|y|, \quad \mathbb{E} \left[ |\xi| + \int_0^T |h(s)| ds \right] < \infty.$$

Then we have  $(Y, Z) \in S^\beta([0, T]) \times M^\beta([0, T])$  for each  $\beta \in (0, 1)$ .

**Proof.** We first prove the following estimate on  $Y$ :

$$|Y_t| \leq e^{A(T-t)} \cdot \mathbb{E} \left[ |\xi| + \int_t^T |h(s)| ds \middle| \mathcal{F}_t \right]. \tag{6}$$

Define the stopping times  $\tau_n$  by

$$\tau_n := \inf \left\{ t \in [0, T], \int_0^t |Z_s|^2 ds \geq n \right\} \wedge T.$$

According to [1, Corollary 2.3], we get

$$\begin{aligned} |Y_{t \wedge \tau_n}| &\leq |Y_{\tau_n}| + \int_{t \wedge \tau_n}^{\tau_n} \operatorname{sgn}(Y_s) f(s, Y_s) ds - \int_{t \wedge \tau_n}^{\tau_n} \operatorname{sgn}(Y_s) Z_s dW_s \\ &\leq |Y_{\tau_n}| + \int_{t \wedge \tau_n}^{\tau_n} [A|Y_s| + |h(s)|] ds - \int_{t \wedge \tau_n}^{\tau_n} \operatorname{sgn}(Y_s) Z_s dW_s. \end{aligned}$$

By taking conditional expectations, it follows that for each  $0 \leq u \leq t \leq T$ ,

$$\mathbb{E} [ |Y_{t \wedge \tau_n}| \middle| \mathcal{F}_u ] \leq \mathbb{E} \left[ |Y_{\tau_n}| + \int_{t \wedge \tau_n}^{\tau_n} [A|Y_s| + |h(s)|] ds \middle| \mathcal{F}_u \right]. \tag{7}$$

Since  $Y$  belongs to class (D), in view of  $Y_{t \wedge \tau_n} \rightarrow Y_t$ ,  $Y_{\tau_n} = Y_{T \wedge \tau_n} \rightarrow Y_T = \xi$ , it follows that  $\mathbb{E}[Y_{t \wedge \tau_n} | \mathcal{F}_u] \rightarrow \mathbb{E}[Y_t | \mathcal{F}_u]$  and  $\mathbb{E}[Y_{\tau_n} | \mathcal{F}_u] \rightarrow \mathbb{E}[\xi | \mathcal{F}_u]$ . Therefore, by sending  $n$  to  $+\infty$  in (7) (extracting a subsequence if necessary), we obtain

$$\begin{aligned} \mathbb{E} [ |Y_t| \middle| \mathcal{F}_u ] &\leq \mathbb{E} \left[ |\xi| + \int_t^T |h(s)| ds + A \int_t^T |Y_s| ds \middle| \mathcal{F}_u \right] \\ &= \mathbb{E} [ |\xi| \middle| \mathcal{F}_u ] + \mathbb{E} \left[ \int_t^T |h(s)| ds \middle| \mathcal{F}_u \right] + A \int_t^T \mathbb{E} [ |Y_s| \middle| \mathcal{F}_u ] ds. \end{aligned}$$

By the virtue of Gronwall inequality (see e.g. [4, Lemma 4.7]), we have

$$\begin{aligned} \mathbb{E} [ |Y_t| \middle| \mathcal{F}_u ] &\leq e^{A(T-t)} \mathbb{E} [ |\xi| \middle| \mathcal{F}_u ] + \mathbb{E} \left[ \int_t^T |h(s)| ds \middle| \mathcal{F}_u \right] + \mathbb{E} \left[ \int_t^T A e^{A(r-t)} \left( \int_r^T |h(s)| ds \right) dr \middle| \mathcal{F}_u \right] \\ &= e^{A(T-t)} \mathbb{E} [ |\xi| \middle| \mathcal{F}_u ] + \mathbb{E} \left[ \int_t^T e^{A(s-t)} |h(s)| ds \middle| \mathcal{F}_u \right] \\ &\leq e^{A(T-t)} \mathbb{E} \left[ |\xi| + \int_t^T |h(s)| ds \middle| \mathcal{F}_u \right]. \end{aligned}$$

In particular, we obtain at  $u = t$ ,

$$|Y_t| \leq e^{A(T-t)} \mathbb{E} \left[ |\xi| + \int_t^T |h(s)| ds \middle| \mathcal{F}_t \right].$$

Therefore, (6) is proved. Using [1, Lemma 6.1], we have for each  $\beta \in (0, 1)$ ,

$$\mathbb{E} \left[ \sup_{t \in [0, T]} |Y_t|^\beta \right] \leq \frac{e^{\beta A(T-t)}}{1 - \beta} \mathbb{E} \left[ |\xi| + \int_0^T |h(s)| ds \right]^\beta < \infty.$$

On the other hand, for any  $y \in \mathbb{R}$ ,

$$y f(s, y) \leq |y| (h(s) + A|y|) \leq |h(s)| \cdot |y| + A|y|^2.$$

By [5, Proposition 2], it follows that for each  $\beta \in (0, 1)$ ,

$$\mathbb{E} \left[ \left( \int_0^T |Z_s|^2 ds \right)^{\beta/2} \right] \leq C \left( \mathbb{E} \left[ \sup_{t \in [0, T]} |Y_t|^\beta \right] + \mathbb{E} \left[ \left( \int_0^T |h(s)| ds \right)^\beta \right] \right) < \infty,$$

for some constant  $C$  depending on  $A, \beta, T$ . So the result follows. □

We now define the real function  $\psi$ :

$$\psi(x, \mu) := x \exp\left(\mu\sqrt{2\log(1+x)}\right), \quad (x, \mu) \in [0, \infty) \times (0, +\infty),$$

which is closely connected to our integrability assumption.

The function  $\psi$  has the following useful properties ([10]).

**Lemma 3.** For any  $x \in \mathbb{R}$ ,  $y \geq 0$  and  $\mu > 0$ , we have

$$e^x y \leq e^{\frac{x^2}{2\mu^2}} + e^{2\mu^2} \psi(y, \mu).$$

**Lemma 4.** Let  $(q_t)_{t \in [0, T]}$  be a  $d$ -dimensional progressively measurable process with  $|q_t| \leq b$  almost surely. For each  $t \in [0, T]$ , if  $0 \leq \lambda < \frac{1}{2b^2(T-t)}$ , then

$$\mathbb{E}\left[e^{\lambda\left|\int_t^T q_s dW_s\right|^2} \middle| \mathcal{F}_t\right] \leq \frac{1}{\sqrt{1 - 2\lambda b^2(T-t)}}.$$

Moreover, the following lemma holds (see [3, Proposition 2.3 and the proof of Theorem 2.5]).

**Lemma 5.** We have the following assertions on  $\psi$ :

- (i) For each  $x \geq 0$ ,  $\psi(x, \cdot)$  is non-decreasing on  $[0, +\infty)$ .
- (ii) For  $\mu \geq 0$ ,  $\psi(\cdot, \mu)$  is a positive, strictly increasing and strictly convex function on  $[0, +\infty)$ .
- (iii) For  $c \geq 1$ , we have  $\psi(cx, \mu) \leq \psi(c, \mu)\psi(x, \mu)$ , for all  $x, \mu \geq 0$ .
- (iv) For all  $x_1, x_2, \mu \geq 0$ , we have  $\psi(x_1 + x_2, \mu) \leq \frac{1}{2}\psi(2, \mu) [\psi(x_1, \mu) + \psi(x_2, \mu)]$ .

Fix some constant  $T_0 \in [0, T]$  and consider the following BSDE with a null generator:

$$y_t = Y_{T_0} - \int_t^{T_0} z_s dW_s, \quad t \in [0, T_0], \tag{8}$$

where  $Y_{T_0}$  is an  $\mathcal{F}_{T_0}$ -measurable random variable.

**Lemma 6.** If  $\mathbb{E}[\psi(|Y_{T_0}|, \mu)] < +\infty$  for some constant  $\mu > 0$ , then the BSDE (8) has a unique solution  $(y, z)$  such that  $\psi(|y|, \mu)$  belongs to class (D).

**Proof.** We first note that  $\mathbb{E}[|Y_{T_0}|] \leq \mathbb{E}[\psi(|Y_{T_0}|, \mu)] < +\infty$ . Set  $y_t := \mathbb{E}[Y_{T_0} | \mathcal{F}_t]$ ,  $t \in [0, T_0]$ . By the martingale representation theorem (see e.g. [13, Theorem 2.46]), there exists a process  $\{z_t, t \in [0, T_0]\}$  such that

$$\mathbb{E}[Y_{T_0} | \mathcal{F}_t] = \mathbb{E}[Y_{T_0}] + \int_0^t z_s dW_s, \quad t \in [0, T_0].$$

Hence, the pair  $(y, z)$  satisfies (8). On the other hand, one has for any  $t \in [0, T_0]$ ,

$$\psi(|y_t|, \mu) \leq \psi(\mathbb{E}[|Y_{T_0}| | \mathcal{F}_t], \mu) \leq \mathbb{E}[\psi(|Y_{T_0}|, \mu) | \mathcal{F}_t],$$

where we used the convexity of  $\psi(\cdot, \mu_0)$ . Therefore,  $\psi(|y|, \mu)$  belongs to class (D). In particular, if  $Y_{T_0} = 0$ , then we notice that  $(y, z) = (0, 0)$ . This proves the uniqueness.  $\square$

### 3. Main result

We consider the BSDE:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad t \in [0, T]. \tag{9}$$

To reach our goal, we first study the relation between the solutions of BSDEs with  $L_{exp}(\mu_0\sqrt{2\log(1+L)})$ -integrable terminal values and the  $L^1$ -solutions of BSDEs with integrable parameters.

**Proposition 7.** *Let the generator  $f$  satisfies (A3). Suppose that*

$$\mathbb{E} \left[ \psi \left( |\xi| + \int_0^T |f(s, 0, 0)| ds, \mu_0 \right) \right] < +\infty, \text{ with } \mu_0 = b\sqrt{T}.$$

*Let  $(Y, Z)$  be a solution of the BSDE (9) such that  $\{\psi(|Y_t|, b\sqrt{t})\}$  belongs to class (D). Set  $\delta := T/4$  and define a pair  $(\mathcal{Y}, \mathcal{Z})$  by*

$$(\mathcal{Y}_t, \mathcal{Z}_t) := I_{[0, T-\delta)}(t) (y_t, z_t) + I_{[T-\delta, T]}(t) (Y_t, Z_t), \tag{10}$$

*where  $(y, z)$  is a solution of the BSDE (8) with  $T_0 = T - \delta$  and terminal condition  $y_{T-\delta} = Y_{T-\delta}$ , and  $\{\psi(|y_t|, b\sqrt{T-\delta}), t \in [0, T-\delta]\}$  belongs to class (D). Then, there exists a probability measure  $\mathbb{Q}$  equivalent to  $\mathbb{P}$  under which  $(\mathcal{Y}, \mathcal{Z})$  is an  $L^1$ -solution to the BSDE:*

$$\mathcal{Y}_t = \xi + \int_t^T I_{[T-\delta, T]}(s) f(s, \mathcal{Y}_s, 0) ds - \int_t^T \mathcal{Z}_s dW_s^{\mathbb{Q}}, \mathbb{Q} - a.s., \quad t \in [0, T], \tag{11}$$

*where  $W^{\mathbb{Q}}$  is a  $\mathbb{Q}$ -Brownian motion. Moreover, one has  $\mathcal{Z} \in \cap_{\beta \in (0, 1)} M^{\beta}([0, T])$ .*

**Proof.** From definition, it is obvious that  $(\mathcal{Y}, \mathcal{Z})$  satisfies the BSDE:

$$\mathcal{Y}_t = \xi + \int_t^T I_{[T-\delta, T]}(s) f(s, \mathcal{Y}_s, \mathcal{Z}_s) ds - \int_t^T \mathcal{Z}_s dW_s, \quad t \in [0, T]. \tag{12}$$

Let us define

$$g_s := g(s, \mathcal{Y}_s, \mathcal{Z}_s), \quad g(s, y, z) := \mathbf{1}_{|z| \neq 0} \frac{f(s, y, z) - f(s, y, 0)}{|z|^2} z, \quad (y, z) \in \mathbb{R} \times \mathbb{R}^{1 \times d}.$$

Since  $I_{[T-\delta, T]}(s)|g_s| \leq |g_s| \leq b$ ,  $\{\mathcal{E}(I_{[T-\delta, T]} g \bullet W)_s, 0 \leq s \leq T\}$  is a uniformly integrable martingale which has moments of all orders (see e.g. [11, Theorem 1.5]).

By the virtue of Girsanov theorem, we have

$$\mathcal{Y}_t = \xi + \int_t^T I_{[T-\delta, T]}(s) f(s, \mathcal{Y}_s, 0) ds - \int_t^T \mathcal{Z}_s dW_s^{\mathbb{Q}}, \quad t \in [0, T], \tag{13}$$

where

$$\mathbb{Q} := [\mathcal{E}(I_{[T-\delta, T]} g \bullet W)]_T \cdot \mathbb{P} = \exp \left( \int_{T-\delta}^T g_r dW_r - \frac{1}{2} \int_{T-\delta}^T |g_r|^2 dr \right) \cdot \mathbb{P}$$

and  $W^{\mathbb{Q}} := W - \int_0^\cdot I_{[T-\delta, T]}(s) g_s ds$ .

Note that  $\mathbb{Q}$  is a probability measure equivalent to  $\mathbb{P}$  and  $W^{\mathbb{Q}}$  is a  $\mathbb{Q}$ -Brownian motion. Recalling that  $\delta = T/4$ , we notice that  $T > 3\delta$  and  $(1 - \frac{2\delta}{T-\delta})^{-1/4} = 3^{1/4}$ .

From Lemmata 3 and 4, it follows that for any  $\tau \in \mathcal{F}[T-\delta, T]$  and  $A \in \mathcal{F}_T$ ,

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}(|\mathcal{Y}_\tau|; A) &= \mathbb{E}^{\mathbb{Q}}(|Y_\tau|; A) \leq \mathbb{E} \left[ |Y_\tau| \exp \left( \int_{T-\delta}^T g_s dW_s \right); A \right] \\ &\leq \mathbb{E} \left[ \exp \left( \frac{\left| \int_{T-\delta}^T g_s dW_s \right|^2}{2(b\sqrt{\tau})^2} \right); A \right] + \mathbb{E} \left[ e^{2(b\sqrt{\tau})^2} \psi(|Y_\tau|, b\sqrt{\tau}); A \right] \\ &\leq \mathbb{E} \left[ \exp \left( \frac{\left| \int_{T-\delta}^T \sqrt{2} g_s dW_s \right|^2}{2(b\sqrt{T-\delta})^2} \right) \right]^{1/2} \cdot \mathbb{P}(A)^{1/2} + \mathbb{E} \left[ e^{2(b\sqrt{\tau})^2} \psi(|Y_\tau|, b\sqrt{\tau}); A \right] \\ &\leq \left( 1 - \frac{2\delta}{T-\delta} \right)^{-1/4} \cdot \mathbb{P}(A)^{1/2} + \mathbb{E} \left[ e^{2b^2\tau} \psi(|Y_\tau|, b\sqrt{\tau}); A \right] \\ &\leq 3^{1/4} \cdot \mathbb{P}(A)^{1/2} + e^{2b^2T} \cdot \mathbb{E} \left[ \psi(|Y_\tau|, b\sqrt{\tau}); A \right]. \end{aligned} \tag{14}$$

On the other hand, one has for any  $\tau \in \mathcal{F}[0, T - \delta]$  and  $A \in \mathcal{F}_T$ ,

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}(|\mathcal{Y}_\tau|; A) &= \mathbb{E}^{\mathbb{Q}}(|Y_\tau|; A) \leq \mathbb{E} \left[ |y_\tau| \exp \left( \int_{T-\delta}^T g_s dW_s \right); A \right] \\ &\leq \mathbb{E} \left[ \exp \left( \frac{\left| \int_{T-\delta}^T g_s dW_s \right|^2}{2b^2(T-\delta)} \right); A \right] + \mathbb{E} \left[ e^{2b^2(T-\delta)} \psi(|y_\tau|, b\sqrt{T-\delta}); A \right] \\ &\leq \mathbb{E} \left[ \exp \left( \frac{\left| \int_{T-\delta}^T \sqrt{2} g_s dW_s \right|^2}{2b^2(T-\delta)} \right) \right]^{1/2} \cdot \mathbb{P}(A)^{1/2} + \mathbb{E} \left[ e^{2b^2(T-\delta)} \psi(|y_\tau|, b\sqrt{T-\delta}); A \right] \tag{15} \\ &\leq \left( 1 - \frac{2\delta}{T-\delta} \right)^{-1/4} \cdot \mathbb{P}(A)^{1/2} + e^{2b^2(T-\delta)} \cdot \mathbb{E} \left[ \psi(|y_\tau|, b\sqrt{T-\delta}); A \right] \\ &\leq 3^{1/4} \cdot \mathbb{P}(A)^{1/2} + e^{2b^2T} \cdot \mathbb{E} \left[ \psi(|y_\tau|, b\sqrt{T-\delta}); A \right]. \end{aligned}$$

We also observe that

$$\begin{aligned} \sup_{\tau \in \mathcal{F}[0, T]} \mathbb{E}^{\mathbb{Q}}(|\mathcal{Y}_\tau|; A) &\leq \sup_{\tau \in \mathcal{F}[0, T]} \mathbb{E}^{\mathbb{Q}}(|\mathcal{Y}_{\tau \wedge (T-\delta)}| + |\mathcal{Y}_{\tau \vee (T-\delta)}|; A) \\ &\leq \sup_{\tau \in \mathcal{F}[0, T]} \mathbb{E}^{\mathbb{Q}}(|\mathcal{Y}_{\tau \wedge (T-\delta)}|; A) + \sup_{\tau \in \mathcal{F}[0, T]} \mathbb{E}^{\mathbb{Q}}(|\mathcal{Y}_{\tau \vee (T-\delta)}|; A) \\ &\leq \sup_{\tau \in \mathcal{F}[0, T-\delta]} \mathbb{E}^{\mathbb{Q}}(|\mathcal{Y}_\tau|; A) + \sup_{\tau \in \mathcal{F}[T-\delta, T]} \mathbb{E}^{\mathbb{Q}}(|\mathcal{Y}_\tau|; A). \end{aligned}$$

Using (14), (15), and the last inequality, we can prove that  $\mathcal{Y}$  belongs to class (D) under  $\mathbb{Q}$ . Indeed, since both  $\{\psi(|Y_t|, b\sqrt{t}), t \in [T - \delta, T]\}$  and  $\{\psi(|y_t|, b\sqrt{T - \delta}), t \in [0, T - \delta]\}$  belong to class (D),

$$\begin{aligned} &\sup_{\tau \in \mathcal{F}[0, T]} \mathbb{E}^{\mathbb{Q}}(|\mathcal{Y}_\tau|) \\ &\leq \sup_{\tau \in \mathcal{F}[0, T-\delta]} \mathbb{E}^{\mathbb{Q}}(|\mathcal{Y}_\tau|) + \sup_{\tau \in \mathcal{F}[T-\delta, T]} \mathbb{E}^{\mathbb{Q}}(|\mathcal{Y}_\tau|) \\ &\leq 2 \cdot 3^{1/4} + e^{2b^2T} \cdot \left( \sup_{\tau \in \mathcal{F}[0, T-\delta]} \mathbb{E} \left[ \psi(|y_\tau|, b\sqrt{T-\delta}) \right] + \sup_{\tau \in \mathcal{F}[T-\delta, T]} \mathbb{E} \left[ \psi(|Y_\tau|, b\sqrt{\tau}) \right] \right) < +\infty. \tag{16} \end{aligned}$$

On the other hand, for any  $\varepsilon > 0$ ,

$$\exists \theta_1 > 0, \forall A \in \mathcal{F}_T (\mathbb{P}(A) < \theta_1); \quad \sup_{\tau \in \mathcal{F}[T-\delta, T]} \mathbb{E} \left[ \psi(|Y_\tau|, b\sqrt{\tau}); A \right] < \frac{\varepsilon}{4e^{2b^2T}},$$

and

$$\exists \theta_2 > 0, \forall A \in \mathcal{F}_T (\mathbb{P}(A) < \theta_2); \quad \sup_{\tau \in \mathcal{F}[0, T-\delta]} \mathbb{E} \left[ \psi(|y_\tau|, b\sqrt{T-\delta}); A \right] < \frac{\varepsilon}{4e^{2b^2T}}.$$

Set  $\theta := \theta_1 \wedge \theta_2 \wedge \frac{\varepsilon^2}{16\sqrt{3}}$ . Then we have  $3^{1/4} \cdot \theta^{1/2} \leq \frac{\varepsilon}{4}$ , and

$$\begin{aligned} \forall A \in \mathcal{F}_T (\mathbb{P}(A) < \theta); \quad \sup_{\tau \in \mathcal{F}[0, T-\delta]} \mathbb{E} \left[ \psi(|y_\tau|, b\sqrt{T-\delta}); A \right] \\ + \sup_{\tau \in \mathcal{F}[T-\delta, T]} \mathbb{E} \left[ \psi(|Y_\tau|, b\sqrt{\tau}); A \right] < \frac{\varepsilon}{2e^{2b^2T}}. \end{aligned}$$

Since  $\mathbb{Q}$  is equivalent to  $\mathbb{P}$ , we also have

$$\exists \vartheta > 0, \forall A (\mathbb{Q}(A) < \vartheta); \mathbb{P}(A) < \theta.$$



Consequently, we obtain

$$\forall \varepsilon > 0, \exists \vartheta > 0 \forall A (\mathbb{Q}(A) < \vartheta)$$

$$\sup_{\tau \in \mathcal{T}[0, T]} \mathbb{E}^{\mathbb{Q}} (|\mathcal{Y}_\tau|; A) \leq 2 \cdot 3^{1/4} \cdot \mathbb{P}(A)^{1/2} + e^{2b^2 T} \cdot \frac{\varepsilon}{2e^{2b^2 T}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon. \quad (17)$$

In view of (16) and (17), we deduce that  $\mathcal{Y}$  belongs to class (D) under  $\mathbb{Q}$ .

Using Lemmata 3 and 4 again, we obtain

$$\mathbb{E}^{\mathbb{Q}} \left[ |\xi| + \int_0^T I_{[T-\delta, T]}(s) |f(s, 0, 0)| ds \right]$$

$$\leq \left( 1 - \frac{b^2}{\mu_0^2} \delta \right)^{-1/2} + e^{2\mu_0^2} \mathbb{E} \left[ \psi \left( |\xi| + \int_{T-\delta}^T |f(s, 0, 0)| ds, \mu_0 \right) \right] < \infty.$$

By Lemma 2, it follows that  $(\mathcal{Y}, \mathcal{Z}) \in S^\beta([0, T]; \mathbb{Q}) \times M^\beta([0, T]; \mathbb{Q})$  for any  $\beta \in (0, 1)$ . So  $(\mathcal{Y}, \mathcal{Z})$  is an  $L^1$ -solution to the BSDE (13) under  $\mathbb{Q}$ . It remains to show that  $\mathcal{Z} \in \bigcap_{\beta \in (0, 1)} M^\beta([0, T])$ . Using Hölder’s inequality, one has for any  $\beta \in (0, 1)$ ,

$$\mathbb{E} \left[ \left( \int_0^T |\mathcal{Z}_s|^2 ds \right)^{\beta/2} \right] = \mathbb{E}^{\mathbb{Q}} \left[ \mathcal{E}(I_{[T-\delta, T]} g \bullet W)_T^{-1} \left( \int_0^T |\mathcal{Z}_s|^2 ds \right)^{\beta/2} \right]$$

$$\leq \mathbb{E}^{\mathbb{Q}} \left[ \left( \mathcal{E}(I_{[T-\delta, T]} g \bullet W)_T^{-1} \right)^\alpha \right]^{1/\alpha} \cdot \mathbb{E}^{\mathbb{Q}} \left[ \left( \int_0^T |\mathcal{Z}_s|^2 ds \right)^{\beta'/2} \right]^{\beta/\beta'},$$

where  $\beta'$  and  $\alpha$  be the numbers such that  $\beta < \beta' < 1$  and  $\frac{1}{\alpha} + \frac{\beta}{\beta'} = 1$ . On the other hand,

$$\mathbb{E}^{\mathbb{Q}} \left[ \left( \mathcal{E}(I_{[T-\delta, T]} g \bullet W)_T^{-1} \right)^\alpha \right]$$

$$= \mathbb{E} \left\{ \left[ \exp \left( - \int_{T-\delta}^T g_s dW_s + \frac{1}{2} \int_{T-\delta}^T |g_s|^2 ds \right) \right]^\alpha \right\}$$

$$= \mathbb{E} \left\{ \left[ \exp \left( - \int_{T-\delta}^T g_s dW_s - \frac{1}{2} \int_{T-\delta}^T |g_s|^2 ds \right) \cdot \exp \left( \int_{T-\delta}^T |g_s|^2 ds \right) \right]^\alpha \right\}$$

$$\leq \mathbb{E} \left[ \mathcal{E}((-g I_{[T-\delta, T]}) \bullet W)_T^\alpha \right] \cdot e^{b^2 \delta \alpha} < +\infty.$$

Hence, we obtain  $\mathbb{E} \left[ \left( \int_0^T |\mathcal{Z}_s|^2 ds \right)^{\beta/2} \right] < +\infty$ . The proof of Proposition 7 is then complete. □

We first recover the existence of Fan and Hu [8] under our general assumptions.

**Theorem 8.** *Suppose that the generator  $f$  satisfies (A1)–(A3). We further assume that with  $\mu_0 := b\sqrt{T}$ ,*

$$\mathbb{E} \left[ \psi \left( |\xi| + \int_0^T |f(t, 0, 0)| dt, \mu_0 \right) \right] < +\infty.$$

*Then the BSDE (9) admits a solution  $(Y, Z)$  such that  $\psi(|Y_t|, b\sqrt{t})$  belongs to class (D) and  $\mathbb{P}$  – a.s., for each  $t \in [0, T]$ ,*

$$|Y_t| \leq \psi(|Y_t|, b\sqrt{t}) \leq C \mathbb{E} \left[ \psi \left( |\xi| + \int_0^T |f(t, 0, 0)| dt, b\sqrt{T} \right) \middle| \mathcal{F}_t \right] + C, \quad (18)$$

*for some constant  $C > 0$ . Moreover,  $Z$  belongs to  $\bigcap_{\beta \in (0, 1)} M^\beta([0, T])$ .*

**Proof.** The existence result under our assumptions can be proved by following exactly the same method as in [8] (see Remark 3.7 therein). For the reader’s convenience, we provide a detailed proof here. Let us fix  $n, p \in \mathbb{N}$ . Set

$$\xi^{n,p} := \xi^+ \wedge n - \xi^- \wedge p, \quad f_0 := f(\cdot, 0, 0), \quad f_0^{n,p} := f_0^+ \wedge n - f_0^- \wedge p, \quad f^{n,p} := f - f_0 + f_0^{n,p}.$$

Since  $\xi^{n,p}$  and  $f^{n,p}(s, 0, 0)$  are bounded and  $f^{n,p}$  satisfies assumptions **(A1)**–**(A3)**, in view of the existence result in [9], the BSDE  $(\xi^{n,p}, f^{n,p})$  has a unique  $L^2$ -solution  $(Y^{n,p}, Z^{n,p})$ . Noting that  $f^{n,p}(s, y, z)$  is weakly monotonic in  $y$  and Lipschitz in  $z$ , in view of Remark 1, we have

$$\text{sgn}(y)f^{n,p}(s, y, z) \leq A + |f_0^{n,p}(t)| + A|y| + b|z|.$$

Using [8, Proposition 3.5] (more precisely, regarding its proof), we deduce that

$$|Y_t^{n,p}| \leq \psi(|Y_t^{n,p}|, b\sqrt{t}) \leq C \cdot \mathbb{E} \left[ \psi \left( |\xi^{n,p}| + \int_0^T (A + |f_0^{n,p}(t)|) dt, b\sqrt{T} \right) \middle| \mathcal{F}_t \right] + C,$$

for some constant  $C > 0$ . Using Lemma 5, this is not bigger than

$$C \cdot \mathbb{E} \left[ \frac{1}{2} \psi(2, \mu) \cdot \left( \psi \left( |\xi^{n,p}| + \int_0^T |f_0^{n,p}(t)| dt, b\sqrt{T} \right) + \psi(A \cdot T, b\sqrt{T}) \right) \middle| \mathcal{F}_t \right] + C \tag{19}$$

$$\leq \bar{C} \cdot \mathbb{E} \left[ \psi \left( |\xi^{n,p}| + \int_0^T |f_0^{n,p}(t)| dt, b\sqrt{T} \right) \middle| \mathcal{F}_t \right] + \bar{C}, \tag{20}$$

$$\leq \bar{C} \cdot \mathbb{E} \left[ \psi \left( |\xi| + \int_0^T |f_0(t)| dt, b\sqrt{T} \right) \middle| \mathcal{F}_t \right] + \bar{C}, \tag{21}$$

for some constant  $\bar{C} > 0$ . Henceforth,

$$|Y_t^{n,p}| \leq \psi(|Y_t^{n,p}|, b\sqrt{t}) \leq \bar{C} \cdot \mathbb{E} \left[ \psi \left( |\xi| + \int_0^T |f_0(t)| dt, b\sqrt{T} \right) \middle| \mathcal{F}_t \right] + \bar{C}, \tag{22}$$

for some constant  $\bar{C} > 0$ . Since  $Y^{n,p}$  is non-decreasing in  $n$  and non-increasing in  $p$  thanks to comparison theorem (see [9, Theorem 5.1] or [5, Theorem 3]), by the localization method in [2], there exists some process  $Z \in L^2(0, T)$  such that  $(Y := \inf_p \sup_n Y^{n,p}, Z)$  is an adapted solution of (9). Moreover, sending  $n$  and  $p$  to infinity in (22) yields the inequality (18), and then  $\psi(|Y_t|, b\sqrt{t})$  belongs to class (D). With the help of Proposition 7, we can also prove that  $Z \in \bigcap_{\beta \in (0, 1)} M^\beta([0, T])$  (as far as we know, this fact was not yet observed in the literature). Indeed, one has for any  $\beta \in (0, 1)$ ,

$$\mathbb{E} \left[ \left( \int_0^T |Z_s|^2 ds \right)^{\beta/2} \right] \leq \mathbb{E} \left[ \left( \int_{T-\delta}^T |Z_s|^2 ds \right)^{\beta/2} \right] + \mathbb{E} \left[ \left( \int_{T-2\delta}^{T-\delta} |Z_s|^2 ds \right)^{\beta/2} \right] + \dots,$$

and the each term on the right-hand side is finite thanks to Proposition 7. Hence,

$$\mathbb{E} \left[ \left( \int_0^T |Z_s|^2 ds \right)^{\beta/2} \right] < +\infty. \tag{□}$$

**Remark 9.** If we assume the following stronger integrability condition:

$$\mathbb{E} \left[ \psi \left( |\xi| + \int_0^T |f(s, 0, 0)| ds, \mu \right) \right] < +\infty, \text{ with } \mu > \mu_0,$$

then, we can follow the argument in [3] to show that  $\psi(|Y|, c_0)$  belongs to class (D) for some constant  $c_0 > 0$ . Indeed, in view of the estimate (18), one has for  $b\sqrt{T} < a < \mu$ ,

$$|Y_t| \leq C \mathbb{E} \left[ \psi \left( |\xi| + \int_0^T |f(t, 0, 0)| dt, a \right) \middle| \mathcal{F}_t \right] + C,$$

Using this estimate and Lemma 5, we then deduce that

$$\psi(|Y_t|, c_0) \leq \psi \left( C \mathbb{E} \left[ \psi \left( |\xi| + \int_0^T |f(t, 0, 0)| dt, a \right) \middle| \mathcal{F}_t \right] + C, c_0 \right) \tag{23}$$

$$\leq \frac{1}{2} \psi(2, c_0) \left[ \psi \left( C \mathbb{E} \left[ \psi \left( |\xi| + \int_0^T |f(t, 0, 0)| dt, a \right) \middle| \mathcal{F}_t \right], c_0 \right) + \psi(C, c_0) \right] \tag{24}$$

$$\leq \frac{1}{2} \psi(2, c_0) \left[ \psi(C, c_0) \psi \left( \mathbb{E} \left[ \psi \left( |\xi| + \int_0^T |f(t, 0, 0)| dt, a \right) \middle| \mathcal{F}_t \right], c_0 \right) + \psi(C, c_0) \right] \tag{25}$$

$$\leq \frac{1}{2} \psi(2, c_0) \left[ \psi(C, c_0) \mathbb{E} \left[ \psi \left( \psi \left( |\xi| + \int_0^T |f(t, 0, 0)| dt, a \right), c_0 \right) \middle| \mathcal{F}_t \right] + \psi(C, c_0) \right] \tag{26}$$

$$\leq \frac{1}{2} \psi(2, c_0) \left[ e^{\frac{a \cdot c_0^2}{d}} \psi(C, c_0) \mathbb{E} \left[ \psi \left( |\xi| + \int_0^T |f(t, 0, 0)| dt, a + c_0 + d \right) \middle| \mathcal{F}_t \right] + \psi(C, c_0) \right] \tag{27}$$

$$\leq \bar{C} \mathbb{E} \left[ \psi \left( |\xi| + \int_0^T |f(t, 0, 0)| dt, \mu \right) \middle| \mathcal{F}_t \right] + \bar{C}, \tag{28}$$

where  $c_0 > 0, d > 0$  is chosen so that  $a + c_0 + d = \mu$ . Hence, it follows that  $\psi(|Y_t|, c_0)$  belongs to class (D).

We are now in a position to state the main result in this paper.

**Theorem 10.** *Suppose that the generator  $f$  satisfies assumptions (A1)-(A3). We further assume that*

$$\mathbb{E} \left[ \psi \left( |\xi| + \int_0^T |f(t, 0, 0)| dt, \mu_0 \right) \right] < +\infty,$$

with  $\mu_0 := b\sqrt{T}$ . Then the BSDE (9) admits a unique solution  $(Y, Z)$  such that  $\psi(|Y_t|, b\sqrt{t})$  belongs to class (D). Moreover,  $Z$  belongs to  $\cap_{\beta \in (0,1)} M^\beta([0, T])$ .

**Proof.** The existence was already proved in Theorem 8. We now prove the uniqueness which is the main contribution in this paper. For  $i = 1, 2$ , let  $(Y^i, Z^i)$  be a solution to BSDE (9) such that  $\psi(|Y_t^i|, b\sqrt{t})$  belongs to the class (D). Define  $(\bar{Y}, \bar{Z}) := (Y^1 - Y^2, Z^1 - Z^2)$ . We have thanks to Lemma 5,

$$\psi(|\bar{Y}_t|, b\sqrt{t}) \leq \psi(|Y_t^1| + |Y_t^2|, b\sqrt{t}) \leq \frac{1}{2} \psi(2, b\sqrt{t}) [\psi(|Y_t^1|, b\sqrt{t}) + \psi(|Y_t^2|, b\sqrt{t})].$$

Therefore,  $\psi(|\bar{Y}_t|, b\sqrt{t})$  belongs to class (D). Obviously,  $(\bar{Y}, \bar{Z})$  satisfies the following equation.

$$\bar{Y}_t = \int_t^T \bar{f}(s, \bar{Y}_s, \bar{Z}_s) ds - \int_t^T \bar{Z}_s dW_s, \quad t \in [0, T], \tag{29}$$

where  $\bar{f}(s, y, z) := f(s, y + Y_s^2, z + Z_s^2) - f(s, Y_s^2, Z_s^2)$ . It is easy to check that  $\bar{f}$  satisfies assumptions (A1) and (A3). Moreover, one has  $\bar{f}(s, 0, 0) = 0$ . Set  $\delta := T/4$ . Let  $\{(y_t, z_t), t \in [0, T - \delta]\}$  be a solution to BSDE (8), with  $T_0 := T - \delta$  and the terminal condition  $y_{T-\delta} = \bar{Y}_{T-\delta}$ . By Proposition 7, there exists a probability measure  $\mathbb{Q}$  equivalent to  $\mathbb{P}$  such that  $(\mathcal{Y}_t, \mathcal{Z}_t) := I_{[0, T-\delta]}(t)(y_t, z_t) + I_{[T-\delta, T]}(t)(\bar{Y}_t, \bar{Z}_t)$  is an  $L^1$ -solution to the BSDE:

$$\mathcal{Y}_t = \int_t^T I_{[T-\delta, T]}(s) \bar{f}(s, \mathcal{Y}_s, 0) ds - \int_t^T \mathcal{Z}_s dW_s^\mathbb{Q}, \quad t \in [0, T], \tag{30}$$

where  $W^\mathbb{Q}$  is a  $\mathbb{Q}$ -Brownian motion. On the other hand, one can easily check that  $(0, 0)$  also becomes a solution to BSDE (30). Therefore, according to the uniqueness of  $L^1$ -solutions to BSDEs with weakly monotonic generators (see [6, Theorem 6.5] or [7, Theorem 1]), we have  $(\mathcal{Y}_t, \mathcal{Z}_t) = (0, 0)$  for all  $t \in [0, T]$ ,  $\mathbb{Q}$ -a.s. Hence,  $(\bar{Y}_t, \bar{Z}_t) = (0, 0)$  for all  $t \in [T - \delta, T] = [3/4T, T]$ ,  $\mathbb{Q}$ -a.s. Since  $\mathbb{Q}$  is equivalent to  $\mathbb{P}$ , it follows that  $(\bar{Y}_t, \bar{Z}_t) = (0, 0)$  for all  $t \in$

$[3/4T, T]$ ,  $\mathbb{P} - a.s.$  In an identical way, we successively have the uniqueness on the intervals  $[3^2T/4^2, 3T/4]$ ,  $[3^3T/4^3, 3^2T/4^2]$ ,  $\dots$ ,  $[3^pT/4^p, 3^{p-1}T/4^{p-1}]$ ,  $\dots$ . Finally, in view of the continuity of process  $\tilde{Y}_t$  with respect to  $t$ , we obtain the uniqueness on the whole interval  $[0, T]$  by sending  $p$  to infinity. The proof of Theorem 10 is then complete.  $\square$

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