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Complex analysis and geometry / *Analyse et géométrie complexes*

# Levi Problem: Complement of a closed subspace in a Stein space and its applications

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**Abstract.** Let  $Y$  be an open subset of a Stein space  $X$ . We show that if  $Y$  is locally Stein and the complement  $X - Y$  is a closed subspace of  $X$ , then  $Y$  is Stein. We also discuss the applications of the theorem to open subsets  $Y$  whose boundaries in  $X$  are not closed subspaces of  $X$ . For example, we show that if for every boundary point  $P \in \partial Y$ , there is a closed subspace  $H$  of pure codimension 1 in  $X$  such that  $P \in H$ ,  $H \cap Y = \emptyset$  and  $X - H$  is locally Stein, then  $Y$  is Stein.

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## 1. Introduction

We consider the following Levi Problem in this paper. The detailed discussions and historic developments of the Levi Problem can be found in many literatures, e.g., [8, 29, 31].

**Levi Problem.** Is a locally Stein open subset of a Stein space Stein?

Let  $Y$  be an open subset of a complex space  $X$ .  $Y$  is locally Stein if for every point  $x$  on the boundary  $\partial Y$  in  $X$ , there is an open neighborhood  $U$  of  $x$  in  $X$  such that  $Y \cap U$  is Stein. A complex space  $Y$  is Stein if it is holomorphically separable (i.e., for any two distinct points  $y_1$  and  $y_2$  in  $Y$ , there is a holomorphic function  $f \in H^0(Y, \mathcal{O}_Y)$  on  $Y$  such that  $f(y_1) \neq f(y_2)$ ) and holomorphically convex (i.e., for any discrete sequence on  $Y$ , there is a holomorphic function  $f \in H^0(Y, \mathcal{O}_Y)$  such that  $f$  is not bounded on the sequence) ([16, pp. 230, 293–294]).

Many mathematicians have made major contributions and proved several important special cases, e.g., [1, 4, 6–9, 19–21, 26, 29, 32] (The literature is vast and this is not a complete list). In 1953, Oka observed that the local property of the boundary of a complex manifold  $Y$  determines the Steinness of  $Y$  [21]. Docquier and Grauert proved that a locally Stein open subset of a Stein

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manifold is Stein [7]. Andreotti and Narasimhan proved that if  $X$  is a Stein space with isolated singularities, then a locally Stein open subset  $Y$  in  $X$  is Stein [1]. Their proof heavily relies on the fact that  $-\log d$  is a pseudoconvex function, where  $d$  is the distance function ([1, 29]). It seems that it is very difficult to find an analogue of  $-\log d$  if the singular points of  $X$  are not isolated. Norguet and Siu showed that if there is a continuous exhaustion function  $\phi$  with two properties, then the space is Stein [19]. Diederich and Ohsawa proved that if  $Y$  is a relatively compact domain in a smooth complex surface  $X$  such that the boundary of  $Y$  is smooth, real analytic, connected and strongly pseudoconvex at some point, then  $Y$  is holomorphically convex [6]. The counterexample in the book of Grauert and Remmert showed that the theorem for surfaces proved by Simha [26] does not hold for higher dimensional complex spaces ([10, p. 130]). Fornæss and Narasimhan considered the Levi problem with any singularities and gave several sufficient conditions such that a locally Stein open subset of a Stein space is Stein [8]. There are many variations of their theorems later.

Since the dimension  $\dim_P(X - Y)$ ,  $P \in X - Y$  is upper semi-continuous on  $X - Y$  ([11, p. 94]), if  $X - Y$  is a closed subspace of  $X$  and  $Y$  is locally Stein, then every irreducible component of  $X - Y$  is of pure codimension 1 ([10, p. 128]), i.e.,  $X - Y$  is a complex analytic hypersurface on  $X$ . To prove that  $Y$  is Stein, we only need to show that  $Y$  is holomorphically convex. We know that with some conditions on the domain, by Runge Approximation Theorem, holomorphic functions or maps can be approximated by polynomial functions or maps ([10, p. 90]). This is an important relationship between analytic objects and algebraic objects. On the other hand, J.-P. Serre constructed a nonsingular complex open algebraic surface and showed that it has no any nonconstant regular functions even though it is a Stein surface ([13, p. 232]). If we consider all holomorphic functions on  $Y$ , then the sheaf of meromorphic functions with singularities on  $X - Y$  is not coherent so we cannot apply Cartan's Theorem A and B. Also it seems that all known methods in the past do not work for singular spaces. In order to construct global holomorphic functions on  $Y$ , we use theory of holomorphic functions with polynomial growth on open subsets and connect them to global sections of suitable coherent analytic sheaves (divisorial sheaves) on  $X$  associated to a Weil divisor  $D$  with support in  $X - Y$ .

We identify a point  $z = (z_1, \dots, z_n) \in \mathbb{C}^n$  with  $x = (x_1, \dots, x_{2n}) \in \mathbb{R}^{2n}$  by  $z_k = x_k + ix_{k+n}$ ,  $1 \leq k \leq n$  and define

$$|z| = |x| = \left( \sum_{k=1}^{2n} |x_k|^2 \right)^{1/2}.$$

Let  $\Omega$  be a bounded open subset in  $\mathbb{C}^n$  and  $\bar{\Omega}$  the closure of  $\Omega$  in  $\mathbb{C}^n$ . For a point  $z \in \mathbb{C}^n$ , define

$$d(z, \Omega) = \inf_{w \in \Omega} |z - w|$$

and the distance from  $z$  to the boundary  $\partial\Omega$  to be

$$d_{\Omega}(z) = d(z, \mathbb{C}^n - \Omega).$$

Following Narasimhan and Siu, we use the following definition [18, 28].

**Definition 1.** A holomorphic function  $f$  in a bounded domain  $\Omega \subset \mathbb{C}^n$  is of polynomial growth if there are positive constants  $C, \alpha$  (may depend on  $f$ ) such that for every point  $z \in \Omega$ , we have

$$|f(z)| \leq Cd_{\Omega}(z)^{-\alpha}.$$

Siu's theory is [28]: Given a matrix  $(\phi_{ij})_{1 \leq i \leq r, 1 \leq j \leq s}$  of holomorphic functions on a neighborhood of the closure  $\bar{\Omega}$  of a bounded open subset  $\Omega$  in  $\mathbb{C}^n$ , let  $(f_i)_{1 \leq i \leq r}$  be an  $r$ -tuple of holomorphic functions on  $\Omega$  having polynomial growth. Assume for some  $s$ -tuple holomorphic functions  $(g_j)_{1 \leq j \leq s}$  on  $\Omega$ , we have

$$f_i = \sum_{j=1}^s \phi_{ij} g_j, \quad 1 \leq i \leq r.$$

If  $\Omega$  is Stein, then there are  $s$ -tuple holomorphic functions  $(h_j)_{1 \leq j \leq s}$  of polynomial growth such that

$$f_i = \sum_{j=1}^s \phi_{ij} h_j, \quad 1 \leq i \leq r.$$

An analytic subset of a complex space  $X$  is not locally biholomorphic to a domain in  $\mathbb{C}^n$  in general. To apply Siu's theory, we use the proper finite holomorphic map from a suitable Stein open subset containing an analytic block in  $X$  to  $\mathbb{C}^n$  and consider the bounded Stein domain containing the Stein subvariety in the bounded open subset of  $\mathbb{C}^n$ . Since every Stein subvariety in a bounded domain in  $\mathbb{C}^n$  admits a Stein neighborhood [30], Siu's theory can be applied to the holomorphic functions in the Stein neighborhood then we consider their restrictions to the Stein subvariety. The traditional and general method to approach the Levi problem is to construct strictly plurisubharmonic exhaustion functions by distance functions. The singularities of  $X$  play an important role in this method. The more singular  $X$  is, the more complicated the construction of the distance function is [1, 8, 31]. It seems that there is no known method to directly deal with bad singularities.

In this paper, we will use a different approach to investigate the Levi problem. With generalized version of polynomial growth theory for multiple holomorphic functions in a bounded Stein domain due to Siu [28], Stein exhaustion theory for Stein spaces [10], and algebraic geometry techniques [10, 11, 14, 35–39], we settle the case when  $X - Y$  is a closed subspace of  $X$ .

**Theorem 2.** *Let  $Y$  be a locally Stein open subset of a Stein space  $X$  such that the complement  $X - Y$  is a closed subspace of  $X$ , then  $Y$  is Stein.*

If  $X - Y$  is locally defined by one holomorphic function, Theorem 2 is claimed to be true without a proof ([10, p. 130]). Locally Stein condition in Theorem 2 is necessary. The following is a counter-example of Grauert and Remmert ([10, p. 130]). Let  $X \in \mathbb{C}^4$ ,

$$X = \{z = (z_1, z_2, z_3, z_4) \in \mathbb{C}^4, p(z) = z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\}.$$

The structure sheaf

$$\mathcal{O}_X = \mathcal{O}_{\mathbb{C}^4} / p(z) \mathcal{O}_{\mathbb{C}^4} |_X.$$

$X$  is a normal Stein space with a unique isolated singularity at 0. Define a hypersurface through 0 by

$$H = \{z = (z_1, z_2, z_3, z_4) \in X, z_1 = iz_2, z_3 = iz_4\}.$$

$H$  cannot be defined by a single holomorphic function.  $X - H$  is not Stein. By Andreotti and Narasimhan's theorem, the open subset  $Y = X - H$  is not locally Stein [1].

Theorem 2 can be stated in the following form.

**Corollary 3.** *Let  $X$  be a Stein space and  $Y$  an open subset of  $X$ . If the boundary  $X - Y$  is a closed subspace in  $X$ , then  $Y$  is Stein if and only if  $Y$  is locally Stein.*

**Corollary 4 (Grauert, Remmert).** *Let  $X$  be a Stein space and  $Y$  an open subset of  $X$ . If the boundary  $X - Y$  is a closed subspace in  $X$  such that locally at every point in  $X - Y$ ,  $X - Y$  is defined by one holomorphic function, then  $Y$  is Stein.*

In Corollary 4,  $X - Y$  is a Cartier divisor. Andreotti and Narasimhan proved that on a  $K$ -complete space, any relatively compact open set which is pseudoconvex with a globally defined boundary is a Stein space [1]. If  $X - Y$  is not a closed subspace of  $X$ , we can apply Theorem 2 to prove the following theorem.

**Theorem 5.** *Let  $X$  be a Stein space and  $Y$  an open subset of  $X$ . If for every boundary point  $P \in \partial Y$  in  $X$ , there is a closed subspace  $H$  of pure codimension 1 in  $X$  such that  $P \in H$ ,  $H \cap Y = \emptyset$  and  $X - H$  is locally Stein, then  $Y$  is Stein.*

The theorem also can be applied to open subsets  $Y$  with real analytic boundaries.

**Theorem 6.** *Let  $X$  be a Stein space and  $Y$  an open subset of  $X$ . If for every boundary point  $P \in \partial Y$  in  $X$ , there is a holomorphic function  $h$  in a neighborhood  $U$  of  $P$  such that  $\partial Y \cap U$  is defined by vanishing of  $h(z) + \overline{h(z)}$  in  $U$  and  $h(z) + \overline{h(z)}$  does not vanish on  $Y \cap U$ , then  $Y$  is Stein.*

Proof of Theorem 2 occupies almost the entire paper. The outline of the proof of Theorem 2 is the following. First, we may assume that  $X$  is a reduced space by the Reduction Theorem ([10, p. 154]) and only need to show that the normalization of  $X$  is Stein ([10, pp. 22, 45]; [16, p. 313]). To show that  $Y$  is holomorphically convex, for any discrete sequence on  $Y$  with an accumulation point  $P_0 \in X - Y$ , we will construct a holomorphic function on  $Y$  such that it is not bounded near  $P_0$ .

Since  $X$  is Stein,  $X = \cup_{r=1}^{\infty} X_r$ , where every  $X_r$  is an analytic block so a compact Stein set such that for every  $r \geq 1$ ,  $X_r$  is contained the analytic interior  $X_{r+1}^0$  of  $X_{r+1}$  and  $(X_r, X_{r+1})$  satisfies Runge Approximation Theorem ([10, pp. 122]): for every coherent sheaf  $\mathcal{F}$  on  $X$ , the space  $H^0(X_{r+1}, \mathcal{F})|_{X_r}$  is dense in  $H^0(X_r, \mathcal{F})$ . Since  $X_r$  is compact, it has a finite Stein open cover  $\{U_i\}_{i=1}^N$ . Choose a suitable complex analytic hypersurface  $H$  in  $X$ . By mathematical induction, we may assume that  $Z = Y \cap H$  is Stein [35–39]. For a holomorphic function  $f$  on  $Z \cap X_r$  with polynomial growth,  $r \in \mathbb{N}$ , it can be extended to a holomorphic function on each  $U_i \cap Y \cap X_r$  with polynomial growth. Patch them together by sheaf cohomology, we can construct a holomorphic function on  $Y_r = X_r \cap Y$  and meromorphic on  $X_r$  such that it is not bounded near the accumulation point  $P_0 \in X - Y_r$ . It shows that each intersection space  $Y_r$  is holomorphically convex. Next we introduce a semi-norm on the space of meromorphic functions with poles on the boundary  $X_r - Y$  ([10, p. 119]) and construct a Cauchy sequence with respect to this semi-norm by Runge Approximation Theorem ([10, p. 122]). Then there is a holomorphic function  $\phi$  on  $Y$  which is the limit of the Cauchy sequence such that  $\phi$  is not bounded on the discrete sequence in  $Y$  with the accumulation point  $P_0 \in X - Y$ . This shows that  $Y$  is holomorphically convex.

We organize this paper as follows. In Section 2, we will extend holomorphic functions with polynomial growth to meromorphic functions with (locally) finite order of pole singularities and connect them with divisorial sheaves associated to Weil divisors in complex spaces. In Section 3, we will construct meromorphic functions in analytic blocks with pole singularities on boundary hypersurfaces  $X_r - Y$ . In Section 4, we will use techniques of coherent sheaves and cohomology to extend holomorphic functions with polynomial growth on a hypersurface to meromorphic functions in the open subspace with pole singularities on the boundary. In Section 5, we will prove the theorems.

The terminology in this paper can be found in [5, 10, 11, 14, 22, 33] and the dimension is the complex dimension. Since the Levi problem for curves and surfaces in Theorem 2 has an affirmative answer [26], we assume that the complex space  $X$  is of dimension at least 3.

## 2. Polynomial Growth Theory for Stein Spaces

**Theorem 7 (Narasimhan).** *Let  $g$  be holomorphic in  $\bar{\Omega}$  (i.e. in a neighborhood of  $\bar{\Omega}$ ) and  $f$  a holomorphic function of polynomial growth in  $\Omega$ . If  $\frac{f}{g}$  is holomorphic in  $\Omega$ , then  $\frac{f}{g}$  has polynomial growth in  $\Omega$ .*

**Theorem 8 (Siu).** *Let  $\Omega$  be a bounded Stein open subset of  $\mathbb{C}^n$  and  $(\phi_{ij})_{1 \leq i \leq r, 1 \leq j \leq s}$  a matrix of holomorphic functions defined in some open neighborhood  $\tilde{\Omega}$  of  $\bar{\Omega}$ . Let*

$$\Phi: H^0(\Omega, \mathcal{O}_{\Omega}^s) \longrightarrow H^0(\Omega, \mathcal{O}_{\Omega}^r)$$

be induced by  $(\phi_{ij})_{1 \leq i \leq r, 1 \leq j \leq s}$ . Then there exist a positive real number  $C \geq 1$  and a nonnegative integer  $p$  with the following property: if  $f \in \text{Im } \Phi$  and for some nonnegative integer  $\alpha$  and  $A > 0$ ,

$$|f(z)| \leq Ad_{\Omega}(z)^{-\alpha}$$

for  $z \in \Omega$ , then there exists  $g \in H^0(\Omega, \mathcal{O}_{\Omega}^s)$  such that  $\Phi(g) = f$  and

$$|g(z)| \leq C^{1+\alpha} Ad_{\Omega}(z)^{-\alpha-p}.$$

Let  $X$  be an irreducible normal reduced complex space and  $Y$  an open subset of  $X$  such that  $X - Y$  is a closed subspace of pure codimension 1. At every point  $x$  of  $X$ , there is an open subset  $U$  of  $x$  which is biholomorphic (by  $\xi$ ) to a closed analytic subspace  $A$  of a bounded domain  $B$  in some  $\mathbb{C}^n$ : let  $\{f_i\}_{i \in I}$  be a finite set of holomorphic functions on  $B$  and  $\mathcal{J}$  the subsheaf of  $\mathcal{O}_B$  generated by the functions  $f_i$  as an  $\mathcal{O}_B$ -module. Let  $A = \{b \in B, f_i(b) = 0, i \in I\}$  and  $\mathcal{O}_A = \mathcal{O}_B / \mathcal{J}$ . A holomorphic function on  $A$  is an element of  $H^0(A, \mathcal{O}_A)$ . Two holomorphic functions  $f$  and  $\tilde{f}$  on  $B$  give the same holomorphic function on  $A$  if  $f - \tilde{f} \in \mathcal{J}$ .

**Definition 9.**

- (1)  $f \in H^0(V, \mathcal{O}_V)$  has polynomial growth in an open subset  $V = Y \cap U$  if in the above notation, for all  $z \in \xi(Y \cap U) \subset B \subset \mathbb{C}^n$ , there are constants  $C > 0$  and  $\alpha \geq 0$  such that

$$|f(\xi^{-1}(z))| \leq Cd_{B-\xi((X-Y) \cap U)}(z)^{-\alpha}.$$

- (2)  $f \in H^0(Y, \mathcal{O}_Y)$  has polynomial growth on  $Y$  if  $f$  has polynomial growth on every open subset  $V_i$ , where  $V_i = Y \cap U_i$  and  $\{U_i\}_{i \in I}$  is a bounded open cover of  $X$ .

For simplicity, we identify the open subset  $U$  with the closed subspace  $A$  of a bounded domain  $B$  in some  $\mathbb{C}^n$  and omit the biholomorphic map  $\xi$ . Then  $f \in H^0(V, \mathcal{O}_V)$  has polynomial growth in  $V$  if for all  $z \in V$ , there are constants  $C > 0$  and  $\alpha \geq 0$  such that

$$|f(z)| \leq Cd_V(z)^{-\alpha},$$

where

$$d_V(z) = d_{B-\xi((X-Y) \cap U)}(z).$$

**Definition 10.** A Weil divisor on an irreducible reduced complex space  $X$  is a locally finite linear combination with integral coefficients of irreducible reduced analytic subspaces of codimension 1 in  $X$  such that every subspace is not contained in the singular locus of  $X$ .

The set of all Weil divisors form an abelian group. If  $D$  is a Weil divisor, then we can write  $D = \sum_{i=1}^{\infty} n_i D_i$ , where  $n_i \in \mathbb{Z}$  and each  $D_i$  is an irreducible reduced analytic subspace of codimension 1 in  $X$  which is not contained in the singular locus of  $X$  ([2]; [10, pp. 139–140]; [14, pp. 130–143]; [33, pp. 35–36]).

The support of a Weil divisor  $D$  is the union of all closed subspaces  $D_i$  such that  $n_i \neq 0$ .  $D$  is an effective divisor, written  $D > 0$ , if every coefficient  $n_i \geq 0$  and  $D$  is not a zero divisor. Two Weil divisors  $D \geq D'$  if  $D - D' \geq 0$ , i.e.,  $D - D'$  is an effective divisor or a zero divisor in  $X$ . When every coefficient  $n_i = 1$ ,  $D = \sum D_i$  is called a reduced divisor.

When  $X$  is a compact normal reduced complex space, then a Weil divisor  $D$  is a finite sum on  $X$ :  $D = \sum_{i=1}^N n_i D_i$  ([33, p. 35]).

If  $X$  is normal, then the singular locus of  $X$  is a closed subspace of codimension at least 2 in  $X$ . A Weil divisor is well-defined as a linear combination of irreducible codimension one closed subspaces on a normal complex space  $X$ . Every Cartier divisor on a normal reduced complex space  $X$  defines a Weil divisor and if  $X$  is nonsingular, then every Weil divisor is Cartier, i.e., locally it is defined by one holomorphic function. But if  $X$  is not a complex manifold, then the Weil divisor  $D$  is not a Cartier divisor in general, i.e., it is not locally defined by one equation ([2]; [33, p. 36]).

A coherent sheaf  $\mathcal{F}$  on a complex space  $X$  is  $\mathcal{O}_X$ -reflexive if the natural map from  $\mathcal{F}$  to the double-dual,  $\mathcal{F} \rightarrow (\mathcal{F}^\vee)^\vee$  is an isomorphism, where

$$\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X).$$

For every coherent sheaf  $\mathcal{F}$  on  $X$ ,  $\mathcal{F}^\vee$  and  $(\mathcal{F}^\vee)^\vee$  are coherent ([11, p. 240]). A reflexive sheaf of rank one is called a divisorial sheaf.

**Definition 11.** Let  $D$  be a Weil divisor on a reduced irreducible normal space  $X$ . Then  $\mathcal{O}_X(D)$ , called the divisorial sheaf of  $X$  associated with  $D$ , is defined to be the sheaf associated to the following property:

$$\mathcal{O}_X(D)(U) = H^0(U, \mathcal{O}_X(D)) = \{\phi \in \mathcal{M}_X(U), \text{div}(\phi)|_U + D|_U \geq 0\},$$

where  $\mathcal{M}_X$  is the sheaf of meromorphic functions on  $X$ ,  $\text{div}(\phi)|_U$  is the principal divisor defined by  $\phi$  on the open subset  $U$  of  $X$  and if  $D = \sum n_i D_i$ , then  $D|_U = \sum n_i D_i|_U$ .

By definition, a section of  $\mathcal{O}_X(D)(U)$  is a meromorphic function on  $U$  with a pole along each  $D_i|_U$  up to order  $n_i$  if  $n_i > 0$ .

The following lemma is known in complex algebraic geometry for normal integral separated noetherian schemes of finite type [24] and irreducible reduced normal compact spaces of dimension 2 [23]. For higher dimensional complex spaces, we need the existence of resolution of singularities [15, 34] and Serre’s extension theorem of reflexive sheaves on complex spaces [25].

**Lemma 12.** For a Weil divisor  $D$  on a reduced irreducible normal space  $X$ , the sheaf  $\mathcal{O}_X(D)$  is coherent and reflexive.

**Proof.** Let  $\pi : X' \rightarrow X$  be a resolution of singularities and  $D'$  the strict transform of  $D$  [15, 34]. Then  $D'$  is a Cartier divisor on  $X'$  since  $X'$  is a complex manifold ([33, p. 36]). Let  $(f_i, U'_i)$  be the local equations of  $D'$  where  $\{U'_i\}_{i \in I}$  is an open cover of  $X'$ . For every point  $P \in U'_i$ , the stalk of the coherent sheaf  $\mathcal{O}_{X'}(D')$  for the Cartier divisor  $D'$  is defined by ([10, p. 146]; [33, p. 30])

$$\mathcal{O}_{X'}(D')_P = f_i^{-1} \mathcal{O}_{X', P}.$$

Since  $\pi$  is a proper holomorphic map [15, 34], by Grauert’s direct image theorem ([11, p. 207]), the zero direct image  $\pi_* \mathcal{O}_{X'}(D')$  is a coherent sheaf on  $X$ . Let  $S$  be the set of singular points of  $X$ , then  $S$  has codimension at least 2 in  $X$  since  $X$  is normal and  $\pi : X' - \pi^{-1}(S) \rightarrow X - S$  is a biholomorphic map (isomorphism). The restriction  $D|_{X-S}$  is a Cartier divisor on the nonsingular complex space  $X - S$ . The coherent sheaf  $\mathcal{O}_{X-S}(D|_{X-S})$  given by the Cartier divisor  $D|_{X-S}$  is equal to the restriction  $\pi_* \mathcal{O}_{X'}(D')|_{X-S}$  since their stalks at every smooth point of  $X$  are equal. In other words,  $\pi_* \mathcal{O}_{X'}(D')$  is a coherent extension of  $\mathcal{O}_{X-S}(D|_{X-S})$  in  $X$ .  $\mathcal{O}_{X-S}(D|_{X-S})$  is an invertible sheaf on  $X - S$  so is a reflexive sheaf on  $X - S$ . We show that  $\mathcal{O}_{X-S}(D|_{X-S})$  is extendible as a coherent reflexive sheaf.

Let

$$i : X - S \hookrightarrow X$$

be the natural inclusion map. By Serre’s theorem, if  $\mathcal{F}$  is an extendible reflexive sheaf on  $X - S$ , then  $i_* \mathcal{F}$  is a reflexive sheaf on  $X$ , which is unique as a reflexive extension of  $\mathcal{F}$  [25]. So  $i_*(\mathcal{O}_{X-S}(D|_{X-S}))$  is the unique coherent reflexive sheaf on  $X$  as a reflexive extension of  $\mathcal{O}_{X-S}(D|_{X-S})$  and

$$i_*(\mathcal{O}_{X-S}(D|_{X-S})) = \pi_* \mathcal{O}_{X'}(D').$$

Removing a codimension at least 2 subset from  $X$ , the sections of the sheaf  $\mathcal{O}_X(D)$  defined by a Weil divisor  $D$  do not change. More precisely, for any open subset  $W \subset X$ ,

$$H^0(W, \mathcal{O}_X(D)) = H^0(W - S, \mathcal{O}_X(D)) = H^0(W - S, \mathcal{O}_{X-S}(D|_{X-S})).$$

It implies that the natural map

$$\mathcal{O}_X(D) \rightarrow i_* (\mathcal{O}_X(D)|_{X-S}) = i_* (\mathcal{O}_{X-S}(D|_{X-S}))$$

is an isomorphism. So  $\mathcal{O}_X(D)$  is a coherent reflexive sheaf. □

**Remark 13.** On the complex manifold  $X - S$ , the sheaf  $\mathcal{F}$  given by the Weil divisor  $D|_{X-S}$  in above Definition 11 is equal to the sheaf  $\mathcal{G}$  defined by the corresponding Cartier divisor  $D|_{X-S}$ . In fact, let  $P \in X - S$  be a smooth point on  $X$  and  $U \subset X - S$  be an open subset such that  $D|_U$  is defined by a holomorphic function  $f$  in  $U$ . If  $\phi \in \mathcal{F}(U)$ , then

$$div(\phi)|_U + D|_U = div(\phi)|_U + div(f)|_U = div(\phi f) \geq 0,$$

which implies that the product function  $\phi f = g$  is a holomorphic function on  $U$ . So on  $U$ ,  $\phi = \frac{g}{f} \in f^{-1}\mathcal{O}_{P,X}$ , which shows that the stalk  $\mathcal{F}_{P,X} \subset f^{-1}\mathcal{O}_{P,X} = \mathcal{G}_{P,X}$ .

On the other hand, if  $\alpha = \frac{g}{f} \in f^{-1}\mathcal{O}_{P,X}$ , where  $g$  is holomorphic on  $U$ , then

$$div(\alpha)|_U + D|_U = [div(g) - div(f) + div(f)]|_U = div(g)|_U \geq 0.$$

It shows that the stalk  $\mathcal{G}_{P,X} \subset \mathcal{F}_{P,X}$ .

**Remark 14.** Let  $D$  and  $D'$  be two Weil divisors on an irreducible reduced complex space  $X$ . If  $D \leq D'$ , then we have  $H^0(X, \mathcal{O}_X(D)) \subset H^0(X, \mathcal{O}_X(D'))$ .

A holomorphic function with polynomial growth on  $Y$  can be extended to a meromorphic function on  $X$  with poles in  $X - Y$  by Lemmas 15 and 16, where we assume that the Weil divisor  $D$  with support in  $X - Y$  is a finite sum.

**Lemma 15.** *Let  $Y$  be an open subset of a connected Stein manifold  $X$  such that  $X - Y$  is a reduced closed subspace of pure codimension 1 in  $X$ . If  $f$  is a holomorphic function with polynomial growth on  $Y$ , then there is a meromorphic function  $g \in H^0(X, \mathcal{O}_X(D))$  for an effective Weil divisor  $D = \sum n_i D_i$  such that  $g|_Y = f$ .*

**Proof.** First, since  $X$  is a Stein manifold,  $H^1(X, \mathcal{O}_X) = 0$ . For any effective Weil divisor  $D$  on  $X$ ,  $\mathcal{O}_X$  is a subsheaf of the line bundle  $\mathcal{O}_X(D)$ . From the exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(D) \rightarrow \mathcal{O}_X(D)/\mathcal{O}_X \rightarrow 0,$$

we have

$$0 \rightarrow H^0(X, \mathcal{O}_X) \rightarrow H^0(X, \mathcal{O}_X(D)) \rightarrow H^0(X, \mathcal{O}_X(D)/\mathcal{O}_X) \rightarrow 0.$$

So the line bundle  $\mathcal{O}_X(D)$  has a lot of global sections on  $X$ .

We will show that there is a Weil divisor  $D$  on  $X$  with support in  $X - Y$  and a meromorphic function  $g \in H^0(X, \mathcal{O}_X(D))$  such that  $g|_Y = f$ .

Let  $\{U_i\}_{i \in I}$  be a Stein open cover of  $X$  such that each  $U_i$  is biholomorphic to a bounded Stein domain  $\Omega_i$  in  $\mathbb{C}^n$  and  $f$  is a holomorphic function on  $Y$  with polynomial growth on every  $V_i = U_i \cap Y$ .  $X - Y$  is a closed subspace with pure codimension 1 in  $X$ . Let  $S$  be the set of singular points of the closed subspace  $X - Y$ . Then  $S$  is a closed subspace with codimension at least 2 in  $X$  and  $(X - Y) - S$  is a complex submanifold of codimension 1 in  $X$  ([11, p. 117]). Let  $P_0 \in (X - Y) - S$  be a smooth point of the closed subspace  $X - Y$ . Then there is an  $U_i$  such that  $P_0 \in U_i$  and  $f$  is of polynomial growth in  $V_i = U_i \cap Y$ : there are two constants  $C_i > 0, \alpha_i > 0$  such that for all  $z \in V_i$ ,

$$|f(z)| \leq \frac{C_i}{d_{V_i}(z)^{\alpha_i}}.$$

On a complex manifold, a Weil divisor is a Cartier divisor ([33, p. 36]). We may change the coordinates and assume  $P_0 = 0$  is the origin and  $X - Y$  is defined by  $z_n = 0$  in a small



neighborhood of  $P_0$ . Let  $U \subset U_i$  be an open block centered at  $P_0 = 0$  such that  $(X - Y) \cap U$  is defined by  $z_n = 0$  in  $U$ :

$$(X - Y) \cap U = \{(z_1, \dots, z_n) \in U, z_n = 0\},$$

where  $(z_1, \dots, z_n)$  are the local coordinates. We may shrink  $U$  and assume that  $U$  is a small block: for every point  $z = (z_1, \dots, z_n) \in U, j = 1, \dots, n$ ,

$$z_j = x_j + iy_j, \quad |x_j| < a_j < 1, \quad |y_j| < b_j < 1.$$

$V = U \cap Y$  is an open subset of  $V_i = U_i \cap Y$ . For every point  $z \in V$ , we have  $d_V(z) \leq d_{V_i}(z)$ , and

$$|f(z)| \leq \frac{C_i}{d_{V_i}(z)^{\alpha_i}} \leq \frac{C}{d_V(z)^\alpha},$$

where for the simplicity, we write  $C = C_i, \alpha = \alpha_i$ . So  $f$  has polynomial growth on  $V = U \cap Y$ .

The open subset  $V$  is obtained by removing the submanifold  $z_n = 0$  from the open block  $U$ . For every point  $z = (z_1, \dots, z_n) \in V = U - (X - Y)|_U$ , the distance from  $z$  to the analytic set  $(X - Y)|_U \subset \partial V$  is  $|z_n|$  since  $(X - Y)|_U$  is defined by  $z_n = 0$ . It implies

$$d_V(z) = \min(d_U(z), |z_n|)$$

is the minimum distance from  $z$  to the real boundary hyperplanes.

The distance functions  $d_U(z)$  and  $d_V(z)$  are continuous. Let  $0 < \epsilon \leq 1/2$  and

$$\Omega_U = \{z \in U, d_U(z) > \epsilon |z_n|\}.$$

Then  $\Omega_U \subset U$  is an open neighborhood of  $(X - Y) \cap U$  [18] and

$$\Omega_V = \{z \in V, d_V(z) > \epsilon |z_n|\} = \{z \in V, d_U(z) > \epsilon |z_n|\} = \Omega_U - \{z_n = 0\}.$$

Let  $\phi(z) = (\epsilon z_n)^{[\alpha]+1}$ , where  $[\alpha]$  is the biggest integer less than or equal to  $\alpha$ , then for every point  $z \in \Omega_V$ , we have

$$|\phi(z)| \leq |\epsilon z_n|^\alpha \leq d_V(z)^\alpha$$

and

$$|\phi(z)f(z)| \leq d_V(z)^\alpha |f(z)| \leq C.$$

The function  $\phi$  is holomorphic on  $\bar{U}$  and  $f$  is holomorphic on  $V$ . The inequality implies that  $\phi f$  is bounded on  $\Omega_V = \Omega_U - \{z_n = 0\}$ . Since  $\Omega_U - \Omega_V$  is an analytic subset of  $U$ , by the First Riemann Removable Singularity Theorem ([11, p. 131]),  $\phi(z)f(z)$  can be extended to a holomorphic function  $\psi$  on  $U$ . On  $U, f(z) = \psi(z)/\phi(z)$  is a meromorphic function which is holomorphic on  $V$  and has pole singularity in  $U \cap (X - Y)$ .

We have shown that for each smooth point  $P \in (X - Y) - S, f$  can be extended to a meromorphic function as a quotient of two holomorphic functions at  $P$

$$g_P = \frac{\psi_P}{\phi_P}$$

in an open subset  $U_P$  of  $P$  with pole singularities in  $U_P \cap (X - Y)$  and  $g_P|_{U_P \cap Y} = f$ . Since  $P$  is a nonsingular point of  $X$ , the stalk  $\mathcal{O}_{X,P}$  is a unique factorization domain. We may choose  $\psi_P$  and  $\phi_P$  such that they are relatively prime. If  $n_P$  is the vanishing order of  $\phi_P$  at  $P$ , then the order of pole of  $g_P$  at  $P$  is  $n_P$  and  $g_P$  has pole singularity of order  $n_P$  along the irreducible component  $D_P = (X - Y) \cap U_P$ .

Let  $Q \in (X - Y) - S$ . If  $U_P \cap U_Q$  is not empty, then  $U_P \cap U_Q \cap Y$  is not empty and is an open subset of  $X$ . Since on  $U_P \cap U_Q \cap Y$

$$g_P = g_Q = f,$$

by the Identity Theorem of Meromorphic Functions ([11, p. 170]; [16, p. 241])  $g_P = g_Q$  on  $U \cap V$  or  $g_Q$  is the extension of  $g_P$  from  $U_P$  to  $U_Q$ . Patching these local meromorphic functions together,

$\{(g_P, U_P)\}_{P \in X-S}$  gives a global meromorphic function on the manifold  $X - S$  and  $g_P|_{U_P \cap Y} = f$ . Since  $S$  is a closed subspace of codimension at least 2 in  $X$ , by Levi Extension Theorem ([11, p. 185]; [16, p. 243]),  $f$  can be extended to a meromorphic function  $g$  on  $X$ .

Next we need to show  $g \in H^0(X, \mathcal{O}_X(D))$  for some Weil divisor  $D$ . We will show that  $g$  has a constant order of pole on every irreducible component  $D_i$  of  $X - Y$ . We only need to show the claim is true for  $D_1$  since the proof works for every component of  $X - Y$ .

The component  $D_1$  is an irreducible reduced closed subspace of codimension 1 in  $X$ .  $D_1 - S$  is a nonsingular open subspace of  $D_1$ . Let  $P, Q \in D_1 - S$  be two distinct smooth points such that  $(g_P, U_P)$  and  $(g_Q, U_Q)$  are extensions of  $f$  in  $U_P$  and  $U_Q$  respectively and  $U_P \cap U_Q = U_{PQ} \neq \emptyset$ . Then on  $U_{PQ}$ , there are holomorphic functions  $\psi_P, \phi_P, \psi_Q$ , and  $\phi_Q$  such that

$$\frac{\psi_P}{\phi_P} = g_P = g_Q = \frac{\psi_Q}{\phi_Q} = f,$$

is a holomorphic function on  $U_{PQ} \cap Y$  and meromorphic on  $U_{PQ}$ , where  $\psi_P$  and  $\phi_P$  are relatively prime and  $\psi_Q$  and  $\phi_Q$  are relatively prime. If  $\phi_P$  and  $\phi_Q$  do not vanish on  $U_{PQ} \cap D_1$ , then  $g_P = g_Q$  is holomorphic in  $U_{PQ}$  and we define  $n_1 = 0$ . If  $\phi_P$  and  $\phi_Q$  vanish on  $U_{PQ} \cap D_1$ , then by Hilbert's zero theorem ([12, v. II, p. 53]),  $\psi_P$  and  $\psi_Q$  do not vanish on entire hypersurface  $U_{PQ} \cap (X - Y) = U_{PQ} \cap D_1$  since  $\mathcal{O}_P$  and  $\mathcal{O}_Q$  are unique factorization domains ([12, v. II, p. 7]). In  $U_{PQ}$ , the two product holomorphic functions are equal:

$$\psi_P \phi_Q = \psi_Q \phi_P.$$

It implies that  $\phi_Q$  and  $\phi_P$  have the same vanishing order on  $U_{PQ} \cap D_1$ . We define the order of pole for the meromorphic function  $g$  along  $D_1$  to be the vanishing order  $n_1$  of  $\phi_P$  and  $\phi_Q$  on  $U_{PQ} \cap D_1$ . ([12, v. II, p. 76-78, 166-168]).

So the meromorphic function  $g = (g_P, U_P)$  on  $X$  has a constant order  $n_1$  of pole along  $D_1$ . Let  $n_i$  be the order of pole for  $g$  on  $U_{PQ} \cap D_i$  and  $D = \sum n_i D_i$ , then  $g|_Y = f$  and  $g \in H^0(X, \mathcal{O}_X(\sum_i n_i D_i))$ . □

**Lemma 16.** *Let  $Y$  be an open subset of an irreducible normal reduced Stein space  $X$  such that  $X - Y$  is a reduced closed subspace of pure codimension 1 in  $X$ . If  $f$  is a holomorphic function with polynomial growth on  $Y$ , then there is a meromorphic function  $g \in H^0(X, \mathcal{O}_X(D))$  for an effective Weil divisor  $D = \sum n_i D_i$  such that  $g|_Y = f$ .*

**Proof.** Let  $S_X$  be the singular set in  $X$  and  $S_{X-Y}$  be the singular set in  $X - Y$ . Let  $X' = X - (S_X \cup S_{X-Y})$ . Then  $X'$  is a complex manifold and  $S = X - X'$  is of codimension at least 2 in  $X$ . By Levi Extension Theorem ([11, p. 185]; [16, p. 243]),  $H^0(X', \mathcal{O}_{X'}(D)) = H^0(X, \mathcal{O}_X(D))$ . Since  $X$  is Stein,  $X'$  has lots of meromorphic functions with poles in  $X' - Y$ . Every holomorphic function  $f$  on  $Y$  with polynomial growth on  $Y$  is a holomorphic function on  $Y' = Y - S$  with polynomial growth on  $Y'$ . By a proof similar to the one of Lemma 15,  $f$  can be extended to a meromorphic function  $g$  on  $X'$  and  $g \in H^0(X', \mathcal{O}_{X'}(D)) = H^0(X, \mathcal{O}_X(D))$ .

The lemma is proved. □

The proof of Lemma 16 works for a Stein space  $X$  if the singular subspace  $S$  is of codimension at least 2.

**Theorem 17.** *Let  $Y$  be an open subset of an irreducible reduced Stein space  $X$  such that  $X - Y$  is a reduced closed subspace of pure codimension 1 in  $X$ . If the closed subspace of singular points on  $X$  is of codimension at least 2 and  $f$  is a holomorphic function with polynomial growth on  $Y$ , then there is a meromorphic function  $g \in H^0(X, \mathcal{O}_X(D))$  for an effective Weil divisor  $D = \sum n_i D_i$  such that  $g|_Y = f$ .*

In order to use mathematical induction, we will show that complex analytic hypersurfaces defined by one equation in Theorem 2 satisfy the locally Stein condition.

**Lemma 18.** *Let  $Y$  be an open subset of an irreducible reduced Stein space  $X$  such that  $Y$  is locally Stein and  $X - Y$  is a closed subspace of  $X$ . Let  $f \in H^0(X, \mathcal{O}_X)$  be a nonconstant holomorphic function on  $X$  and  $Z = \{x \in X, f(x) = 0\}$  such that  $Z \cap Y \neq \emptyset$  and  $Z \cap (X - Y) \neq \emptyset$ . Then  $Z \cap Y$  is a locally Stein open subset in the hypersurface  $Z$ .*

**Proof.** Since  $Y$  is locally Stein and  $X - Y$  is a closed subspace of  $X$ ,  $X - Y$  is of pure codimension 1 ([10, p. 128]). If  $X$  is a curve, then  $Y$  is Stein. Let  $d \geq 2$  be the dimension of  $X$ .  $Z$  is a hypersurface of dimension  $d - 1$  on  $X$  since  $X$  is irreducible and reduced ([11, pp. 100, 106]).  $Z \cap (X - Y)$  is of pure dimension  $d - 2$  in  $Z$  and  $Z \cap Y$  is an open subset of  $Z$ .

Let  $z$  be a point on the boundary  $Z \cap (X - Y)$  of  $Z \cap Y$ . Because  $Z \cap Y \neq \emptyset$  and  $Z \cap (X - Y) \neq \emptyset$ , there is an open subset  $U_z \ni z$  in  $Z$  such that  $U_z \cap Y \neq \emptyset$  and  $U_z \cap (X - Y) \neq \emptyset$ . Let  $W_z$  be an open subset in  $X$  such that  $z \in W_z$ . We may replace  $U_z$  by  $U_z \cap W_z$  if it is necessary and assume that  $U_z \subset W_z$ . Then  $z$  is also a boundary point of  $Y$  in  $X$ . Since  $Y$  is locally Stein, for every boundary point  $z \in \partial(Z \cap Y)$  on  $Z$ , there is an open subset  $U \subset X$  containing  $z$  such that  $Y \cap U$  is Stein.  $Z$  is a hypersurface of the Stein space  $X$  defined by one equation, so  $Z$  is Stein. Then  $Z \cap (Y \cap U)$  is a closed subspace of Stein space  $U \cap Y$  so is Stein ([10, pp. 125–127]). Now  $U \cap Z$  is an open subset of  $Z$  and  $(U \cap Z) \cap (Y \cap Z) = Z \cap (Y \cap U)$  is a Stein open subset of  $Z \cap Y$ . Therefore  $Z \cap Y$  is a locally Stein open subset of  $Z$ . □

**Remark 19.** If  $D$  and  $D'$  are two effective Weil divisors on  $X$  and  $D' \leq D$ , then  $H^0(X, \mathcal{O}_X(D')) \subset H^0(X, \mathcal{O}_X(D))$ . In Lemma 15, Lemma 16 and Theorem 17, if the Weil divisor has finitely many irreducible components, then we may choose  $D$  to be a reduced divisor and for sufficiently large  $N \in \mathbb{N}$ , the meromorphic function  $g \in H^0(X, \mathcal{O}_X(ND))$ .

### 3. Analytic Blocks

A pair  $(A, \pi)$  is an analytic stone in a complex space  $X$  if the following conditions are satisfied ([10, p. 111]):

- (1)  $A$  is a non-empty compact subset in  $X$  and  $\pi : X \rightarrow \mathbb{C}^m$  is a holomorphic map.
- (2) There exists a compact block  $B$  in  $\mathbb{C}^m$ , and an open set  $W$  in  $X$  such that  $A = \pi^{-1}(B) \cap W$ .

A stone  $(A, \pi)$  is an analytic block ([10, p. 116]) if there are two open neighborhoods  $U \subset X$  and  $V \subset \mathbb{C}^m$  of  $A$  and  $B$  respectively such that  $\pi(U) \subset V$ ,  $A = \pi^{-1}(B) \cap U$  and the induced map  $\pi|_U : U \rightarrow V$  is a proper and finite holomorphic map, where the compact block

$$B = \{z = (z_1, \dots, z_m) \in \mathbb{C}^m, z_j = x_j + iy_j, a_j \leq x_j \leq b_j, c_j \leq y_j \leq d_j, j = 1, \dots, m\}.$$

If  $X$  is an irreducible normal reduced Stein space, since  $A$  is compact, there is a holomorphic map  $e$  from  $X$  to  $\mathbb{C}^n$ , for some  $n \in \mathbb{N}$  and an open subset  $U'$  such that  $A \subset U' \subset U$  and  $e|_{U'} : U' \rightarrow \mathbb{C}^n$  is an embedding ([16, pp. 167, 233]), where  $e(U')$  is an analytic subset of a bounded open subset  $V'$  in  $\mathbb{C}^n$ . Let  $W$  be an open block in  $\mathbb{C}^m$  such that  $B \subset W \subset V$  and  $U'' = (\pi|_U)^{-1}(W) \subset U'$ . Then  $\pi|_{U''} : U'' \rightarrow W$  is a proper finite holomorphic map ([11, p. 48]) and  $e|_{U''} : U'' \rightarrow \mathbb{C}^n$  is an embedding. Replace  $U$  by  $U''$  and  $V$  by  $W$ , we may assume that  $U$  is a Stein open subset of  $X$ ,  $A \subset U$ ,  $\pi|_U : U \rightarrow V$  is a finite holomorphic map and  $e|_U : U \rightarrow \mathbb{C}^n$  is an embedding. We have

$$\begin{array}{ccccc} A & \subset & U & \subset & X \\ \downarrow \pi|_A & & \downarrow \pi|_U & & \downarrow \pi \\ B & \subset & V & \subset & \mathbb{C}^m, \end{array}$$

and

$$\begin{array}{ccccc} A & \subset & U & \subset & X \\ \downarrow e|_A & & \downarrow e|_U & & \downarrow e \\ e(A) & \subset & e(U) & \subset & \mathbb{C}^n, \end{array}$$

where  $B$  is a compact compact block and  $V$  is a bounded Stein open subset of  $\mathbb{C}^m$ .

A complex space  $X$  is holomorphically convex if the holomorphically convex hull

$$\widehat{K}_X = \left\{ x \in X, |h(x)| \leq \sup_{y \in K} |h(y)|, \forall h \in H^0(X, \mathcal{O}_X) \right\}$$

of any compact subset  $K \subset X$  is compact in  $X$  ([10, Introduction, pp. 108–109]). An equivalent definition is that a complex space is holomorphically convex if for any discrete sequence on  $X$ , there is a holomorphic function  $f$  on  $X$  such that  $f$  is not bounded on the sequence ([10, pp. 110, 114]).

A complex space  $X$  is weakly holomorphically convex if for every compact set  $K \subset X$ , there exists an open subset  $U \subset X$  such that the holomorphically convex hull

$$\widehat{K}_X = \left\{ x \in X, |h(x)| \leq \sup_{y \in K} |h(y)|, \forall h \in H^0(X, \mathcal{O}_X) \right\} \subset U$$

is compact ([10, p. 113]). If  $X$  is holomorphically convex, then it is weakly holomorphically convex ([16, p. 301]).

A sequence  $\{K_r\}_{r \geq 1}$  of compact subsets of a topological space  $X$  is an exhaustion of  $X$  if  $K_r$  is contained in the interior  $K_{r+1}^\circ$  of  $K_{r+1}$  and  $X$  is the union of all  $K_r$  ([10, p. 102]):

$$X = \cup_{r=1}^\infty K_r.$$

There is an exhaustion by analytic blocks for every weakly holomorphically convex space  $X$  if every compact analytic subset of  $X$  is finite ([10, p. 118]). A complex space is Stein if and only if it is weakly holomorphically convex and every compact analytic subset is finite ([16, pp. 293–294]). So a Stein space has an exhaustion by analytic blocks.

A subset  $A$  in a complex space  $X$  is a Stein set if for every coherent analytic sheaf  $\mathcal{F}$  defined in a small open neighborhood of  $A$ , and every  $j > 0$  ([10, p. 96]; [16, p. 230]),

$$H^j(A, \mathcal{F}|_A) = 0.$$

For the simplicity, we write the coherent sheaf  $\mathcal{F}$  on  $A$  in stead of  $\mathcal{F}|_A$ . The equation

$$H^j(A, \mathcal{F}) = 0$$

means that if  $\mathcal{F}$  is defined in an open neighborhood  $U$  of  $A$  and  $\xi \in H^j(V, \mathcal{F})$  with  $A \subset V \subset U$ , then there is an open neighborhood  $W$  with  $A \subset W \subset V$  such that  $\xi|_W = 0$  in  $H^j(W, \mathcal{F})$  ([16, pp. 228, 230]). The space of global sections of the sheaf  $\mathcal{F}$  on  $A$  is the direct limit

$$\mathcal{F}(A) = H^0(A, \mathcal{F}) = \lim_{A \subset \overline{U} \subset X} \mathcal{F}(U).$$

Particularly, we say that a holomorphic function  $f$  is of polynomial growth on a set  $A$  if  $f$  is of polynomial growth on a small open neighborhood of  $A$ . If  $A$  is an analytic subset of a connected Stein space  $X$  of dimension  $d$ , then  $A$  has a fundamental system of open  $\mathcal{O}(X)$ -convex neighborhoods ([16, p. 296]). An open subset  $U$  in  $X$  is  $\mathcal{O}(X)$ -convex if for any compact set  $K \subset U$ ,  $\widehat{K}_X \cap U$  is compact. By Oka–Weil approximation Theorem [32], if  $X$  is a Stein space,  $U$  is  $\mathcal{O}(X)$ -convex if and only if  $(U, X)$  is a Runge pair ([16, p. 295]).

A Stein set is holomorphically separable and holomorphically convex ([16, p. 232]). If  $A_1$  and  $A_2$  are two holomorphically convex subsets of a complex space, then  $A_1 \cap A_2$  is a holomorphically convex subset ([10, p. 127]). If  $(A, \pi)$  is an analytic block in  $X$ , then  $A$  is a compact Stein subset in  $X$  ([10, p. 116]). But an analytic block is not a Stein space since a compact complex space is Stein if and only if its dimension is zero ([16, p. 224]). Let  $(A', \pi')$  be an analytic block with map  $\pi' : X \rightarrow \mathbb{C}^{m'}$  and block  $B' \subset \mathbb{C}^{m'}$ . The block  $(A, \pi)$  is contained in  $(A', \pi')$ ,  $(A, \pi) \subset (A', \pi')$ , if the following conditions are satisfied ([10, p. 112])

- (1) The set  $A$  is contained in the analytic interior of  $A'$ :  $A \subset A^0$ , where  $A^0$  is the preimage of the interior  $B'^\circ$  with respect to  $\pi'|_{U'}$ ,  $A^0 = \pi'^{-1}(B'^\circ) \cap U'$ ,  $U'$  is an open subset of  $X$ . The analytic interior  $A^0$  is contained in the interior  $A^\circ$  and they may not be equal ([10, p. 111]).
- (2)  $\mathbb{C}^{m'} = \mathbb{C}^m \times \mathbb{C}^n$  and there is a point  $q \in \mathbb{C}^n$  such that  $B \times \{q\} \subset B'$ .
- (3) There is a holomorphic map  $\phi : X \rightarrow \mathbb{C}^n$  such that for every  $x \in X$ ,  $\pi'(x) = (\pi(x), \phi(x))$ .

Let  $X$  be an irreducible, normal and reduced Stein space. Then there is an exhaustion  $X = \cup_{r=1}^\infty X_r$  by analytic blocks such that each  $X_r$  is a compact Stein set in  $X$ ,  $(X_r, \pi_r) \subset (X_{r+1}, \pi_{r+1})$  and  $(X_r, X_{r+1})$  is a Runge pair, i.e., for every coherent sheaf  $\mathcal{F}$  on  $X$ , the space  $H^0(X_{r+1}, \mathcal{F})|_{X_r}$  is dense in  $H^0(X_r, \mathcal{F})$  ([10, pp. 122–123]). If  $Y$  is an open subset of  $X$ , then each  $Y_r = X_r \cap Y$  is an open subset of the analytic block  $X_r$  with the induced topology on  $X_r$  from  $X$ . Moreover, if  $Y$  is locally Stein in  $X$ , then  $Y_r$  is locally Stein in  $X_r$ . In fact, let  $x \in X_r - Y_r$  be a boundary point of  $Y_r$  in  $X_r$ , then  $x \in X_r - Y \subset X - Y$ . Let  $U$  be an open subset in  $X$  such that  $x \in U$  and  $U \cap Y = V$  is Stein. Since  $X_r$  is also Stein, we have

$$(U \cap X_r) \cap Y_r = U \cap (Y \cap X_r) = (U \cap Y) \cap X_r = V \cap X_r,$$

where  $U \cap X_r \ni x$  is an open subset of  $X_r$  and  $V \cap X_r$  is a Stein set ([10, p. 127]; [16, p. 231]). In fact, let  $\{P_n\}_{n=1}^\infty$  be a discrete set in  $V \cap X_r$ . If it has an accumulation point  $v_0 \in V$ , since  $X_r$  is a compact set,  $v_0 \in X_r$ , then  $\{P_n\}_{n=1}^\infty$  is not a discrete set in  $V \cap X_r$ . So  $\{P_n\}_{n=1}^\infty$  is also a discrete set in  $V$ . But  $V$  is a Stein open subset of  $X$ , there is a holomorphic function on  $V$  such that it is not bounded on  $\{P_n\}_{n=1}^\infty$ . We show that  $V \cap X_r$  is holomorphically convex so the intersection set of an open Stein subset with a compact Stein subset is a Stein set. Therefore each  $Y_r$  is a locally Stein open subset of  $X_r$ , for all  $r \geq 1$ .

An analytic subset in a domain in  $\mathbb{C}^n$  is not smooth in general. Since every Stein subvariety in a complex space admits a Stein neighborhood [30], we can apply Siu's Theorem 8 to a Stein subvariety in a bounded domain in  $\mathbb{C}^n$ .

**Lemma 20.** *Let  $V$  be a bounded Stein domain in  $\mathbb{C}^n$ ,  $E$  an irreducible closed analytic subset in  $V$  and  $F$  is a closed analytic subset of codimension 1 in  $E$  such that  $E - F$  is locally Stein in  $E$ . Then for every point  $q_0 \in F$  and every hypersurface  $H$  passing through  $q_0$  in  $V$  such that  $E \not\subset H$ ,  $(E \setminus F) \cap H \neq \emptyset$ , there is a holomorphic function  $g$  on a Stein open subset  $W$  containing  $H \cap (E \setminus F)$  in  $V \setminus F$  such that  $g$  is not bounded on any sequence in  $W$  with the accumulation point  $q_0$  and is of polynomial growth on  $W$ .*

**Proof.**  $H \cap E$  is a closed Stein subspace in  $E$  ([16, p. 224]) of codimension 1 in  $E$  ([11, p. 170]). By Lemma 18,  $N = H \cap (E - F)$  is a locally Stein open subset of the Stein space  $H \cap E$ . If  $N$  is a curve, then  $N$  has no compact components so it is Stein ([16, p. 224]). By mathematical induction, we may assume that  $N$  is Stein.

There are holomorphic (polynomial) functions  $f_1, \dots, f_m$  in  $\mathbb{C}^n$  such that  $q_0$  is the unique common zero in  $\mathbb{C}^n$ . Since  $N$  is a Stein analytic subset in  $V \setminus F$  [30], there is a Stein open neighborhood  $W$  of  $N$  in  $V \setminus F$  and  $f_1, \dots, f_m$  have no common zeros on  $W$ . So there are holomorphic functions  $g_1, \dots, g_m$  on  $W$  such that ([10, p. 161])

$$f_1 g_1 + \dots + f_m g_m = 1.$$

The open set  $W$  is a bounded open Stein domain in  $\mathbb{C}^n$ . By Theorem 8, there are  $g_1, \dots, g_m$  with polynomial growth on  $W$  such that they satisfy the above equation. Since each  $f_i$  is continuous and  $f_i(q_0) = 0$  for all  $i = 1, \dots, m$ , at least one  $g_i$  is not bounded on any discrete sequence in  $W$  with the accumulation point  $q_0$ . □

**Corollary 21.** *In Lemma 16, let  $(A, \pi)$  be an analytic block in  $X$  such that  $A$  is not contained in  $Y$ . For every point  $p_0 \in A - A \cap Y = A \cap (X - Y)$ , there is a hypersurface  $Z_{p_0}$  defined by a holomorphic*

function  $h$  in  $X$  such that  $Z = Y \cap Z_h$  is locally Stein in  $Z_h$  and there is a holomorphic function  $G$  on a Stein open subset  $W' \supset Z \cap A$  with polynomial growth on  $W'$  such that  $G$  is not bounded on any sequence in  $W'$  with the accumulation point  $p_0$ .

**Proof.** Since  $A$  is a compact subset in the irreducible Stein space  $X$ , there is a holomorphic map  $e$  from  $X$  to  $\mathbb{C}^n$  for some  $n \in \mathbb{N}$  such that  $e|_U : U \rightarrow \mathbb{C}^n$  is an embedding ([10, p. 126]; [16, p. 233]), where  $U \supset A$  is a Stein open neighborhood of  $A$  in  $X$  such that  $e(U)$  is an irreducible analytic subset (because  $U$  is irreducible and  $e|_U$  is an embedding) of a bounded Stein open subset  $V$  in  $\mathbb{C}^n$ . In Lemma 20,  $F = e(U \cap (X - Y)) \subset e(U) = E$  is a closed subspace of pure codimension 1 in  $E$  such that  $E - F$  is locally Stein in  $E$ . By Lemma 20, pull the unbounded holomorphic function  $g$  in the Stein open subset  $W \setminus F$  back to  $X$  by the holomorphic map  $e$ , the corollary 21 is proved.  $\square$

**Remark 22.** In Corollary 21, given finitely many points in the bounded Stein domain  $V$ , we may choose a hypersurface  $H$  passing through these points and the fixed point  $q_0 \in e((X - Y) \cap U)$  such that it is defined by a polynomial  $h$  in  $\mathbb{C}^n$  and there is a holomorphic function on  $W$  such that it is of polynomial growth and is not bounded near  $q_0$ . Pull this hypersurface  $H$  and corresponding holomorphic functions back to  $A$  by  $e$ , Corollary 21 is true, i.e., there is a hypersurface  $Z$  defined by a holomorphic function  $h$  in  $X$  such that  $Z$  passes through finitely many points including  $p_0 = e^{-1}(q_0)$  in  $A \cap (X - Y)$ ,  $Z_Y = Y \cap Z$  is locally Stein in  $Z$  and there is a holomorphic function  $G$  in a neighborhood of  $Z \cap A$  with polynomial growth and not bounded on any discrete sequence in  $W'$  with an accumulation point  $p_0$ .

#### 4. Extension of holomorphic functions with polynomial growth

Let  $Y$  be a proper open subset of an irreducible normal reduced Stein space  $X$  such that  $Y$  is locally Stein and  $X - Y$  is a closed subspace of  $X$ . Let  $f \in H^0(X, \mathcal{O}_X)$  be a nonconstant holomorphic function with zeros in  $X$ . The holomorphic hypersurface  $Z = \{x \in X, f(x) = 0\}$  is a closed subspace of the Stein space  $X$  therefore is Stein ([16, p. 224]). By Lemma 18,  $Z \cap Y$  is a locally Stein subspace of the Stein space  $Z$ . If  $Z$  is a curve, then  $Z \cap Y$  is Stein. By mathematical induction, we may assume that the hypersurface  $Z \cap Y$  is Stein.

If  $Y$  is locally Stein, then there is a bounded Stein open cover  $\{W_i\}$  of  $X$  such that  $W_i \cap Y$  is Stein. In fact, for every boundary point  $z \in X - Y$  in  $X$ , there is an open subset  $U_z$  in  $X$  such that  $z \in U_z$  and  $V_z = Y \cap U_z \subset Y$  is Stein. Let  $U'_z \subset U_z$  be a bounded Stein open subset of  $U_z$  such that  $z \in U'_z$ , then  $V_z \cap U'_z = Y \cap U'_z$  is Stein ([10, p. 127]). We may add more bounded Stein open subsets  $\{W_r\}$  in  $Y$  such that  $\{W_r\} \cup \{U'_z\}$  is a bounded Stein open cover of  $X$  and both  $U'_z \cap Y$  and  $W_r \cap Y$  are Stein. So the condition that  $Y$  is locally Stein is equivalent to the condition that the natural inclusion  $\epsilon : Y \hookrightarrow X$  is a Stein morphism, i.e., for every point  $x \in X$ , there is a bounded Stein open subset  $U \subset X$  such that  $x \in U$  and  $\epsilon^{-1}(U) = U \cap Y$  is Stein.

In the following Lemmas 23-27, we assume

- (1)  $Y$  is a locally Stein open subset of an irreducible normal reduced Stein space  $X$  such that  $X - Y$  is a closed subspace in  $X$ .
- (2)  $(X_1, \pi)$  is an analytic block (defined in Section 3) in  $X$ ,  $X_1 \not\subset Y$  and  $Y_1 = X_1 \cap Y$ .
- (3) The dimension of  $X$  is greater than or equal to 3 since the curve case is trivial and surface case was proved by Simha ([10, p. 130]; [26]).

By conditions (1) and (2),  $Y_1$  is a locally Stein open subset of the compact Stein set  $X_1$  such that  $X_1 - Y_1$  is of pure codimension 1 in  $X_1$ ,  $X_1 - Y_1$  has finitely many irreducible components of codimension 1 and  $X_1$  can be covered by finitely many bounded Stein open subsets  $\{U_i\}_{1 \leq i \leq N}$  in  $X$  such that every  $U_i \cap Y_1$  is Stein.

Since  $X_1$  is compact and  $X$  is an irreducible normal reduced Stein space, there is a holomorphic map  $e$  from  $X$  to  $\mathbb{C}^n$ , for some  $n \in \mathbb{N}$  such that  $e|_U : U \rightarrow \mathbb{C}^n$  is an embedding ([10, p. 126];

[16, p. 233]), where  $U$  is a Stein open subset of  $X$ ,  $X_1 \subset U$  and  $e(U)$  is an analytic subset of a bounded Stein open subset  $V$  in  $\mathbb{C}^n$ . We use the notation in the proof of Lemma 20 and Corollary 21 and define

$$N = (e(U) - e(X - Y)) \cap H,$$

which is a nonempty open subset of the hypersurface section  $e(U) \cap H$ .

Any compact Stein set in a complex space  $X$  admits a neighborhood basis in  $X$  consisting of Stein open subsets of  $X$  [27]. We may choose a finite Stein open cover  $\{U_i\}_{1 \leq i \leq N}$  of the compact Stein set  $X_1$  such that  $\cup_{i=1}^N U_i = U$ , where  $U$  is the above Stein open subset. In fact, let  $\{U_i\}_{1 \leq i \leq N}$  be a Stein open cover of  $X_1$  such that  $U_i \subset U$ . The finite union  $\cup_{i=1}^N U_i = U'$  is an open subset of  $U \subset X$  and  $X_1 \subset U' \subset U$ .  $e(U') \supset e(X_1)$  is a closed subspace of an open subset  $V' \subset V$  in  $\mathbb{C}^n$ . Let  $B_1 \subset V' \subset V$  be a compact block in  $\mathbb{C}^n$  such that  $e^{-1}(B_1) \cap U = X_1$ . Then  $e(X_1) \subset B_1$ . Choose a Stein open subset  $V''$  such that  $B_1 \subset V'' \subset V'$  in  $\mathbb{C}^n$ .  $(e|_U)^{-1}(V'') \supset X_1$  is a Stein open subset of  $U$ . Replace  $U$  by the smaller Stein open subset  $(e|_U)^{-1}(V'') \supset X_1$  and each  $U_i$  by a smaller Stein open subset  $U_i \cap e^{-1}(V'')$ , we may assume that the finite union  $\cup_{i=1}^N U_i = U$  is a Stein open subset containing the compact block  $X_1$  in the Stein space  $X$ . We have

$$\begin{array}{ccc} X_1 & \subset & U \subset X \\ \downarrow e|_{X_1} & & \downarrow e|_U \quad \downarrow e \\ B & \subset & V \subset \mathbb{C}^n, \end{array}$$

where  $B$  is a compact subset of  $V$ . Let  $\mathcal{C} = X - Y$  be the closed analytic subset of  $X$  and  $\mathcal{I}_{\mathcal{C}}$  be the ideal sheaf of  $\mathcal{C}$  on  $X$ .  $X_1$  is compact therefore we may choose  $U$  such that  $\mathcal{C}$  has finitely many irreducible components on  $U$ . When restricted to the open subset  $U$ ,  $\mathcal{C}|_U$  is the support of effective Weil divisors  $D = \sum_{i=1}^k n_i D_i$ , where each  $D_i$  is a prime divisor, i.e., an irreducible reduced closed analytic subset of codimension 1 on  $U$ .

Every statement in Lemmas 23-27 for the analytic block  $X_1$  means that the statement holds for an appropriate open Stein neighborhood of  $X_1$  in  $X$ .

**Lemma 23.** *Let  $P_0$  be a fixed point in  $X_1 - Y_1$  and  $d \geq 3$  the dimension of  $X$ . Then*

- (1) *There is a hypersurface  $Z$  on  $X$  defined by a nonconstant holomorphic function  $f \in H^0(X, \mathcal{O}_X)$  such that  $P_0 \in Z$ ,  $Z \cap Y_1 \neq \emptyset$ ,  $Z \cap U_i \neq \emptyset$ , for  $i = 1, 2, \dots, N$ ,  $Z \cap D_j \neq \emptyset$  for  $j = 1, \dots, k$ , the co-dimension of the singular locus of  $Z \cap U$  is at least 2 on  $Z \cap U$  and  $D|_Z$  is a Weil divisor on  $Z \cap U$  such that  $D_i \cap Z$  is irreducible for all irreducible components  $D_i$  with  $D_i \cap U \neq \emptyset$ .*
- (2) *There are holomorphic functions  $f_1, \dots, f_m$  in  $X$  and  $g_1, \dots, g_m$  on a Stein open subset  $W' \subset U \setminus (X - Y)$  such that  $Z \cap Y_1 \subset W' \subset U$ ,*

$$f_1(P_0) = f_2(P_0) = \dots = f_m(P_0) = 0$$

and on  $W'$ ,

$$f_1 g_1 + \dots + f_m g_m = 1.$$

- (3) *There is a holomorphic function  $g$  on  $W'$  with polynomial growth such that it is not bounded on any discrete sequence in  $W'$  with the accumulation point  $P_0$ .*

**Proof.**

- (1) Let  $P_{ij}$  be a smooth point in the open subset  $V_{ij} = U_i \cap U_j \cap Y \neq \emptyset$ ,  $i \neq j$ , such that  $P_{ij} \neq P_{i'j'}$  if  $(i, j) \neq (i', j')$ ,  $i, j, i', j' = 1, \dots, N$ , where  $V_{ij}$  are Stein sets by choice of open cover  $\{U_i\}_{i=1}^N$ . There are at most  $N(N-1)/2$  points in the set  $\{P_{ij}\}$ . For each prime divisor  $D_l$  such that  $P_0 \notin D_l$ ,  $1 \leq l \leq k$ , let  $P_l \in D_l$  be a smooth point of  $X$  and  $D_l$ . By the proof of Corollary 21, and Remark 22, we can choose a hypersurface  $Z$  defined by a holomorphic function  $f$  on  $X$ , where  $f$  is the pull back of a holomorphic function  $\psi$  in

$\mathbb{C}^n$  by  $e : X \rightarrow \mathbb{C}^n$  such that  $\psi$  defines a smooth irreducible hypersurface  $H$  in  $\mathbb{C}^n$ ,  $H \cap V$  is irreducible, smooth and  $\psi(e(P_{ij})) = \psi(e(P_0)) = \psi(e(P_l)) = 0$ ,  $i, j = 1, \dots, N, 1 \leq l \leq k$ , and each irreducible divisor  $D_i \cap H \cap U = D_i|_{H \cap U}$  is irreducible. By dimension theory and Bertini's Theorem in algebraic geometry, there is a hypersurface  $Z$  such that the dimension of its singular locus in  $U$  is at most  $d - 3$  and  $D|_Z$  is a Weil divisor on  $Z \cap U$ . We include a proof here for completeness.

Let  $\alpha_1, \alpha_2, \dots, \alpha_K$  be polynomials in  $\mathbb{C}^n$  for sufficiently large  $K$  such that

- (a) every hypersurface  $H_i$  defined by  $\alpha_i$  is irreducible and smooth and the intersection  $H_i \cap e(U)$  is a analytic subset and smooth at  $e(P_{ij})$  and  $e(P_l)$ ,  $i, j = 1, \dots, N, l = 1, \dots, k$  ([10, p. 150]);
- (b) Every  $H_i$  does not contain any  $D_i$  and any irreducible component of the singular set  $S$  of  $e(X)$  with positive dimension since  $S$  is an analytic subset of closed subspace  $e(X)$  in  $\mathbb{C}^n$  ([11, p. 170]);
- (c) Their zero set

$$\bigcap_{i=1}^K H_i = \{z \in \mathbb{C}^n, \alpha_i(z) = 0, i = 1, \dots, K\} = \{e(P_{ij})\}_{i,j=1}^N \cup \{e(P_l)\}_{l=0}^k.$$

Then every point in the set  $\{e(P_{ij})\}_{i,j=1}^N \cup \{e(P_l)\}_{l=0}^k$  is an isolated common zero of these  $K$  holomorphic functions and they have no other common zeros.

Let  $c = (c_1, \dots, c_K) \in \mathbb{C}^K$  and  $C_0$  be the set of points  $c \in \mathbb{C}^K$  such that  $\alpha = c_1\alpha_1 + \dots + c_K\alpha_K$  vanishes on some  $D_i$  or an irreducible component of  $S$ . Then  $C_0$  is a proper  $\mathbb{C}$ -vector subspace of  $\mathbb{C}^K$ . So there is a dense open subset  $C = \mathbb{C}^K \setminus C_0$  in  $\mathbb{C}^K$  such that for all points  $c \in C$ ,  $\alpha_c = c_1\alpha_1 + \dots + c_K\alpha_K$  does not vanish on every irreducible component of  $S$  with positive dimension and the hypersurface  $H_c$  defined by  $\alpha_c = c_1\alpha_1 + \dots + c_K\alpha_K$  with  $c = (c_1, \dots, c_K) \in C$  does not contain any  $D_i$  and any irreducible component of  $S$  with positive dimension.

Let  $\{W_i\}_{i \geq 1}$  be a countable Stein open cover of complex manifold  $e(X)_{reg} = e(X) \setminus S$  such that every  $W_i$  is biholomorphic to an open ball  $B_i$  in  $\mathbb{C}^d$ . For hypersurfaces  $H_c$  corresponding to  $c = (c_1, \dots, c_K) \in \mathbb{C}^K$  in each  $W_i$ , there is an affine variety  $C_i \subset \mathbb{C}^K$  such that  $Z_c = H_c \cap W_i$  is nonsingular for all  $c \in \mathbb{C}^K \setminus C_i$  in  $W_i$ . Since  $\mathbb{C}^K$  is a Baire space,  $T = \bigcap_{i=0}^\infty (\mathbb{C}^K \setminus C_i)$  is a dense subset of  $\mathbb{C}^K$  ([3, Ch. IX, Section 5, Baire Spaces]). Choose  $c \in T$ , the hypersurface section  $H_c \cap e(X)_{reg}$  is nonsingular, does not contain any  $D_i$  and any irreducible component of  $S$  with positive dimension, and  $e(D)|_{H_c}$  is a Weil divisor on  $H_c \cap V$ . Let  $Z = e^{-1}(H_c \cap X)$ , it satisfies all conditions in (1).

- (2) Similar to the proof of Corollary 21 and Remark 22, the statement is true.
- (3) By Part (2), since  $f_1(P_0) = f_2(P_0) = \dots = f_m(P_0)$ , for some  $i$ ,  $1 \leq i \leq m$ , and  $f_1g_1 + \dots + f_mg_m = 1$  on  $W'$ , there is a holomorphic function  $g = g_i$  on  $W' = e^{-1}(W) \cap U$  such that  $g$  is not bounded on any discrete sequence in  $W'$  with the accumulation point  $P_0$ . We may assume that  $g = g_1$  is this function and not bounded near  $P_0$ . □

In the proof of Lemma 23, we may assume that  $g = g_1$  is not bounded near  $P_0$ . In the rest of this section,  $Z$  is the hypersurface defined by holomorphic function  $f$  in Lemma 23.

**Lemma 24.** *Every holomorphic function  $g_k, k = 1, \dots, m$ , on  $W' = e^{-1}(W) \cap U$ , a Stein open neighborhood of  $Z \cap Y_1 = \{y \in Y_1, f(y) = 0\}$  in the above Lemma 23, can be extended to a holomorphic function on  $U_i \cap Y_1$  for all  $i = 1, \dots, N$ . Particularly,  $g = g_1$  can be extended to a holomorphic function  $G_i$  on  $U_i \cap Y_1$  and is not bounded on any discrete sequence in  $U_i \cap Y_1$  with the accumulation point  $P_0 \in X_1 - Y_1$ .*

**Proof.** By the choice of the hypersurface  $Z$  on  $Y_1$ ,  $U_i \cap Z \neq \emptyset$  for all  $i = 1, \dots, N$ . On  $Y_1 \cap Z$ , by Lemma 23, we have

$$f_1g_1 + \dots + f_mg_m = 1,$$



where every holomorphic function  $g_k, k = 1, \dots, m$ , has polynomial growth on every analytic subset  $Z_i = U_i \cap Y_1 \cap Z$  and  $g = g_1$  is not bounded near  $P_0$ . Because  $f\mathcal{O}_{Y_1 \cap U_i}$  is a subsheaf of  $\mathcal{O}_{Y_1 \cap U_i}$ , there is a natural injective holomorphic map from  $f\mathcal{O}_{Y_1 \cap U_i}$  to  $\mathcal{O}_{Y_1 \cap U_i}$ . Therefore we have a short exact sequence

$$0 \longrightarrow f\mathcal{O}_{Y_1 \cap U_i} \xrightarrow{\sigma} \mathcal{O}_{Y_1 \cap U_i} \xrightarrow{r} \mathcal{O}_{Y_1 \cap U_i} / f\mathcal{O}_{Y_1 \cap U_i} \longrightarrow 0,$$

where the first map  $\sigma$  is natural inclusion and  $\mathcal{O}_{Y_1 \cap U_i} / f\mathcal{O}_{Y_1 \cap U_i} = \mathcal{O}_{Z_i}$ .

$f\mathcal{O}_{Y_1 \cap U_i}$  is a coherent ideal sheaf on the Stein set  $Y_1 \cap U_i$ . By Cartan's Theorem B, for all  $j > 0$ ,

$$H^j(Y_1 \cap U_i, f\mathcal{O}_{Y_1 \cap U_i}) = H^j(Y_1 \cap U_i, \mathcal{O}_{Y_1 \cap U_i}) = 0.$$

The corresponding long exact sequence is

$$0 \longrightarrow H^0(Y_1 \cap U_i, f\mathcal{O}_{Y_1 \cap U_i}) \longrightarrow H^0(Y_1 \cap U_i, \mathcal{O}_{Y_1 \cap U_i}) \longrightarrow H^0(Z_i, \mathcal{O}_{Z_i}) \longrightarrow 0.$$

By the second surjective holomorphic map from  $H^0(Y_1 \cap U_i, \mathcal{O}_{Y_1 \cap U_i})$  to  $H^0(Z_i, \mathcal{O}_{Z_i})$ , any holomorphic function on  $Z_i$  can be lifted to a holomorphic function on  $Y_1 \cap U_i$ . Let

$$G_{i,j} \in H^0(Y_1 \cap U_i, \mathcal{O}_{Y_1 \cap U_i})$$

such that the restriction  $G_{i,j}|_{Z_i} = g_j, j = 1, \dots, m, i = 1, \dots, N$ . By our notation,  $G_{i,1}|_{Z_i} = g_1 = g$ .

On  $Z_i$ , we have

$$(f_1 G_{i,1} + \dots + f_m G_{i,m})|_{Z_i} = (f_1 g_1 + \dots + f_m g_m)|_{Z_i} = 1,$$

or  $(f_1 G_{i,1} + \dots + f_m G_{i,m})|_{Z_i} - 1 = 0$ . In the above exact sequence,

$$H^0(Y_1 \cap U_i, f\mathcal{O}_{Y_1 \cap U_i}) = fH^0(Y_1 \cap U_i, \mathcal{O}_{Y_1 \cap U_i}),$$

and  $f_1 G_{i,1} + \dots + f_m G_{i,m} - 1$  is the zero element in  $H^0(Z_i, \mathcal{O}_{Z_i})$ . It is in the kernel of the second map in the exact sequence so is contained in the image of the first map, i.e., there is a  $h \in H^0(Y_1 \cap U_i, \mathcal{O}_{Y_1 \cap U_i})$  such that on  $Y_1 \cap U_i$ ,

$$f_1 G_{i,1} + \dots + f_m G_{i,m} - 1 = fh.$$

In the Stein subset  $Y_1 \cap U_i$ , it is equivalent to the equation

$$fh + f_1 G_{i,1} + \dots + f_m G_{i,m} = 1.$$

If  $P_0 \in U_i$ , then near  $P_0$  in the hypersurface  $Z_i \subset U_i$ , the first term  $fh = 0$  and  $f_1(x), \dots, f_m(x)$  all approach zero if  $x$  approaches  $P_0$ . So at least one function among  $G_{i,1}, \dots, G_{i,m}$  is not bounded on  $Z_i$  near  $P_0$ . As before, we may assume  $G_i = G_{i,1}$ , i.e.,  $G_i|_{Z_i} = G_{i,1}|_{Z_i} = g$  is not bounded near  $P_0, i = 1, \dots, N$ . By Lemma 23, when restricted to  $Z_i, G_{i,1}, \dots, G_{i,m}$  have polynomial growth on  $Z_i$ . □

Next we will show that a holomorphic function on the hypersurface  $Z_i = U_i \cap Y_1 \cap Z$  on  $V_i = U_i \cap Y_1$  with polynomial growth can be extended to a meromorphic function on  $U_i$ , holomorphic on the open subset  $U_i \cap Y_1$  and has pole singularities in  $U_i - U_i \cap Y \subset X - Y$ .

**Lemma 25.** *In Lemma 24, for all  $i = 1, \dots, N$ , there is a holomorphic function  $G_i$  on  $U_i \cap Y$  such that  $G_i|_{Z_i} = g, G_i$  has polynomial growth and  $G_i \in H^0(U_i, \mathcal{O}_X(l_i D))$  for some  $l_i \in \mathbb{N}$ , where  $D = \sum D_j$  is the reduced divisor.*

**Proof.** By the choice in Lemma 23,  $D|_Z$  is a Weil divisor on  $Z$  such that the intersection of each prime divisor  $D_i$  with  $Z \cap U$  is also a prime divisor on  $Z \cap U$ . This choice guarantees that every reduced Weil divisor on  $Z \cap U$  can be lifted to a reduced Weil divisor on the open subset  $U$  in  $X$ . Now  $g = g_1$  is a holomorphic function with polynomial growth in a Stein open neighborhood  $W' = e^{-1}(W)$  of the hypersurface  $Z \cap Y_1$  and is not bounded near the fixed point  $P_0 \in X_1 - Y_1$ .

By Theorem 17,  $g$  can be extended to a meromorphic function on  $Z \cap U$  and for the reduced divisor  $D = \sum D_j$ , we may choose a sufficiently large  $l_i \in \mathbb{N}$  such that  $div(g) + l_i D|_{Z \cap U_i} > 0$ . It

implies  $g \in H^0(U_i \cap Z, \mathcal{O}_{U_i \cap Z}(l_i D))$ . Let  $D_{U_i} = D|_{U_i}$ .  $Z_i = Z \cap U_i \subset Z \cap e^{-1}(W)$  is a Cartier divisor on  $U_i$  so  $\mathcal{O}_{U_i}(-Z_i) = f\mathcal{O}_{U_i}$  is a locally free sheaf of rank 1, i.e., an invertible sheaf on  $U_i$  ([14, p. 143]; [33, p. 30]). There are isomorphisms ([17, p. 609]; [24])

$$\mathcal{O}_{U_i}(l_i D_{U_i} - Z_i) \cong \mathcal{O}_{U_i}(l_i D_{U_i}) \otimes \mathcal{O}_{U_i}(-Z_i) = \mathcal{O}_{U_i}(l_i D_{U_i}) \otimes f\mathcal{O}_{U_i} \cong f\mathcal{O}_{U_i}(l_i D_{U_i}),$$

and the following short exact sequence

$$0 \longrightarrow f\mathcal{O}_{U_i}(l_i D_{U_i}) \longrightarrow \mathcal{O}_{U_i}(l_i D_{U_i}) \longrightarrow \mathcal{O}_{Z \cap U_i}(l_i D_{U_i}) \longrightarrow 0.$$

$U_i$  is Stein and  $f\mathcal{O}_{U_i}(l_i D_{U_i})$  is coherent on  $U_i$ , so  $H^1(U_i, f\mathcal{O}_{U_i}(l_i D_{U_i})) = 0$ . The short exact sequence gives the following exact sequence of global sections

$$0 \rightarrow H^0(U_i, f\mathcal{O}_{U_i}(l_i D_{U_i})) \rightarrow H^0(U_i, \mathcal{O}_{U_i}(l_i D_{U_i})) \rightarrow H^0(Z \cap U_i, \mathcal{O}_{Z \cap U_i}(l_i D_{U_i})) \rightarrow 0.$$

As an element of  $H^0(Z \cap U_i, \mathcal{O}_{Z \cap U_i}(l_i D_{U_i}))$ ,  $g$  can be extended to an element  $G_i$  of  $H^0(U_i, \mathcal{O}_{U_i}(l_i D_{U_i}))$ , i.e., a holomorphic function on  $U_i \cap Y_1$  with poles in  $U_i - Y_1 \subset X_1 - Y_1$  up to order  $l_i$  along  $D_i$ . So  $G_i$  has polynomial growth in  $U_i$  and  $G_i|_{Z_i} = g$ . □

**Lemma 26.** *In Lemma 25, if  $U_i \cap U_j \cap Y_1 \cap Z$  is not an empty set, then the meromorphic function*

$$\frac{G_i - G_j}{f}$$

*is holomorphic on  $U_i \cap U_j \cap Y_1$ , has polynomial growth and there is an  $L \in \mathbb{N}$  such that for all  $i, j = 1, \dots, N$ ,  $U_i \cap U_j \cap Y_1 \cap Z \neq \emptyset$ ,*

$$\frac{G_i - G_j}{f} \in H^0(U_i \cap U_j, \mathcal{O}_X(LD)).$$

**Proof.** By the choice of bounded Stein open cover  $\{U_i\}_{i=1}^N$  of  $X_1$ , the subsets  $V_i = Y_1 \cap U_i$  are Stein. So the intersections  $V_{ij} = V_i \cap V_j$  are Stein ([10, Page 127]).

On  $V_{ij}$ , both  $G_i$  and  $G_j$  are holomorphic and when restricted to  $Z_{ij} = Z \cap V_{ij}$ ,

$$G_i|_{Z_{ij}} = G_j|_{Z_{ij}} = g,$$

where  $g = g_1$  is the function in Lemmas 23-25 which is not bounded near the fixed point  $P_0 \in X_1 - Y_1$ . So  $G_i - G_j = 0$  on the hypersurface  $Z_{ij}$  in  $V_{ij}$ .  $Z = \{x \in X, f(x) = 0\}$  is a closed subspace of  $X$  with structure sheaf  $\mathcal{O}_X/f\mathcal{O}_X$ , where  $f$  is the holomorphic function in Lemma 23.

From the short exact sequence

$$0 \longrightarrow f\mathcal{O}_{V_{ij}} \longrightarrow \mathcal{O}_{V_{ij}} \longrightarrow \mathcal{O}_{V_{ij}}/f\mathcal{O}_{V_{ij}} \longrightarrow 0,$$

and  $H^p(V_{ij}, f\mathcal{O}_{V_{ij}}) = 0$  for all  $p > 0$ , we have exact sequence

$$0 \longrightarrow H^0(V_{ij}, f\mathcal{O}_{V_{ij}}) \longrightarrow H^0(V_{ij}, \mathcal{O}_{V_{ij}}) \longrightarrow H^0(Z_{ij}, \mathcal{O}_{V_{ij}}/f\mathcal{O}_{V_{ij}}) = H^0(Z_{ij}, \mathcal{O}_{Z_{ij}}) \longrightarrow 0.$$

A zero function  $(G_i - G_j)|_{Z_{ij}} = 0$  on  $Z_{ij}$  is contained in the image of the first map, which is the natural inclusion. Since

$$H^0(V_{ij}, f\mathcal{O}_{V_{ij}}) = fH^0(V_{ij}, \mathcal{O}_{V_{ij}}),$$

on  $V_{ij}$ ,  $G_i - G_j$  is an element in the ideal  $(f)$  generated by  $f$  in  $H^0(V_{ij}, \mathcal{O}_{V_{ij}})$ . So the function

$$\frac{G_i - G_j}{f}$$

is holomorphic on  $V_{ij}$ .

By Lemma 25, there are  $l_i, l_j \in \mathbb{N}$  such that

$$G_i \in H^0(U_i, \mathcal{O}_X(l_i D))$$

and

$$G_j \in H^0(U_j, \mathcal{O}_X(l_j D)).$$

Choose sufficiently large  $l \geq \max(l_i, l_j)$ , then  $G_i \in H^0(U_i, \mathcal{O}_X(lD))$  and  $G_j \in H^0(U_j, \mathcal{O}_X(lD))$ .

Because  $G_i - G_j$  has polynomial growth in  $V_{ij}$ ,  $f$  is holomorphic on  $V_{ij}$  and  $(G_i - G_j)/f$  is holomorphic on  $V_{ij}$ , by Theorem 7,  $(G_i - G_j)/f$  has polynomial growth in  $V_{ij}$  [18].

By Lemma 16, there is an  $m_{ij} \in \mathbb{N}$  such that

$$G_{ij} = \frac{G_i - G_j}{f} \in H^0(U_i \cap U_j, \mathcal{O}_X(m_{ij} D)).$$

Choose sufficiently large  $L \geq m_{ij}$  for all  $i, j = 1, \dots, N$ , then for  $U_i \cap U_j \cap Y_1 \cap Z \neq \emptyset$ , we have

$$G_{ij} = \frac{G_i - G_j}{f} \in H^0(U_i \cap U_j, \mathcal{O}_X(LD)). \quad \square$$

**Lemma 27.** *There is a meromorphic function  $\beta \in H^0(X_1, \mathcal{O}_X(LD))$  such that it is holomorphic on  $Y_1$  and is not bounded on any discrete sequence in  $Y_1$  with the accumulation point  $P_0 \in X_1 - Y_1$ .*

**Proof.** We will construct a meromorphic function  $\beta \in H^0(U, \mathcal{O}_X(LD)|_U)$  such that it is holomorphic on  $Y_1 = X_1 \cap Y$  and is not bounded on any discrete sequence in  $Y_1$  with the accumulation point  $P_0 \in X_1 - Y_1$ , where the Stein open subset  $U = \cup_{i=1}^N U_i$  and  $\mathcal{U} = \{U_i\}_{i=1}^N$  is a bounded Stein open cover of the compact Stein set  $X_1$  such that  $V_i = U_i \cap Y_1$  is Stein for  $i = 1, \dots, N$ . By Lemmas 25 and 26, there is a positive integer  $L$  such that  $G_i \in H^0(U_i, \mathcal{O}_X(LD)|_{U_i})$  and

$$G_{ij} = \frac{G_i - G_j}{f} \in H^0(U_i \cap U_j, \mathcal{O}_X(LD))$$

for all  $1 \leq i, j \leq N$ . By Leray's Theorem ([12, III, p. 56]; [11, p. 35]), since  $\mathcal{O}_X(LD)$  is coherent, the first Čech cohomology

$$H^1(\mathcal{U}, \mathcal{O}_X(LD)) = 0.$$

Consider the cochain complex

$$C^0(\mathcal{U}, \mathcal{O}_X(LD)) \xrightarrow{\delta^0} C^1(\mathcal{U}, \mathcal{O}_X(LD)) \xrightarrow{\delta^1} C^2(\mathcal{U}, \mathcal{O}_X(LD)) \xrightarrow{\delta^2} \dots,$$

the kernel of  $\delta^1$  is equal to the image of  $\delta^0$  since the first cohomology vanishes. In Lemma 26, the meromorphic functions  $G_{ij}$  are 1-cochains in  $C^1(\{U_i\}_{i=1}^N, \mathcal{O}_X(LD))$ . In fact,  $\{G_{ij}\}$  is a 1-cocycle:

$$G_{ij} - G_{ik} + G_{jk} = \frac{(G_i - G_j) - (G_i - G_k) + (G_j - G_k)}{f} = 0.$$

Since  $\{G_{ij}\}$  is an element of the kernel of  $\delta^1$ , it is a 1-coboundary, i.e., there is a 0-cochain  $(H_i) \in C^0(\mathcal{U}, \mathcal{O}_X(LD)|_U)$  such that on  $U_i \cap U_j$ ,

$$\frac{G_i - G_j}{f} = H_i - H_j.$$

Then on  $U_i \cap U_j$ ,

$$G_i - fH_i = G_j - fH_j \in H^0(U_i \cap U_j, \mathcal{O}_X(LD)).$$

Let  $\beta = (U_i, G_i - fH_i)$ . When restricted to each hypersurface section  $U_i \cap Z$  on  $U_i$ ,  $fH_i$  is a meromorphic function and is zero on the open subset  $U_i \cap Z \cap Y$  in  $U_i \cap Z$ . Since the set of zeros of a meromorphic function cannot be an open subset ([12, v. II, p. 179]),  $fH_i$  is identically zero on the hypersurface  $U_i \cap Z$ . Then  $\beta \in H^0(U, \mathcal{O}_X(LD))$  and is not bounded near  $P_0 \in X - Y$ . In this way, we obtain a global meromorphic function on  $X_1$  with pole singularities on  $X_1 - Y_1$  up to order  $L$  such that it is not bounded on any discrete sequence in  $Y_1$  with the accumulation point  $P_0 \in X_1 - Y_1$ . □

By the locally Stein condition and the vanishing of the first cohomology, using polynomial growth theory for bounded Stein domains, we extend the holomorphic function on  $Z \cap Y_1$  with polynomial growth to a meromorphic function on  $Z \cap X_1$  and lift it to  $X_1$ . Then we obtain a meromorphic function on the compact Stein set  $X_1$  with poles in  $X_1 - Y_1$ . With this construction, we can show that  $Y_1$  is holomorphically convex in Section 5.

### 5. Proof of Main Theorems

Let  $Y$  be a locally Stein open subset of an irreducible normal and reduced Stein space  $X$  such that  $X - Y$  is a closed subspace in  $X$ . Let  $(X_1, \pi)$  be an analytic block in  $X$ ,  $X_1 \not\subseteq Y$  and  $Y_1 = X_1 \cap Y$ . Then  $X_1 - Y_1$  has finitely many irreducible components in  $X_1$  and  $Y_1$  is locally Stein in  $X_1$ . Let  $D = \sum_{j=1}^K D_j$  be the reduced Weil divisor with support  $X_1 - Y_1$  in  $X_1$  which implies that there is a Stein open subset  $U$  such that  $X_1 \subset U \subset X$  and  $D = \sum_{j=1}^K D_j$  is a Weil divisor in  $U$  with support  $U \cap (X - Y)$  (Section 4 or [27]).

Let  $P_0 \in X_1 - Y_1$  be a point and  $f \in H^0(X, \mathcal{O}_X)$  be the nonconstant holomorphic function in  $X$  such that  $f(P_0) = 0$ ,  $Z = \{x \in X, f(x) = 0\}$ ,  $Z \cap Y_1 \neq \emptyset$  and  $Z_1 = Z \cap X_1$ , where  $f$  is the pull back of the holomorphic function from  $\mathbb{C}^n$  by  $e$  in Lemma 23.

**Proposition 28.** *Let  $Y$  be a locally Stein open subset of an irreducible, normal and reduced Stein space  $X$  such that  $X - Y$  is a closed subspace of  $X$ . Let  $\{X_r\}_{r \geq 1}$  be an exhaustion of  $X$  by analytic blocks such that  $Y_r = X_r \cap Y \subsetneq X_r$ . Then  $Y_r$  is holomorphically convex for all  $r \in \mathbb{N}$ .*

**Proof.** We only need to show that the claim is true for  $r = 1$ . The same proof works for all  $r \in \mathbb{N}$ .

Let  $p_0$  be a point in  $X_1 \setminus Y_1$  and  $P = \{p_1, p_2, \dots\}$  a set of points in  $Y_1$  such that  $p_0$  is an accumulation point of  $P$ . By Lemma 27, there is a holomorphic function  $g$  on  $Y_1$  such that it is not bounded on  $P$ . So  $Y_1$  is a holomorphically convex subset of the compact Stein set  $X_1$ .  $\square$

Since  $Y$  is holomorphically separable, we have the following Corollary.

**Corollary 29.** *Let  $Y$  be a locally Stein open subset of an irreducible, normal and reduced Stein space  $X$  such that  $X - Y$  is a closed subspace of  $X$ . Let  $\{X_r\}_{r \geq 1}$  be an exhaustion of  $X$  by analytic blocks such that  $Y_r = X_r \cap Y \subsetneq X_r$ . Then  $Y_r$  is a Stein set for every  $r \in \mathbb{N}$ .*

We now recall Stein exhaustion for a Stein space ([10, Chapter IV, pp. 105–108]).

#### Definition 30.

(1) *Let  $X$  be a complex space and  $\mathcal{F}$  an analytic coherent sheaf on  $X$ . An exhaustion  $\{X_r\}_{r \geq 1}$  of  $X$  by compact Stein sets is called a Stein exhaustion of  $X$  relative to  $\mathcal{F}$  if the following four conditions are satisfied:*

- (a) *There is a semi-norm  $\|\cdot\|_r$  on every  $\mathbb{C}$ -vector space  $H^0(X_r, \mathcal{F})$  such that the subspace  $H^0(X_{r+1}, \mathcal{F})|_{X_r} \subset H^0(X_r, \mathcal{F})$  is dense in  $H^0(X_r, \mathcal{F})$ .*
- (b) *Every restriction map  $H^0(X_{r+1}, \mathcal{F}) \subset H^0(X_r, \mathcal{F})$  is bounded: for all  $s \in H^0(X_{r+1}, \mathcal{F})$ ,  $r \geq 1$ , there is a positive real number  $M_r$  such that*

$$\|s|_{X_r}\|_r \leq M_r \|s\|_{r+1}.$$

- (c) *If  $(s_j)_{j \in \mathbb{N}}$  is a Cauchy sequence in  $H^0(X_{r+1}, \mathcal{F})$ , then the restriction sequence  $(s_j|_{X_r})_{j \in \mathbb{N}}$  has a limit in  $H^0(X_r, \mathcal{F})$  for all  $r \geq 1$ .*
- (d) *If  $s \in H^0(X_r, \mathcal{F})$  and  $\|s\|_r = 0$ , then  $s|_{X_{r-1}} = 0$ ,  $r \geq 2$ .*

(2) *An exhaustion  $\{X_r\}_{r \geq 1}$  of  $X$  by compact Stein sets is called a Stein exhaustion of  $X$  if it is a Stein exhaustion relative to every analytic coherent sheaf  $\mathcal{E}$  ([10, p. 108]).*

Since  $(X_r, \pi_r)$  is an analytic block in  $X$ , there is an open subset  $U_r$  in  $X$  and a bounded Stein domain  $V_r \subset \mathbb{C}^{n_r}$  such that  $\pi_r : X \rightarrow \mathbb{C}^{n_r}$  is a holomorphic map,  $\pi_r|_{U_r} : U_r \rightarrow V_r$  is a proper finite holomorphic map and  $X_r = \pi_r^{-1}(B_r) \cap U_r$ , where  $B_r$  is a compact block in  $V_r$ :

$$\begin{array}{ccc} X_r & \subset & U_r \subset X \\ \downarrow \pi_r|_{X_r} & & \downarrow \pi_r|_{U_r} \downarrow \pi_r \\ B_r & \subset & V_r \subset \mathbb{C}^{n_r}. \end{array}$$

Let  $\tau_r = \pi_r|_{U_r}$ . For every analytic coherent sheaf  $\mathcal{F}$  on  $X$ , by the Direct Image Theorem of Grauert ([11, p. 207]), the image sheaf

$$\mathcal{G}_r = \tau_{r*}(\mathcal{F}|_{U_r})$$

is coherent on  $V_r$ . By Theorem A for compact blocks in  $\mathbb{C}^{n_r}$  ([10, p. 96]), there is a positive integer  $l_r \in \mathbb{N}$  and a surjective morphism

$$\phi_r : \mathcal{O}^{l_r}|_{B_r} \rightarrow \mathcal{G}_r|_{B_r}.$$

Let  $\mathcal{K}_r$  be the kernel of  $\phi_r$ , there is a short exact sequence

$$0 \rightarrow \mathcal{K}_r \rightarrow \mathcal{O}^{l_r}|_{B_r} \rightarrow \mathcal{G}_r|_{B_r} \rightarrow 0.$$

By Serre's Three Lemma ([10, p. 11]),  $\mathcal{K}_r$  is a coherent sheaf on  $B_r$  since other two sheaves are coherent in the short exact sequence. By Theorem B ([10, p. 97]),  $H^1(B_r, \mathcal{K}_r) = 0$ . The short exact sequence induces a surjective morphism of modules of sections

$$\phi_{B_r} : H^0(B_r, \mathcal{O}^{l_r}) \rightarrow H^0(B_r, \mathcal{G}_r).$$

Since  $X_r = \tau_r^{-1}(B_r) = \pi_r^{-1}(B_r) \cap U_r$  and  $\tau_r$  is a finite holomorphic map, there is a  $\mathbb{C}$ -vector space isomorphism ([10, p. 47])

$$\alpha_r : H^0(X_r, \mathcal{F}) \rightarrow H^0(B_r, \mathcal{G}_r).$$

Now we can define a semi-norm on  $H^0(X_r, \mathcal{F})$  ([10, p. 108, p. 119]).

**Definition 31.** For every section  $s \in H^0(X_r, \mathcal{F})$ , the semi-norm

$$\|s\|_r = \inf \left\{ \|f\|_{B_r} = \sup_{z \in B_r} |f(z)|, f \in H^0(B_r, \mathcal{O}^{l_r}), \phi_{B_r}(f) = \alpha_r(s) \right\}.$$

$X$  is Stein if and only if  $X$  is weakly holomorphically convex and every compact analytic subset of  $X$  is finite ([16, p. 293, Theorem 63.2]). Therefore  $X$  has an exhaustion  $\{(X_r, \pi_r)\}_{r \geq 1}$  by analytic blocks ([10, p. 118, Theorem 7]) which is a Stein exhaustion of  $X$  ([10, p. 123, Theorem 5]).

**Lemma 32.** Let  $Y$  be an open subset of an irreducible, normal and reduced Stein space  $X$  such that  $X - Y$  is a closed subspace of  $X$ . Let  $\{(X_r, \pi_r)\}_{r \geq 1}$  be an exhaustion of  $X$  by analytic blocks. Then  $Y = \cup_{r=1}^{\infty} Y_r$  is holomorphically convex, where  $Y_r = X_r \cap Y$ .

**Proof.** Since  $X$  is Stein, if a sequence on  $Y$  has no accumulation point in  $X$ , then there is a holomorphic function  $f$  on  $X$  such that it is not bounded on the sequence. Let  $P = \{P_r\}_{r=1}^{\infty}$  be a discrete sequence in  $Y$  such that  $x_0 \in X - Y$  is an accumulation point in  $X$ . By eliminating redundant subsets from  $\{X_r\}$  and reordering the subsets, we may assume that for all  $r \geq 1$ ,  $X_r$  is not a subset of  $Y$ ,  $x_0 \in X_r - Y$  and

$$Y_r \subsetneq Y_{r+1} \subsetneq X_{r+1}.$$

For all  $r = 1, 2, 3, \dots$  and a section  $s \in H^0(X_r, \mathcal{O}_X(LD))$ ,  $L \in \mathbb{N}$ , the semi-norm in the vector space  $H^0(X_r, \mathcal{O}_X(LD))$  is

$$\|s\|_r = \inf \left\{ \|f\|_{B_r}, f \in H^0(B_r, \mathcal{O}^{l_r}), \phi_{B_r}(f) = \alpha_r(s) \right\}.$$

We start with  $Y_1 \subset X_1$ . By Lemma 27, there is a holomorphic function  $g_1$  on  $Y_1$  meromorphic in  $X_1$  such that  $g_1$  is not bounded near  $x_0$  on the sequence and for some  $L \in \mathbb{N}$ ,

$$g_1 \in H^0(X_1, \mathcal{O}_X(LD)).$$

Every analytic block  $X_r$  is a compact Stein subset of the Stein space  $X$  ([10, p. 116]). Since  $(X_r, \pi_r) \subset (X_{r+1}, \pi_{r+1})$  are analytic blocks in  $X$ , in the space  $X_r$ , by definition of the semi-norm and the Runge approximation theorem ([10, p. 122]), for the same integer  $L$  determined by the order of poles for  $g_1$  and every  $\epsilon > 0$ , there is a meromorphic function

$$g_2 \in H^0(X_2, \mathcal{O}_X(LD))$$

such that

$$\|g_2 - g_1\|_1 \leq \frac{\epsilon}{2^2}.$$

For every  $n > 1$ , there is a  $g_n \in H^0(X_n, \mathcal{O}_X(LD))$  such that

$$\|g_n - g_{n-1}\|_{n-1} \leq \frac{\epsilon}{2^n}.$$

We assume that  $g_1, \dots, g_n$  have been obtained and define  $g_{n+1} \in H^0(X_{n+1}, \mathcal{O}_X(LD))$  such that

$$\|g_{n+1} - g_n\|_n \leq \frac{\epsilon}{2^{n+1}}.$$

The sequence  $\{g_j\}_{j>i}$  is a Cauchy sequence in  $X_i$  with respect the semi-norm. By the convergence theorem ([10, p. 121]), the sequence has a limit  $h_i$  in  $H^0(X_i, \mathcal{O}_X(LD))$ . The uniqueness of the limit ([10, p. 121]) implies that

$$h_{i+1}|_{X_i^0} = h_i|_{X_i^0},$$

where  $X_i^0$  is the analytic interior ([10, pp. 111, 123]). Since  $X = \cup_{i \geq 1} X_i^0$ , there is a global section  $h \in H^0(X, \mathcal{O}_X(LD))$  such that  $h|_{X_i} = h_i, i \geq 1$  ([10, pp. 113, 123]).

By our construction,  $h$  is a holomorphic function on  $Y$  such that it is not bounded on  $P$ . The Lemma 32 is proved. □

Now we are ready to prove Theorem 2.

**Theorem 33.** *Let  $Y$  be a locally Stein open subset of a Stein space  $X$  such that the complement  $X - Y$  is a closed subspace of  $X$ , then  $Y$  is Stein.*

**Proof.** By the Reduction Theorem ([10, p. 154]),  $X$  is Stein if and only if its reduction is Stein. We may assume that  $X$  is a reduced Stein space. The normalization of a reduced complex space is a finite surjective holomorphic map ([10, p. 22]). So a complex space is Stein if and only if its normalization space is Stein ([16, p. 313]). If  $\alpha : \tilde{X} \rightarrow X$  is the normalization, and  $\tilde{Y} = \alpha^{-1}(Y)$ , then  $\tilde{X} - \tilde{Y}$  is a closed subspace of  $\tilde{X}$  and we have the commutative diagram

$$\begin{array}{ccc} \tilde{Y} & \hookrightarrow & \tilde{X} \\ \downarrow \alpha|_{\tilde{X}} & & \downarrow \alpha \\ Y & \hookrightarrow & X. \end{array}$$

Let  $\tilde{x} \in \tilde{X} - \tilde{Y}$  be a boundary point of  $\tilde{Y}$ . Then  $x = \alpha(\tilde{x}) \in X - Y$  is a boundary point of  $Y$  in  $X$ . Let  $U$  be an open subset in  $X$  such that  $x \in U$  and  $U \cap Y = V$  is Stein. Then  $\alpha^{-1}(V)$  is a Stein open subset in  $\tilde{Y}$ . A component  $U' \subset \alpha^{-1}(U)$  is an open neighborhood of  $\tilde{x}$  such that  $V' = U' \cap \tilde{Y} \subset \alpha^{-1}(V)$  is Stein since an open subspace is Stein if and only if every component is Stein ([12, v. III, p. 154]). So  $\tilde{Y}$  is locally Stein in the Stein space  $\tilde{X}$ .

The normalization  $\tilde{X}$  is the disjoint union of irreducible components and it is Stein if and only if every irreducible component is Stein ([11, p. 172]; [12, v. III, p. 154]). We may assume that  $X$  is an irreducible normal reduced Stein space. Then  $X$  is of pure finite dimension ([11, p. 106]; [16, p. 196]). It suffices to show that  $Y$  is holomorphically convex.

Let  $X = \cup_{r=1}^{\infty} X_r$  be an exhaustion of the Stein space  $X$  by analytic blocks ([10, p. 118]). Let  $P_0 \in X - Y$ . Then there is a positive integer  $N$  such that  $P_0 \in X_N$ . Since  $X = \cup_{r=N}^{\infty} X_r$  is still an exhaustion of the Stein space  $X$  by analytic blocks, we may reorder the blocks and assume that  $X = \cup_{r=1}^{\infty} X_r$  is an exhaustion of the Stein space  $X$  by analytic blocks and for all  $r \in \mathbb{N}$ ,  $P_0 \in X_r \not\subseteq Y$ . Every  $X_r$  is a compact Stein subset of  $X$ ,  $X_r$  is contained in the analytic interior  $X_{r+1}^0$  of  $X_{r+1}$  ([10, Chapter IV]) and by the Runge approximation theorem, for every coherent analytic sheaf  $\mathcal{F}$  on  $X$ , the restriction map on the spaces of sections

$$H^0(X_{r+1}, \mathcal{F}) \rightarrow H^0(X_r, \mathcal{F})$$

has dense image ([10, p. 123]). Let  $Y_r = X_r \cap Y$ . Then by Lemma 32, the union  $Y = \cup_{r=1}^{\infty} Y_r$  is holomorphically convex. The Theorem 33 is proved.  $\square$

**Corollary 34.** *Let  $X$  be a Stein space and  $Y$  an open subset of  $X$ . If the boundary  $X - Y$  in  $X$  is a closed subspace in  $X$  such that at every point in  $X - Y$ ,  $X - Y$  is locally defined by one holomorphic function, then  $Y$  is Stein.*

**Proof.** Since locally the boundary  $X - Y$  in  $X$  is defined by one holomorphic function and  $X$  is Stein, the open subset  $Y$  is locally Stein and  $X - Y$  is a closed subspace of  $X$ . By Theorem 33,  $Y$  is Stein.  $\square$

We will apply Theorem 33 to two cases when  $X - Y$  is not a closed subspace of  $X$ .

**Theorem 35.** *Let  $X$  be a Stein space and  $Y$  an open subset of  $X$ . If for every boundary point  $P \in \partial Y$ , there is a closed subspace  $H$  of codimension 1 in  $X$  such that  $P \in H$ ,  $H \cap Y = \emptyset$  and  $X - H$  is locally Stein, then  $Y$  is Stein.*

**Proof.** Let  $P = \{P_1, P_2, \dots\}$  be a discrete set in  $Y$ . Since  $X$  is Stein, if  $S$  has no accumulation point in  $X$ , then there is a holomorphic function on  $X$  such that it is unbounded on  $S$ . Assume that  $S$  has a accumulation point  $x_0 \in \partial Y \subset X - Y$ . Let  $H$  be a closed subspace of codimension 1 in  $X$  such that  $x_0 \in H$ ,  $H \cap Y$  is empty and  $X - H$  is locally Stein. By Theorem 33, there is a holomorphic function  $f$  on  $X - H$  such that  $f$  is not bounded near  $x_0$  on  $P$ . So  $Y$  is holomorphically convex thus is Stein.  $\square$

**Corollary 36.** *Let  $X$  be a Stein space and  $Y$  an open subset of  $X$ . If for every boundary point  $P \in \partial Y$ , there is a hypersurface  $H$  locally defined by one holomorphic function in  $X$  such that  $P \in H$ ,  $H \cap Y = \emptyset$ , then  $Y$  is Stein.*

Theorem 35 also can be applied to open subset  $Y$  with real analytic boundary.

**Theorem 37.** *Let  $X$  be a Stein space and  $Y$  an open subset of  $X$ . If for every boundary point  $P \in \partial Y$ , there is a holomorphic function  $h$  in a neighborhood  $U$  of  $P$  such that  $\partial Y \cap U$  is defined by vanishing of  $h(z) + \overline{h(z)}$  in  $U$  and  $h(z) + \overline{h(z)}$  does not vanish on  $Y \cap U$ , then  $Y$  is Stein.*

**Proof.** For any boundary point  $P \in \partial Y$ , define a holomorphic function  $g(z) = h(z) - h(P)$  on  $U$ . Then  $g(P) = 0$ . By assumption, for every point  $Q \in \partial Y \cap U$ ,  $h(Q) + \overline{h(Q)} = 0$ . Then for all  $Q \in \partial Y \cap U$

$$g(Q) + \overline{g(Q)} = h(Q) + \overline{h(Q)} - \left[ h(P) + \overline{h(P)} \right] = 0.$$

Let  $z \in Y \cap U$ . If  $g(z) = 0$ , then

$$g(z) + \overline{g(z)} = h(z) + \overline{h(z)} - \left[ h(P) + \overline{h(P)} \right] = h(z) + \overline{h(z)} = 0.$$

This is not possible by assumption. So the holomorphic function  $g$  does not vanish at every point in  $Y \cap U$ .

Let  $\{U_i\}$  be a Stein open cover of  $\partial Y$  such that on each  $U_i$ ,  $g_i$  is similarly constructed. Then the hypersurface  $H$  defined by  $(g_i, U_i)$  satisfies the condition in Theorem 35. So  $Y$  is Stein.  $\square$

**Example 38.** By Theorem 37, it follows that the following open subsets in  $\mathbb{C}^n$  are Stein.

- (1) Every open ball.
- (2) Every open block.
- (3) Every open polycylinder
- (4) Every open polydisc.
- (5) Every linearly convex domain.
- (6) Let  $h(z)$  be a holomorphic function. Let  $\Omega$  be a domain with real analytic hypersurface boundary  $h(z) + \overline{h(z)} = 0$ . Then  $\Omega$  is Stein. In fact, real analytic hypersurface  $h(z) + \overline{h(z)} = 0$  divides  $\mathbb{C}^n$  into separated Stein domains.

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