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Numerical analysis / *Analyse numérique*

A Fast and Accurate Numerical Method for Radiative Transfer in the Atmosphere

Une méthode numérique rapide et précise pour le transfert radiatif dans l'atmosphère

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Abstract. To solve the radiative transfer equations for the atmosphere, we turn to an equivalent integral equation which has no numerically singular terms. An iterative scheme is proposed for its solution and convergence is proved.

The side effect of this proof is an existence result for the radiative transfer equations, in one spatial variable, with frequency dependent absorption and scattering coefficients.

A numerical study is given with some comments on the effect of greenhouse gases on the temperature in the atmosphere.

Résumé. Pour résoudre les équations du transfert radiatif pour l'atmosphère nous nous tournons vers une formulation intégrale équivalente qui a l'avantage de ne pas contenir de fonction singulière. Une méthode itérative est proposée pour sa résolution et un résultat de convergence est donné.

En corollaire un résultat d'existence et d'unicité est prouvé pour les équations du transfert radiatif, 1D en espace, sous des hypothèses assez générales sur les coefficients d'absorption et de scattering et leurs dépendances en fréquence.

Une étude numérique termine cette étude ainsi que quelques commentaires sur l'effet des gaz à effet de serre sur la température de l'atmosphère.

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1. Introduction

Photons travel from the Sun to the Earth and participate to its thermal balance. The photons may interact with the atoms in the Earth atmosphere before reaching the ground. To simulate sufficiently many photons numerically to reach a numerical equilibrium is currently very difficult [3]; one must recourse to continuum mechanics and write energy conservations, the equations of radiative transfer (see [1, 4, 11] and (1),(2) below). Furthermore one assumes that both Sun and Earth behave as black bodies. Consequently the spectrum of sunlight received is given by the Planck function with the Sun temperature while the Earth emits infrared light according to the Planck function with the Earth temperature. These provides boundary conditions to the radiative transfer partial differential equations. We chose those used in [6, equation 2.16, p. 71], but others can be used too.

For the Earth atmosphere one last simplification is applied: local invariance with respect to the spatial coordinates in a plan tangent to Earth. Hence the only spatial variable of interest is the altitude and only the angle ϕ with the vertical is of interest for the light intensity in that direction (see [12] or [2] for details).

The end result is a system of 2 integro-differential equations in a cylinder $(0, Z) \times (-1, 1) \times \mathbb{R}^+$, for the rescaled altitude τ , for $\mu := \cos \phi$ and the light frequencies ν (see (4),(5),(6)).

There are two important parameters: the absorption κ_ν and the scattering albedo a_ν . When these are constant, it is known as Milne’s problem and existence, uniqueness holds [1, 4]. For the general case partial results are known only (see for example [10]).

Numerically, the problem is difficult because the solution has numerical singularities (see [9] for a review of the numerical methods). To illustrate this point, a finite element solution of the integro-differential system is given in Section 3.

Then in Section 4 we turn to an equivalent integral equation which has no numerical singularities. An iterative scheme is proposed for its solution and convergence is proved.

The side effect of this proof is an existence result for the radiative transfer equations with frequency dependent absorption and scattering coefficients.

The method extends to some very limited non isotropic scattering only.

Numerical tests conclude this study with some comments on the effect of greenhouse gases on the temperature in the atmosphere.

2. The Fundamental equations

Let $I_\nu(\mathbf{x}, \boldsymbol{\omega})$ be the light intensity of frequency ν in the direction $\boldsymbol{\omega}$ at point \mathbf{x} in the Earth atmosphere Ω , and let $T(\mathbf{x})$ be the temperature. In the stationary case and in absence of wind,

$$\boldsymbol{\omega} \cdot \nabla I_\nu + \rho k_\nu a_\nu \left[I_\nu - \frac{1}{4\pi} \int_{\mathbb{S}^2} p(\boldsymbol{\omega}, \boldsymbol{\omega}') I_\nu(\boldsymbol{\omega}') d\boldsymbol{\omega}' \right] = \rho k_\nu (1 - a_\nu) [B_\nu(T) - I_\nu], \tag{1}$$

$$\kappa_T \Delta T = \nabla \cdot \int_0^\infty \int_{\mathbb{S}^2} I_\nu(\boldsymbol{\omega}') \boldsymbol{\omega}' d\boldsymbol{\omega}' d\nu, \quad B_\nu(T) = \frac{2\hbar\nu^3}{c^2 \left[e^{\frac{\hbar\nu}{kT}} - 1 \right]}, \tag{2}$$

$$I_\nu(x, \boldsymbol{\omega}) \text{ given on } \{(\mathbf{x}, \boldsymbol{\omega}) \in \partial\Omega \times \mathbb{S}^2 : \mathbf{n}(\mathbf{x}) \cdot \boldsymbol{\omega} < 0\}, \quad \partial_n T \text{ given on } \partial\Omega, \tag{3}$$

where all differential operators are with respect to \mathbf{x} , where κ_T is the thermal diffusion, \mathbf{n} is the outer unit normal of $\partial\Omega$. \mathbb{S}^2 is the unit sphere, c, \hbar, k the speed of light, the Planck and Boltzmann constants respectively; $\rho(\mathbf{x})$ is the density of air.

- k_ν is the mass extinction coefficient; $k_\nu \rho(\mathbf{x})$ is a dimensionless *absorption* coefficient, which is affected by the presence of *greenhouse gases* in the atmosphere.
- a_ν is the scattering albedo, due to clouds, ice drops, etc.

• $p(\omega, \omega')$ is proportional to the probability that a ray in direction ω' scatters in direction ω . As indicated by their subscripts, both k_ν and a_ν usually depend on ν .

2.1. Simplification

The Earth has an evanescent atmosphere; it is a source of numerical imprecision: $\rho = \rho_0 e^{-r}$ where r is the altitude; it is avoided by a change of coordinate $\tau = 1 - e^{-r}$. So, $\tau = 0$ is the ground and $\tau = Z$ the beginning of the stratosphere. At 12km $Z = 1 - e^{-12} \approx 1$.

By scaling it is possible to adimensionalize the equations. Furthermore when $\kappa_T = 0$, (2) becomes

$$\int_0^\infty \kappa_\nu (1 - a_\nu) B_\nu(T) d\nu = \int_0^\infty \frac{\kappa_\nu}{2} (1 - a_\nu) \int_{-1}^1 I_\nu d\mu d\nu, \quad B_\nu(T) = \frac{\nu^3}{e^{\frac{\nu}{T}} - 1}. \tag{4}$$

From now on $\kappa_\nu = \rho_0 k_\nu$. The atmosphere is very thin with respect to the Earth radius. A common approximation is to say that the light intensity in a direction at an angle ϕ with the vertical $I_\nu(\tau, \mu)$ is a function of $\mu = \cos\phi$ only. Then (1) becomes: $\forall \tau, \mu, \nu \in (0, Z) \times (-1, 1) \times \mathbb{R}^+$,

$$\mu \partial_\tau I_\nu + \kappa_\nu I_\nu - \frac{\kappa_\nu a_\nu}{2} \int_{-1}^1 p(\mu, \mu') I_\nu(\tau, \mu') d\mu' = \kappa_\nu (1 - a_\nu) B_\nu(T). \tag{5}$$

The boundary conditions [6],

$$I(0, \mu)|_{\mu>0} = Q_\nu \mu, \quad I(Z, \mu)|_{\mu<0} = 0, \tag{6}$$

imply that Earth receive from the Sun an energy $\nu \mapsto Q_\nu$ and due to Planck's black body law, Earth radiates (infrared) light which escapes at $\tau = Z$ without back-scattering. Nevertheless what follows applies to other boundary conditions also.

Note that (6) implies that sunlight crosses the atmosphere unaffected, something that should be revisited in the presence of ozone, for instance.

3. Iterative Schemes

Let I_ν^{n+1} be computed by (4),(6) from a known state I_ν^n, T^n :

$$\mu \partial_\tau I_\nu^{n+1} + \kappa_\nu I_\nu^{n+1} = \frac{\kappa_\nu a_\nu}{2} \int_{-1}^1 p(\mu, \mu') I_\nu^n(\tau, \mu') d\mu' + \kappa_\nu (1 - a_\nu) B(T_\tau^n), \tag{7}$$

$$I_\nu^{n+1}(0, \mu)|_{\mu>0} = Q_\nu \mu, \quad I_\nu^{n+1}(Z, \mu)|_{\mu<0} = 0, \tag{8}$$

and then let $\tau \mapsto T_\tau^{n+1}$ be computed by (5):

$$\int_0^\infty \kappa_\nu (1 - a_\nu) B(T_\tau^{n+1}) d\nu = \int_0^\infty \frac{\kappa_\nu}{2} (1 - a_\nu) \int_{-1}^1 I_\nu^{n+1} d\mu d\nu, \quad \forall \tau \in (0, Z). \tag{9}$$

When κ is independent of ν (Milne's problem for a grey atmosphere) and $a_\nu = 0$,

$$\mu \partial_\tau I^{n+1} + \kappa I^{n+1} = \frac{\kappa}{2} \int_{-1}^1 I^n d\mu, \quad I^{n+1}(0, \mu)|_{\mu>0} = \mu, \quad I^{n+1}(Z, \mu)|_{\mu<0} = 0,$$

linear convergence is well known(see [2]): $\|I^{n+1} - I^*\|_0 \leq c \|I^n - I^*\|_0, \quad c \in (0, 1)$. We shall extend this result to the case κ_ν non constant.

Note that when κ_ν is constant (9) is trivially solved by the Stefan-Boltzmann law, directly derived from (4):

$$\int_0^\infty \frac{\nu^3}{e^{\frac{\nu}{T}} - 1} d\nu = \frac{\pi^4}{15} T^4, \Rightarrow \frac{\pi^4}{15} (T_\tau^{n+1})^4 = \frac{1}{2} \int_{-1}^1 I^{n+1} d\mu, \quad \forall \tau \tag{10}$$

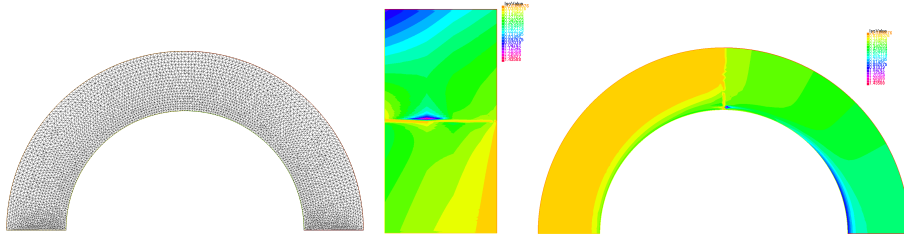


Figure 1. Results of a finite element simulation of Milne's problem (κ constant) with streamline upwinding. The physical domain is on the left: radial altitude and angle at which the intensity is measured. The results are independent of the radius of the circle, (except if a Chandrasekhar correction is added [2], in which case it should be the Earth radius). The mesh is mapped onto the $\tau, \mu \in ((0, Z) \times (-1, 1))$ computational domain on which the variational formulation is set and solve. The color map of I shown on the computational domain at the center exhibits a singularity at $\mu = 0$. The right picture shows the same color map but in the original physical variables. The singularity corresponds to a light intensity at an angle perpendicular to the the incoming sunlight (horizontal from the right), near the ground.

3.1. Discretization with SUPG-FEM

For clarity we assume no scattering $a_v = 0$. Let $\Omega = (0, Z) \times (-1, 1)$; (7), (8) are approximated by

$$\begin{aligned} & \int_{\Omega} \left[(\kappa_v I_h^{n+1} + \mu \partial_{\tau} I_h^{n+1}) \hat{I}_h + h_{\text{SUPG}} (\mu \partial_{\tau} I_h^{n+1} + \kappa_v I_h) (\mu \partial_{\tau} \hat{I}_h^{n+1} + \kappa_v \hat{I}_h) \right] d\tau d\mu \\ & + \int_{\Omega} \left[\gamma(\tau, \mu) \partial_{\mu} I_h^{n+1} \hat{I}_h + h_{\text{SUPG}} \gamma(\tau, \mu) \partial_{\mu} I_h^{n+1} \partial_{\mu} \hat{I}_h \right] d\tau d\mu \\ & = \int_{\Omega} \kappa_v B(T^n) \hat{I}_h d\tau d\mu, \end{aligned}$$

for all $\hat{I}_h \in \{v_h \in C^0(\Omega) : v_h|_{T_h} \in P^1(T_h) \text{ for all triangles } T_h \text{ of a triangulation } \Omega_h \approx \Omega\}$ and with (8) with $Q_v = 0$; the solution I_h^{n+1} is seek in the same space plus (8).

The streamline upwinding parameter h_{SUPG} must be small but not too small to avoid oscillations. The Chandrasekhar correction

$$\gamma(\tau, \mu) = \frac{1 - \mu^2}{(1 - \tau)(R - \log(1 - \tau))}$$

(see [2]) which accounts for the spherical nature of Earth has been added to the formulation because it regularizes the problem.

Implemented with FreeFem++ [8], a typical numerical result is shown on Figure 1. Convergence of the iterative scheme is fast, but the error decrease with the mesh size h is slow, less than expected, and does not decrease beyond a threshold even with automatic mesh adaptivity. It must be due to the singularities at $\mu = 0$, because the same method applied to a regular synthetic solution built from an appropriate right hand-side does not have such limitations. These singularities will appear clearly in the next section.

4. An integral formulation for the temperature

Consider equations (4), (5), (6) without scattering ($a_v = 0$): for all $\tau \in (0, Z)$,

$$\mu \partial_{\tau} I_v + \kappa_v I_v = \kappa_v B_v(T_{\tau}), \quad \forall \mu \in (-1, 1), \quad \forall v \in \mathbb{R}^+, \quad \int_0^{\infty} \kappa_v B_v(T_{\tau}) dv = \int_0^{\infty} \frac{\kappa_v}{2} \int_{-1}^1 I_v d\mu dv, \quad (11)$$

with B given in (4) and $I(0, \mu)|_{\mu>0} = Q_v \mu$ and $I(Z, \mu)|_{\mu<0} = 0$.

If $\tau \mapsto T_\tau$ is known, the first equation has a closed form solution:

$$I(\tau, \mu) = \mathbf{1}_{\mu>0} \left[Q_v \mu e^{-\kappa_v \frac{\tau}{\mu}} + \int_0^\tau \frac{e^{\kappa_v \frac{t-\tau}{\mu}}}{\mu} \kappa_v B_v(T_t) dt \right] - \mathbf{1}_{\mu<0} \int_\tau^Z \frac{e^{\kappa_v \frac{t-\tau}{\mu}}}{\mu} \kappa_v B_v(T_t) dt, \tag{12}$$

Here the singularity at $\tau = \mu = 0$ is visible and great numerical care is needed to resolve them.

Recall some identities for exponential integrals, for integer $n > -2$:

$$E_m(x) := \int_1^\infty \frac{e^{-xt}}{t^m} dt \Rightarrow \int_0^1 \mu^n e^{-\frac{x}{\mu}} d\mu = E_{n+2}(x), \quad - \int_{-1}^0 \mu^n e^{\frac{x}{\mu}} d\mu = (-1)^n E_{n+2}(x), \quad \forall x \in \mathbb{R}^+$$

An integration in μ of (12) gives an integral equation for $G_v(\tau) := \int_{-1}^1 I_v(\tau, \mu) d\mu$ and $B_v(T)$:

$$G_v(\tau) = Q_v E_3(\kappa_v \tau) + \kappa_v \int_0^Z E_1(\kappa_v |\tau - t|) B_v(T_t) dt, \quad \int_0^\infty \kappa_v B_v(T_\tau) dv = \int_0^\infty \frac{\kappa_v}{2} G_v(\tau) dv \tag{13}$$

Computing G_v from the first equation and T from the second leads to a scheme for which convergence will be established:

$$G_v^{n+1}(\tau) = Q_v E_3(\kappa_v \tau) + \kappa_v \int_0^Z E_1(\kappa_v |\tau - t|) B_v(T_t^n) dt, \tag{14}$$

$$\int_0^\infty \kappa_v B_v(T_\tau^{n+1}) dv = \int_0^\infty \frac{\kappa_v}{2} G_v^{n+1}(\tau) dv \tag{15}$$

Remark 1. Note that once $\tau \mapsto T_\tau$ has been computed by using (13), $I_v(\tau, \mu)$ can be computed by (12).

Lemma 2. Assume that κ_v is bounded from below by $\kappa_m > 0$ and above by κ_M . It is always possible to compute a unique $\tau \mapsto T_\tau^{n+1}$ from (15).

Proof. This is because the Stefan–Boltzmann formula (10) leads to

$$F(T) := \int_0^\infty \kappa_v B_v(T) dv, \quad \Rightarrow \kappa_m \frac{\pi^4}{15} T^4 \leq F(T) \leq \kappa_M \frac{\pi^4}{15} T^4, \tag{16}$$

from which we see that $F(0) = 0$ and $F(+\infty) = +\infty$. So, $T \mapsto F(T)$ being continuous, there is at least one solution to the equation

$$F(T) = \int_0^\infty Q_v E_3(\tau) dv + \int_0^\infty \frac{\kappa_v}{2} G_v(\tau) dv,$$

because the right hand-side is always non negative.

Uniqueness holds because $T \mapsto F(T)$ is strictly increasing:

$$\frac{dF}{dT} = \frac{d}{dT} \int_0^\infty \frac{\kappa_v v^3}{e^{\frac{v}{T}} - 1} dv = \int_0^\infty \frac{\kappa_v v^4 e^{\frac{v}{T}}}{T^2 (e^{\frac{v}{T}} - 1)^2} dv > 0. \quad \square$$

Proposition 3 (The homogeneous case). Assume that $\kappa_v \in (\kappa_m, \kappa_M)$, $\kappa_m > 0$. Let

$$c_1(\kappa_v) := \sup_{\tau \in (0, Z)} \int_0^Z E_1(\kappa_v |\tau - t|) dt,$$

(see Figure 2). Consider (14),(15) with $Q_v = 0$. Then, for some constant c_0 ,

$$T_\tau^n \leq c_0 \left(\frac{\kappa_M c_1(\kappa_m)}{2} \right)^{\frac{n}{4}}, \quad \forall \tau \in (0, Z), \quad |T^{n+1}|_\infty \leq \left(\frac{\kappa_M^2 c_1(\kappa_m)}{2\kappa_m} \right)^{\frac{1}{4}} |T^n|_\infty. \tag{17}$$

Proof. Notice that $E_1(\kappa_v|\tau - t) < E_1(\kappa_m|\tau - t)$ because E_1 is a decreasing function. Therefore, from (14),

$$\begin{aligned} & \int_0^\infty \kappa_v G_v^{n+1}(\tau) dv \\ &= \int_0^\infty \kappa_v^2 \int_0^Z E_1(\kappa_v|\tau - t) B_v(T_t^n) dt dv \leq \kappa_M \int_0^Z E_1(\kappa_m|\tau - t) \int_0^\infty \kappa_v B_v(T_t^n) dv dt \\ &= \frac{\kappa_M}{2} \int_0^Z E_1(\kappa_m|\tau - t) \int_0^\infty \kappa_v G_v^n(t) dv dt \leq \frac{\kappa_M c_1(\kappa_m)}{2} \sup_{t \in (0, Z)} \left(\int_0^\infty \kappa_v G_v^n(t) dv \right); \end{aligned} \tag{18}$$

(15) has been used to derive the equality in the second line. Consequently, using (16),

$$\begin{aligned} \kappa_m \frac{\pi^4}{15} (T_\tau^n)^4 &\leq \int_0^\infty \kappa_v B_v(T_\tau^n) dv \leq \frac{1}{2} \sup_{t \in (0, Z)} \left(\int_0^\infty \kappa_v G_v^n(t) dv \right) \\ &\leq \frac{1}{2} \left(\frac{\kappa_M c_1(\kappa_m)}{2} \right)^n \sup_{t \in (0, Z)} \left(\int_0^\infty \kappa_v G_v^0(t) dv \right). \end{aligned} \tag{19}$$

To prove the second inequality in (17) we add a lower bound to the beginning of (18) by making use of (15) and (16):

$$\int_0^\infty \kappa_v G_v^{n+1}(\tau) dv = 2 \int_0^\infty \kappa_v B_v(T_\tau^{n+1}) dv \geq 2\kappa_m \frac{\pi^4}{15} (T_\tau^{n+1})^4.$$

Similarly we add an upper bound to the end of (18):

$$\sup_{t \in (0, Z)} \left(\int_0^\infty \kappa_v G_v^n(t) dv \right) = 2 \sup_{t \in (0, Z)} \int_0^\infty \kappa_v B_v(T_t^n) dv \leq 2\kappa_M \frac{\pi^4}{15} \left(|T_\infty^n|^4 \right).$$

□

Now let us analyse the non homogeneous case with $Q_v > 0$.

Lemma 4. *If $\kappa_M c_1(\kappa_m) < 2$, then (14),(15) generates a uniformly bounded sequence $\{|T^n|_\infty\}_{n \geq 0}$.*

Proof. Let $H(\tau) := \int_0^\infty \kappa_v Q_v E_3(\kappa_v \tau) dv$. By (14),

$$\int_0^\infty \kappa_v G_v^{n+1}(\tau) dv = H(\tau) + \int_0^\infty \int_0^Z \kappa_v^2 E_1(\kappa_v|\tau - t) B_v(T_t^n) dt dv \tag{20}$$

$$\leq H(\tau) + \kappa_M \int_0^Z E_1(\kappa_m|\tau - t) \int_0^\infty \kappa_v B_v(T_t^n) dv dt. \tag{21}$$

Therefore

$$\int_0^\infty \kappa_v B_v(T_\tau^{n+1}) dv \leq \frac{1}{2} H(\tau) + \frac{1}{2} \kappa_M c_1(\kappa_m) \sup_{t \in (0, Z)} \int_0^\infty \kappa_v B_v(T_t^n) dv. \tag{22}$$

Let $J^n = \sup_{t \in (0, Z)} \int_0^\infty \kappa_v B_v(T_t^n) dv$. With $c_2 = \frac{1}{2} \sup_{t \in (0, Z)} H(t)$ and $c_3 = \frac{1}{2} \kappa_M c_1(\kappa_m)$, the above is

$$J^{n+1} \leq c_2 + c_3 J^n, \Rightarrow J^{n+1} \leq c_4 := c_2 \sum_{k=0}^n c_3^k + c_3^{n+1} J^0.$$

Therefore by (22) and (16), $\kappa_m \frac{\pi^4}{15} (T_\tau^{n+1})^4 \leq J^{n+1} \leq c_4$. □

Theorem 5. *If $\kappa_M c_1(\kappa_m) < 2$, and if the iterative scheme (14),(15) is initialized with $T^0 = 0$, it generates a uniformly increasing sequence $T_\tau^{n+1} > T_\tau^n, \forall \tau \in \mathbb{R}^+$, for all n . Consequently there exists T_τ^* solution of (13) and $\lim_{n \rightarrow \infty} T_\tau^n = T_\tau^*, \forall \tau \in \mathbb{R}^+$.*

Proof. Observe, first, that T_τ^1 computed by (14), (15) with $T^0 = 0$, is positive. Assume that $T_\tau^n > T_\tau^{n-1}$ for all $\tau > 0$. Then $T \mapsto B_\nu(T)$ being increasing, $B_\nu(T_\tau^n) > B_\nu(T_\tau^{n-1})$, $\forall \tau, \nu$, and so, with $F(T) = \int_0^\infty \kappa_\nu B_\nu(T) d\nu$,

$$\begin{aligned}
 F(T_\tau^{n+1}) - F(T_\tau^n) &= \int_0^\infty \kappa_\nu (B_\nu(T_\tau^{n+1}) - B_\nu(T_\tau^n)) d\nu = \int_0^\infty \frac{\kappa_\nu}{2} (G_\nu^{n+1}(\tau) - G_\nu^n(\tau)) d\nu \\
 &= \int_0^\infty \frac{\kappa_\nu^2}{2} \int_0^Z E_1(\kappa_\nu |\tau - t|) (B_\nu(T_\tau^n) - B_\nu(T_\tau^{n-1})) dt d\nu > 0, \forall \tau \in \mathbb{R}^+. \quad (23)
 \end{aligned}$$

By the mean value theorem, there exist θ_τ^n such that $F(T_\tau^{n+1}) - F(T_\tau^n) = \frac{dF}{dT}(\theta_\tau^n)(T_\tau^{n+1} - T_\tau^n)$. As $\frac{dF}{dT} > 0$, it implies that $T_\tau^{n+1} > T_\tau^n$. By Lemma 2 T_τ^n is bounded. For each τ fixed, $\{T_\tau^n\}_n$ is a bounded increasing sequence, so it converges to some T_τ^* for all τ . In (14), $T \mapsto B_\nu(T)$ is continuous, so $B_\nu(T_\tau^n) \rightarrow B_\nu(T_\tau^*)$ for all t and ν and the convergence of $E_1(\kappa_\nu |\tau - t|) B_\nu(T_\tau^n) \rightarrow E_1(\kappa_\nu |\tau - t|) B_\nu(T_\tau^*)$ is monotone, the integral converges to the integral of the limit. In (15), for the same reasons, the left hand side integral converges to the integral of the limit and on the right hand-side G_ν^* is defined as the limit of the left hand-side of (14). This shows that T_τ^* is the solution of the problem. □

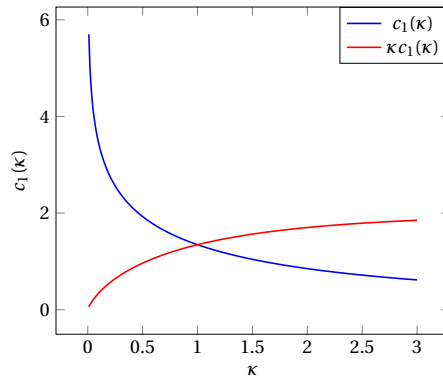


Figure 2. Plots of $\kappa \mapsto c_1(\kappa)$ and $\kappa \mapsto \kappa c_1(\kappa)$.

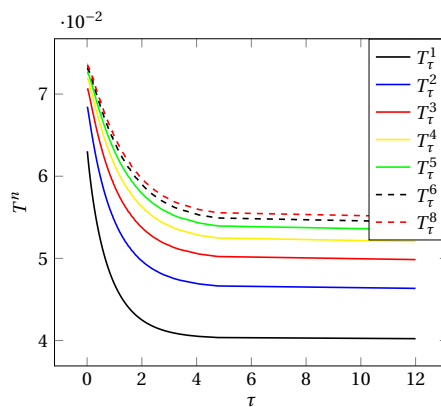


Figure 3. $\tau \mapsto T_\tau^n, n = 1 \dots 8, \kappa = 1.225 - 0.51 \mathbf{1}_{\nu \in (0.2, 0.3)}$ and $a_\nu = 0$.

5. Extension to include scattering

5.1. Isotropic scattering

Let $a_\nu \in (0, 1)$ and $p \equiv 1$ and consider (4),(5). As for (12), a partially closed form solution of (5) is

$$I(\tau, \mu) = \mathbf{1}_{\mu > 0} \left[Q_\nu \mu e^{-\kappa_\nu \frac{\tau}{\mu}} + \int_0^\tau \frac{e^{\kappa_\nu \frac{t-\tau}{\mu}}}{\mu} \kappa_\nu \left((1 - a_\nu) B_\nu(T_t) + \frac{a_\nu}{2} \int_{-1}^1 I_\nu(\tau, \mu) d\mu \right) dt \right] - \mathbf{1}_{\mu < 0} \int_\tau^Z \frac{e^{\kappa_\nu \frac{t-\tau}{\mu}}}{\mu} \kappa_\nu \left((1 - a_\nu) B_\nu(T_t) + \frac{a_\nu}{2} \int_{-1}^1 I_\nu(\tau, \mu) d\mu \right) dt, \tag{24}$$

With the same definition for $G_\nu = \int_{-1}^1 I_\nu d\mu$, the previous method can be extended to isotropic scattering, leading to the iterative scheme:

$$G_\nu^{n+1}(\tau) = Q_\nu E_3(\kappa_\nu \tau) + \kappa_\nu \int_0^Z E_1(\kappa_\nu |\tau - t|) \left(\frac{a_\nu}{2} G_\nu^n(t) + (1 - a_\nu) B_\nu(T_t^n) \right) dt, \tag{25}$$

$$\int_0^\infty \kappa_\nu (1 - a_\nu) B_\nu(T_\tau^{n+1}) = \frac{1}{2} \int_0^\infty \kappa_\nu (1 - a_\nu) G_\nu^{n+1}(\tau) d\nu. \tag{26}$$

The same argument used in Lemma 2 shows that T_τ^{n+1} is uniquely defined by (26).

As for Lemma 4, when $Q_\nu = 0$, it is easy to show that

$$\kappa_m (1 - a_M) \frac{\pi^4}{15} (T_t^{n+1})^4 \leq \frac{\kappa_M^2}{2} c_1(\kappa_m) (1 - a_m) (1 + a_M - a_m) \sup_{t \in (0, Z)} \int_0^\infty B_\nu(T_t^n) d\nu$$

Therefore linear convergence to zero will hold if $(\frac{\kappa_M}{\kappa_m})^2 \kappa_m c_1(\kappa_m) \frac{1 - a_m}{1 - a_M} (1 + a_M - a_m) < 2$.

Theorem 6. *Provided that $\kappa_M c_1(\kappa_m) (1 + a_M - a_m) < 2$, The iterative scheme (25),(26) generates a uniformly bounded increasing $\{\tau \mapsto T_\tau^n\}_n \nearrow \{\tau \mapsto T_\tau^*\}$, solution of (4),(5).*

Proof. The proof is similar to that of Theorem 5.

Assume that $T_\tau^n > T_\tau^{n-1}$ and $G_\nu^n(\tau) > G_\nu^{n-1}(\tau)$ for all $\tau > 0$. Then $T \mapsto B_\nu(T)$ being increasing, $B_\nu(T_\tau^n) > B_\nu(T_\tau^{n-1})$, $\forall \tau, \nu$, and so $F(T_\tau^n) := \int_0^\infty \kappa_\nu (1 - a_\nu) B_\nu(T_\tau^n) d\nu > F(T_\tau^{n-1})$. Hence $\forall \tau \in \mathbb{R}^+$,

$$\begin{aligned} F(T_\tau^{n+1}) - F(T_\tau^n) &= \int_0^\infty \frac{\kappa_\nu}{2} (1 - a_\nu) \int_0^Z E_1(\kappa_\nu |\tau - t|) \kappa_\nu \left(\frac{a_\nu}{2} (G_\nu^n(\tau) - G_\nu^{n-1}(\tau)) + (1 - a_\nu) (B_\nu(T_t^n) - B_\nu(T_t^{n-1})) \right) dt d\nu \\ &\geq \frac{\kappa_m}{2} (1 - a_M) \int_0^Z E_1(\kappa_M |\tau - t|) \int_0^\infty \kappa_\nu \left(\frac{a_\nu}{2} (G_\nu^n(\tau) - G_\nu^{n-1}(\tau)) + (1 - a_\nu) (B_\nu(T_t^n) - B_\nu(T_t^{n-1})) \right) dt d\nu. \end{aligned}$$

So $F(T_\tau^{n+1}) - F(T_\tau^n) > 0$ and by the mean value theorem $T_\tau^{n+1} > T_\tau^n$. Similarly, from (25),

$$G_\nu^{n+1}(\tau) - G_\nu^n(\tau) = \kappa_\nu \int_0^Z E_1(\kappa_\nu |\tau - t|) \left(\frac{a_\nu}{2} (G_\nu^n(t) - G_\nu^{n-1}(t)) + (1 - a_\nu) (B_\nu(T_t^n) - B_\nu(T_t^{n-1})) \right) dt,$$

is uniformly positive. To show that T_τ^n is bounded we proceed as in the proof of Lemma 2 up to (22), which, with $H'(\tau) = \int_0^\infty Q_\nu \kappa_\nu (1 - a_\nu) E_3(\kappa \tau) d\nu$, becomes

$$\begin{aligned} &\int_0^\infty \kappa_\nu (1 - a_\nu) B_\nu(T_\tau^{n+1}) d\nu \\ &\leq \frac{1}{2} H'(\tau) + \frac{1}{2} \kappa_M c_1(\kappa_m) \sup_{t \in (0, Z)} \int_0^\infty \kappa_\nu \left((1 - a_\nu)^2 B_\nu(T_t^n) + \frac{a_\nu}{2} (1 - a_\nu) G_\nu^n(t) \right) d\nu \\ &\leq \frac{1}{2} H'(\tau) + \frac{1}{2} \kappa_M c_1(\kappa_m) \sup_{t \in (0, Z)} \int_0^\infty \kappa_\nu \left((1 - a_\nu)^2 B_\nu(T_t^n) + a_M (1 - a_\nu) B_\nu(T_t^n) \right) d\nu \end{aligned} \tag{27}$$

For the last term, (26) has been used. Define $J^n = \sup_{t \in (0, Z)} \int_0^\infty \kappa_\nu (1 - a_\nu) B_\nu(T_t^n) d\nu$. Then

$$J^{n+1} \leq \frac{1}{2} |H'|_\infty + \frac{1}{2} \kappa_M c_1 (\kappa_m) (1 + a_M - a_m) J^n$$

The rest of the proof is as for Theorem 5. □

5.2. Non isotropic scattering

The most common expression for the probability of scattering from a direction ω' to a direction ω is $p(\omega \cdot \omega')$, which is, in the case of the atmosphere $p(\mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2})$.

Unfortunately, the generalization to non-isotropic scattering seems doable only when $p(\mu, \mu') = \sum_{i=0}^M c_i \mu'^i$; on the other hand a_ν can depend on τ . Recall that if κ_ν depends on τ (24) does not hold. We illustrate our claim with one example.

Let $p = 1 + \beta\mu'$, $\beta \in (0, 1)$. Note that $p \in (1 - \beta, 1 + \beta)$ and $\frac{1}{2} \int_{-1}^1 p(\mu, \mu') d\mu' = 1$. The problem is:

$$\mu \partial_\tau I_\nu + \kappa_\nu I_\nu - \frac{\kappa_\nu a_\nu}{2} \int_{-1}^1 (1 + \beta\mu') I_\nu(\tau, \mu') d\mu' = \kappa_\nu (1 - a_\nu) B_\nu(T_\tau), \tag{28}$$

$$\int_0^\infty \kappa_\nu (1 - a_\nu) B_\nu(T_\tau) d\nu = \int_0^\infty \frac{\kappa_\nu}{2} (1 - a_\nu) \int_{-1}^1 I_\nu d\mu. \tag{29}$$

Let

$$G_\nu(\tau) := \int_{-1}^1 I_\nu(\tau, \mu) d\mu, \quad S_\nu(\tau) = \int_{-1}^1 \mu I_\nu(\tau, \mu) d\mu.$$

Expanding (28) gives

$$\begin{aligned} \mu \partial_\tau I + \kappa_\nu I &= \frac{\kappa_\nu a_\nu}{2} \beta S_\nu + \frac{\kappa_\nu a_\nu}{2} G_\nu + \kappa_\nu (1 - a_\nu) B_\nu, \\ \int_0^\infty \kappa_\nu (1 - a_\nu) B_\nu(T_\tau) d\nu &= \int_0^\infty \frac{\kappa_\nu}{2} G_\nu(\tau) d\nu. \end{aligned} \tag{30}$$

Therefore

$$\begin{aligned} I = \mathbf{1}_{\mu > 0} &\left[Q_\nu \mu e^{-\kappa_\nu \frac{\tau}{\mu}} + \int_0^\tau \frac{e^{\kappa_\nu \frac{t-\tau}{\mu}}}{\mu} \kappa_\nu \left[(1 - a_\nu) B_\nu(t) + \frac{a_\nu}{2} G_\nu(t) + \frac{a_\nu}{2} \beta S_\nu(t) \right] dt \right] \\ &- \mathbf{1}_{\mu < 0} \int_\tau^Z \frac{e^{\kappa_\nu \frac{t-\tau}{\mu}}}{\mu} \kappa_\nu \left[(1 - a_\nu) B_\nu(T_t) + \frac{a_\nu}{2} G_\nu(t) + \frac{a_\nu}{2} \beta S_\nu(t) \right] dt, \end{aligned} \tag{31}$$

An integration in μ gives G_ν and a multiplication by μ followed by an integration in μ gives S_ν :

$$\begin{aligned} G_\nu(\tau) &= Q_\nu E_3(\kappa_\nu \tau) + \kappa_\nu \int_0^Z \left[E_1(\kappa_\nu |\tau - t|) \left(\frac{a_\nu}{2} G_\nu(t) + (1 - a_\nu) B_\nu(T_t) + \beta \frac{a_\nu}{2} S_\nu(t) \right) \right] dt \\ S_\nu(\tau) &= Q_\nu E_4(\kappa_\nu \tau) + \kappa_\nu \int_0^\tau \left[E_2(\kappa_\nu |\tau - t|) \left(\frac{a_\nu}{2} G_\nu(t) + (1 - a_\nu) B_\nu(T_t) + \beta \frac{a_\nu}{2} S_\nu(t) \right) \right] dt \\ &- \kappa_\nu \int_\tau^Z \left[E_2(\kappa_\nu |\tau - t|) \left(\frac{a_\nu}{2} G_\nu(t) + (1 - a_\nu) B_\nu(T_t) + \beta \frac{a_\nu}{2} S_\nu(t) \right) \right] dt \end{aligned} \tag{32}$$

It leads to a formulation involving 2 frequency dependent integrals, $S_\nu(\tau)$ and $G_\nu(\tau)$ for which the iterative scheme is proposed:

$$\begin{aligned} G_\nu^{n+1}(\tau) &= Q_\nu E_3(\kappa_\nu \tau) + \kappa_\nu \int_0^Z \left[E_1(\kappa_\nu |\tau - t|) \left(\frac{a_\nu}{2} G_\nu^n(t) + (1 - a_\nu) B_\nu(T_t^n) + \frac{a_\nu}{2} \beta S_\nu^n(t) \right) \right] dt, \\ S_\nu^{n+1}(\tau) &= Q_\nu E_4(\kappa_\nu \tau) + \kappa_\nu \int_0^\tau \left[E_2(\kappa_\nu |\tau - t|) \left(\frac{a_\nu}{2} G_\nu^n(t) + (1 - a_\nu) B_\nu(T_t^n) + \frac{a_\nu}{2} \beta S_\nu^n(t) \right) \right] dt \\ &- \kappa_\nu \int_\tau^Z \left[E_2(\kappa_\nu |\tau - t|) \left(\frac{a_\nu}{2} G_\nu^n(t) + (1 - a_\nu) B_\nu(T_t^n) + \frac{a_\nu}{2} \beta S_\nu^n(t) \right) \right] dt, \end{aligned} \tag{33}$$

$$\begin{aligned} &\int_0^\infty \kappa_\nu (1 - a_\nu) B_\nu(T_\tau^{n+1}) d\nu = \int_0^\infty \frac{\kappa_\nu}{2} (1 - a_\nu) G_\nu^{n+1}(\tau) d\nu. \end{aligned} \tag{34}$$

6. Numerical Results

The temperature of the Sun is $5800K$; the heat flux on Earth is 1370Watt/m^2 and 75% reaches the ground, approximately. All variables are de-dimensionalized (see [2, Section 3.3]), giving Q_v to 3.042×10^{-5} times the Planck function (2) and $T_{sun} = 1.209$.

A Computer implementation in C++ has been written. With the parameters described below, the code runs in 2.5 seconds on a MacBook Pro i9 at 2.3GHz.

In all tests, there are 120 altitude points and 150 frequencies both geometrically spaced from the origin. In the integrals with E_1 the t-step is 0.005 or smaller to make sure that there are at least 4 quadrature points in each integral. When $|\tau - t| \approx 0$, the log part of E_1 is integrated semi-analytically.

We report on 7 runs which are combinations of 3 cases for κ_v and 3 cases for a_v :

- $\kappa_0 = 1.225$, corresponding to $k_v = 1$ because the density of air is $\rho_0 e^{-r}$, $\rho_0 = 1.225$, and $\kappa_v = \rho_0 k_v$. This choice corresponds to Milne's problem for a grey atmosphere.
- $\kappa_1 = \kappa_0 - 0.5 \mathbf{1}_{(0.2,0.3)}$. When a greenhouse gas (GHG) is added to the atmosphere the infrared range (0.2,0.3) which was transparent before becomes opaque. Thus going from κ_1 to κ_0 tells what happens to the temperature when a GHG is added.
- $\kappa_2 = \kappa_0 - 0.5 \mathbf{1}_{(0.1,0.4)}$. GHG can narrow the infrared transparency window. Thus going from κ_3 to κ_2 is another way to simulate the effect of a GHG on the atmosphere.
- $a_v = 0$ means no scattering.
- Another choice is $a = 0.3 \mathbf{1}_{\tau \in (0.6,0.9)Z}$ which corresponds to a layer of clouds between altitude 6000m and 9600m with 0.3 isotropic scattering intensity.
- $a = 0.3 \mathbf{1}_{\tau \in (0.5,0.8)Z}$ with $p(\mu, \mu') = 1 - \frac{1}{2}\mu'$ which corresponds to anisotropic scattering between 6000m and 9600m with preferred downward scattering.

The results are reported on Figure 4. On the left, scaled temperatures are plotted for the 7 cases. On the right 6 relative temperature changes display more clearly the effect of increasing κ in an infrared region or narrowing the transparent window and/or the effect of adding scattering.

The numbers speak for themselves but let us point to a few unexpected facts:

- Increasing κ_1 to κ_0 cools the atmosphere. This unexpected phenomenon is observed in the high atmosphere where there is no scattering [5].
- Narrowing the transparency window by going from κ_3 to κ_2 also cools the atmosphere.
- When albedo due to a layer of cloud is added, the atmosphere heats up when κ increases from κ_1 to κ_0 and also if κ_3 is changed to κ_2 . It shows a greenhouse effect which wasn't there in absence of scattering.
- Going from isotropic to $1 - \mu$ anisotropic scattering makes little change.

7. Conclusion

To study the effect on the atmosphere due to changes in absorption and albedo coefficients, we propose to use the integral formulation which results from an elimination of the light intensity from the radiative transfer equations. This integral formulation does not contain any numerically singular terms, except for the integral of a log function near zero. In addition of giving results to any desired precision it is computationally very fast. However it does not handle the anisotropic scattering commonly used.

We have shown that the formulation is sound because existence holds when $\kappa_{max}/\kappa_{min}$ is not too large. An iterative scheme has been shown to be monotone and linearly convergent. In a forthcoming publication [7] we will show uniqueness and how to relax the constraints on κ_v by a method which uses the bounds on $\kappa_{c_1}(\kappa)$ rather than $c_1(\kappa)$ alone.

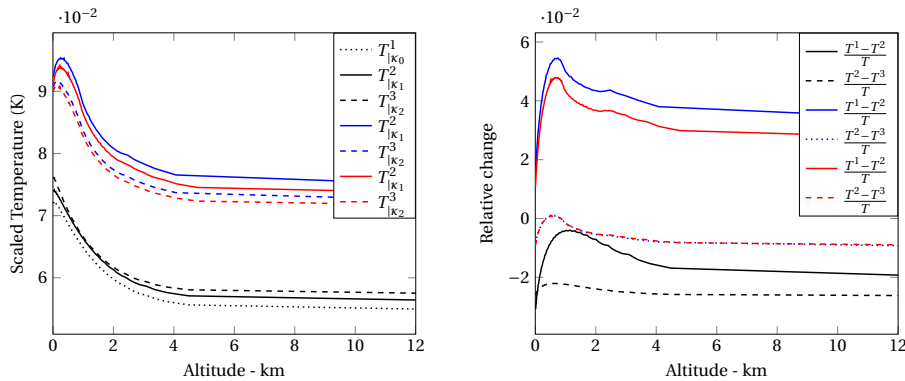


Figure 4. Scaled temperatures (left) and relative temperature changes (right). Black color curves: no scattering and $\kappa_0 = 1.225$ for T^1 , $\kappa_1 = \kappa_0 - 0.5 \mathbf{1}_{(0.2, 0.3)}$ for T^2 , $\kappa_2 = \kappa_0 - 0.5 \mathbf{1}_{(0.1, 0.4)}$ for T^3 . Blue color: isotropic scattering with $a = 0.3 \mathbf{1}_{\tau \in (6, 9.6) km}$. Red color: anisotropic scattering with a as above and $p = 1 - \frac{\mu'}{2}$. Note that, on the right, the dotted blue curve and the dashed red one are on top of each other.

Some numerical tests have been performed to see the effect of greenhouse gases, and demonstrate the power of this new method.

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