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Nef cones of some Quot schemes on a Smooth Projective Curve

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Abstract. Let C be a smooth projective curve over $\mathbb{C}$. Let $n, d \geq 1$. Let $\mathcal{Q}$ be the Quot scheme parameterizing torsion quotients of the vector bundle $\mathcal{O}_C^n$ of degree d. In this article we study the nef cone of $\mathcal{Q}$. We give a complete description of the nef cone in the case of elliptic curves. We compute it in the case when $d = 2$ and C very general, in terms of the nef cone of the second symmetric product of C. In the case when $n \geq d$ and C very general, we give upper and lower bounds for the Nef cone. In general, we give a necessary and sufficient criterion for a divisor on $\mathcal{Q}$ to be nef.

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1. Introduction

Throughout this article we assume that the base field to be $\mathbb{C}$. Let X be a smooth projective variety and let $N^1(X)$ be the $\mathbb{R}$-vector space of $\mathbb{R}$-divisors modulo numerical equivalence. It is known that $N^1(X)$ is a finite dimensional vector space. The closed cone $\mathrm{Nef}(X) \subset N^1(X)$ is the cone of all $\mathbb{R}$-divisors whose intersection product with any curve in X is non-negative. It has been an interesting problem to compute $\mathrm{Nef}(X)$. For example, when $X = \mathbb{P}(E)$ where E is a semistable vector bundle over a smooth projective curve, Miyaoka computed the $\mathrm{Nef}(X)$ in [14]. In [4], $\mathrm{Nef}(X)$ was computed in the case when X is the Grassmann bundle associated to a vector bundle E on a smooth projective curve C, in terms of the Harder Narasimhan filtration of E. Let $C^{(d)}$ denote the dth symmetric product. In [15], the author computed the $\mathrm{Nef}(C^{(d)})$ in the case when C is a very general curve of even genus and $d = \mathrm{gon}(C) - 1$. In [11] $\mathrm{Nef}(C^{(2)})$ is computed in the case when C is very general and g is a perfect square. In [5] $\mathrm{Nef}(C^{(2)})$ was computed assuming the Nagata conjecture. We refer the reader to [12, Section 1.5] for more such examples and details.

The reader is referred to [6] for the definition and details on Quot schemes. Let E be a vector bundle over a smooth projective curve C. Fix a polynomial $P \in \mathbb{Q}[t]$. Let $\mathcal{Q}(E, P)$ denote the Quot

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scheme parametrizing quotients of E with Hilbert polynomial P. In [16], when $C = \mathbb{P}^1$, the quot scheme $\mathcal{Q}(\mathcal{O}_C^n, P)$ is studied as a natural compactification of the set of all maps from C to some Grassmannians of a fixed degree. In this article we will consider the case when $P = d$ a constant, that is, when $\mathcal{Q}(E,d)$ parametrizes torsion quotients of E of degree d. For notational convenience, we will denote $\mathcal{Q}(E,d)$ by $\mathcal{Q}$, when there is no possibility of confusion. It is known that $\mathcal{Q}$ is a smooth projective variety. Many properties of $\mathcal{Q}$ have been studied. In [1], the Betti cohomologies of $\mathcal{Q}(\mathcal{O}_C^n, d)$ are computed, $\mathcal{Q}(\mathcal{O}_C^n, d)$ has been interpreted as the space of higher rank divisors of rank n, and an analogue of the Abel–Jacobi map was constructed. In [2] the automorphism group scheme of $\mathcal{Q}(\mathcal{Q}_C^n, d)$ was computed in the case when the genus of C satisfies $g(C) > 1$ and a Torelli theorem for these Quot schemes was proved. In [3] the Brauer group of $\mathcal{Q}(\mathcal{O}_C^n, d)$ is computed. In [7], the automorphism group scheme of $\mathcal{Q}(E,d)$ was computed in the case when either rk $E \geq 3$ or E is semistable and genus of C satisfies $g(C) > 1$. In [8], the S-fundamental group scheme of $\mathcal{Q}(E,d)$ was computed.

In this article, we address the question of computing Nef($\mathcal{Q}$). Recall that we have a Hilbert–Chow map $\Phi : \mathcal{Q} \to C^{(d)}$ (this map is explained after Definition 9. A precise definition can be found, for example, in [8]). For notational convenience, for a divisor $D \in N^1(C^{(d)})$ we will denote its pullback $\Phi^* D \in N^1(\mathcal{Q})$ by D, when there is no possibility of confusion. The line bundle $\mathcal{O}_{\mathcal{Q}}(1)$ is defined in Definition 9. In Section 2 we recall the results we need on Nef($C^{(d)}$). In Section 3 we compute Pic($\mathcal{Q}$).

Theorem (Theorem 11). $\operatorname{Pic}(\mathcal{Q}) = \Phi^* \operatorname{Pic}(C^{(d)}) \bigoplus \mathbb{Z}[\mathcal{O}_{\mathcal{Q}}(1)]$.

As a corollary (Corollary 13) we get that $N^1(\mathcal{Q}) \cong N^1(C^{(d)}) \oplus \mathbb{R}[\mathcal{O}_{\mathcal{Q}}(1)]$. The computation of $N^1(\mathcal{Q})$ can also be found in [3]. As a result, when $C \cong \mathbb{P}^1$, since $C^{(d)} \cong \mathbb{P}^d$, we have that the $N^1(\mathcal{Q})$ is 2-dimensional and we prove that its nef cone is given as follows.

Theorem (Theorem 34). *Let $C = \mathbb{P}^1$. Let $E = \bigoplus_{i=1}^{k} \mathcal{O}(a_i)$ with $a_i \leq a_j$ for $i < j$. Let $d \geq 1$. Then*

$$\operatorname{Nef}(\mathcal{Q}(E,d)) = \mathbb{R}_{\geq 0}\left(\left[\mathcal{O}_{\mathcal{Q}(E,d)}(1)\right] + (-a_1 + d - 1)\left[\mathcal{O}_{\mathbb{P}^d}(1)\right]\right) + \mathbb{R}_{\geq 0}\left[\mathcal{O}_{\mathbb{P}^d}(1)\right].$$

Note that this theorem was already known in the case when $E = V \otimes \mathcal{O}_{\mathbb{P}^1}$, for a vector space V over k ([16, Theorem 6.2]).

For the rest of the introduction, we will assume $E = V \otimes \mathcal{O}_C$ with $\dim_k V = n$ and denote by $\mathcal{Q} = \mathcal{Q}(n,d)$ the Quot scheme $\mathcal{Q}(E,d)$. Let us consider the case $g = 1$. In this case, $N^1(\mathcal{Q})$ is three-dimensional (see Proposition 14), and we prove that its nef cone is given as follows (see Definition 4 for notations).

Theorem (Theorem 43). *Let $g = 1$, $n \geq 1$ and $\mathcal{Q} = \mathcal{Q}(n,d)$. Then the class $[\mathcal{O}_{\mathcal{Q}}(1)] + [\Delta_d/2] \in N^1(\mathcal{Q})$ is nef. Moreover,*

$$\operatorname{Nef}(\mathcal{Q}) = \mathbb{R}_{\geq 0}([\mathcal{O}_{\mathcal{Q}}(1)] + [\Delta_d/2]) + \mathbb{R}_{\geq 0}[\theta_d] + \mathbb{R}_{\geq 0}[\Delta_d/2].$$

From now on assume that $g \geq 2$ and C is very general. See Definition 9 for the definition of t and α_t. When $d = 2$ we have the following result.

Theorem (Theorem 37). *Let $g \geq 2$ and C be very general. Let $d = 2$. Consider the Quot scheme $\mathcal{Q} = \mathcal{Q}(n,2)$. Then*

$$\operatorname{Nef}(\mathcal{Q}) = \mathbb{R}_{\geq 0}\left([\mathcal{O}_{\mathcal{Q}}(1)] + \frac{t+1}{g+t}[L_0]\right) + \mathbb{R}_{\geq 0}[L_0] + \mathbb{R}_{\geq 0}[\alpha_t].$$

Precise values of t are known for small genus. When $g \geq 9$ it is conjectured that $t = \sqrt{g}$. This is known when g is a perfect square. The precise statements have been mentioned after Theorem 37.

In general (without any assumptions on n and d), we give a criterion for certain line bundle on $\mathscr{Q}$ to be nef in terms of its pullback along certain natural maps from products $\prod_i C^{(d_i)}$, see Subsection 7.1 for notation.

Theorem (Theorem 39). *Let $\beta \in N^1(C^{(d)})$. Then the class $[\mathscr{O}_{\mathscr{Q}}(1)] + \beta \in N^1(\mathscr{Q})$ is nef iff the class $[\mathscr{O}(-\Delta_{\mathbf{d}}/2)] + \pi^*_{\mathbf{d}}\beta \in N^1(C^{(\mathbf{d})})$ is nef for all $\mathbf{d} \in \mathscr{P}^{\leq n}_d$.*

Using the above we show that certain classes are in Nef($\mathscr{Q}$). Define

$$\kappa_1 := [\mathscr{O}_{\mathscr{Q}}(1)] + \mu_0[L_0] + \frac{d+g-2}{dg}[\theta_d] \qquad \kappa_2 = [\mathscr{O}_{\mathscr{Q}}(1)] + \frac{g+1}{2g}[L_0] \in N^1(\mathscr{Q}). \tag{1}$$

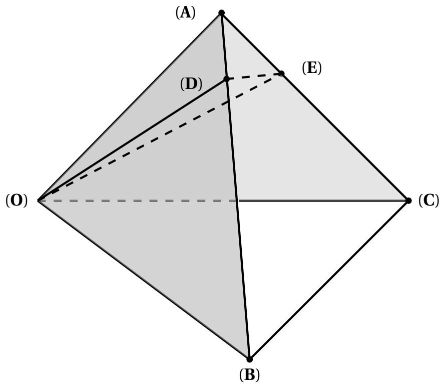
Proposition (Proposition 41). *Let $g \geq 1$, $n \geq 1$ and $\mathscr{Q} = \mathscr{Q}(n,d)$. Then*

$$\mathrm{Nef}(\mathscr{Q}) \supset \mathbb{R}_{\geq 0}\kappa_1 + \mathbb{R}_{\geq 0}\kappa_2 + \mathbb{R}_{\geq 0}[\theta_d] + \mathbb{R}_{\geq 0}[L_0].$$

Now consider the case when $n \geq d \geq \mathrm{gon}(C)$. Then $\mathrm{Nef}(C^{(d)})$ is generated by θ_d and L_0 (see Definitions 1 and 4). In this case we give the following upper bound for the nef cone in Proposition 20. Let $\mu_0 := \frac{d+g-1}{dg}$. Then

$$\mathrm{Nef}(\mathscr{Q}) \subset \mathbb{R}_{\geq 0}\left([\mathscr{O}_{\mathscr{Q}}(1)] + \mu_0[L_0]\right) + \mathbb{R}_{\geq 0}[\theta_d] + \mathbb{R}_{\geq 0}[L_0].$$

When $d \geq \mathrm{gon}(C)$, in Lemma 30 we show that any convex linear combination of the κ_1 and θ_d is nef but not ample. In particular, any such class lies on the boundary of Nef($\mathscr{Q}$). Similarly, in Corollary 42 we show when $n \geq d$, any convex linear combination of the class κ_2 and $L_0^{(d)}$ is nef but not ample. So any such class lies on the boundary of Nef($\mathscr{Q}$).



(2)

(1) $(\mathbf{A}) = [\mathscr{O}_{\mathscr{Q}}(1)] + \mu_0[L_0]$
(2) $(\mathbf{B}) = [\theta_d]$
(3) $(\mathbf{C}) = [L_0]$
(4) $(\mathbf{D}) = \tau\kappa_1 = \tau([\mathscr{O}_{\mathscr{Q}}(1)]/2 + \mu_0[L_0]) + (1-\tau)[\theta_d] \qquad \tau = \frac{1}{1+\frac{d+g-2}{dg}}$
(5) $(\mathbf{E}) = \rho\kappa_2 = \rho([\mathscr{O}_{\mathscr{Q}}(1)] + \mu_0[L_0]) + (1-\rho)[L_0] \qquad \rho = \frac{1}{1+\frac{g+1}{2g}-\frac{d+g-2}{dg}}$

In terms of the above diagram, we have that when $n \geq d \geq \mathrm{gon}(C)$

$$\left\langle \overline{OD}, \overline{OE}, \overline{OC}, \overline{OB} \right\rangle \subset \mathrm{Nef}(\mathscr{Q}) \subset \left\langle \overline{OA}, \overline{OC}, \overline{OB} \right\rangle.$$

We do not know if the inclusion in the right is an equality when $n \geq d \geq \mathrm{gon}(C)$. This is same as saying that $[\mathscr{O}_{\mathscr{Q}}(1)] + \mu_0[L_0]$ is nef when $n \geq d \geq \mathrm{gon}(C)$. In Section 8 we give a sufficient condition for when the pullback of $[\mathscr{O}_{\mathscr{Q}}(1)] + \mu_0[L_0]$ along a map $D \to \mathscr{Q}$ is nef. However, when $d = 3$ we have the following result.

Theorem (Theorem 49). *Let C be a very general curve of genus $2 \le g(C) \le 4$. Let $n \ge 3$ and let $\mathscr{Q} = \mathscr{Q}(n,3)$. Let $\mu_0 = \frac{g+2}{3g}$ Then*

$$\mathrm{Nef}(\mathscr{Q}) = \mathbb{R}_{\ge 0}\left([\mathscr{O}_{\mathscr{Q}}(1)] + \mu_0[L_0]\right) + \mathbb{R}_{\ge 0}[\theta_d] + \mathbb{R}_{\ge 0}[L_0].$$

Some of the results above can be improved in the case when $g = 2k$ using the results in [15]. (See Proposition 32.)

2. Nef cone of $C^{(d)}$

We follow [15, § 2] for this section. Assume that either C is an elliptic curve or is a very general curve of genus $g \ge 2$. Then it is known that the Neron–Severi space is 2-dimensional. So in this case, to compute the nef cone, it is enough to give two classes in $N^1(C)$ which are nef but not ample.

For any smooth projective curve and $d \ge 2$ (not just a very general curve) there is a natural line bundle L_0 on $C^{(d)}$ which is nef but not ample. This line bundle is constructed in the following manner. Consider the map

$$\begin{aligned}\phi : C^d &\to J(C)^{\binom{d}{2}},\\ (x_i) &\mapsto \left(x_i - x_j\right)_{i<j}.\end{aligned}$$

Let p_{ij} denote the projections from $J(C)^{\binom{d}{2}}$. Since ϕ is not finite, as it contracts the diagonal, the line bundle $\phi^*(\otimes p_{ij}^*\Theta)$ is nef but not ample. This line bundle is invariant under the action of S_d on C^d. This follows from the fact that Θ in $J(C)$ is invariant under the involution $L \mapsto L^{-1}$.

Definition 1. *$\phi^*(\otimes p_{ij}^*\Theta)$ descends to a line bundle L_0 on $C^{(d)}$.*

Since ϕ contracts the small diagonal $\delta : C \hookrightarrow C^{(d)}$, we have $\delta^*[L_0] = 0$. Hence L_0 is nef but not ample [15, Lemma 2.2]. Therefore, in the case when C is very general, computing the nef cone of $C^{(d)}$ boils down to finding another class which is nef but not ample.

In the case when $d \ge \mathrm{gon}(C) =: e$, [15, Lemma 2.3] we can easily construct another line bundle which is nef but not ample: Then we have a map $g_e : C \to \mathbb{P}^1$ of degree e. This induces a closed immersion $\mathbb{P}^1 \to C^{(e)}$ with $v \mapsto [(g_e)^{-1}(v)] \in C^{(e)}$. This in turn gives a closed immersion $\mathbb{P}^1 \to C^{(d)}$ with $v \mapsto [(g_e)^{-1}(v) + (d-e)x]$ for some point $x \in C$.

Definition 2. *Denote the class of this $\mathbb{P}^1$ in $N_1(C^{(d)})$ by $[l']$.*

The composition $\mathbb{P}^1 \to C^{(d)} \xrightarrow{u_d} J(C)$ is constant, since there can be no non-constant maps from $\mathbb{P}^1 \to J(C)$. Hence $u_d : C^{(d)} \to J(C)$ is not finite and we get that $u_d^*\Theta$ is nef but not ample.

Definition 3. *Define $\theta_d := u_d^*\Theta$.*

Recall that over $C^{(d)}$ we have natural divisors [15, § 2]:

Definition 4. *Define*

1. θ_d
2. *the big diagonal $\Delta_d \hookrightarrow C^{(d)}$*
3. *If $i_{d-1} : C^{(d-1)} \to C^{(d)}$ is the map given by $D \mapsto D + x$ for a point $x \in C$, then the image $i_{d-1}(C^{(d-1)})$. This divisor will be denoted $[x]$.*

It is known that when $g = 1$ or C is very general of $g \ge 2$, then $N^1(C^{(d)})$ is of dimension 2 and any two of the above three forms a basis.

By abuse of notation, let us denote the class (δ is the small diagonal) $[\delta_*(C)] \in N_1(C^{(d)})$ by δ. We summarise the above discussion in the following theorem.

Proposition 5 ([15, Proposition 2.4]). *When $d \geq \mathrm{gon}(C)$, we have:*

(1) $\mathrm{Nef}(C^{(d)}) = \mathbb{R}_{\geq 0}[L_0] \oplus \mathbb{R}_{\geq 0}[\theta_d]$,
(2) $\overline{NE}(C^{(d)}) = \mathbb{R}_{\geq 0}[l'] \oplus \mathbb{R}_{\geq 0}[\delta]$.

The above basis are dual to each other.

We will need to write $[L_0]$ in terms of $[x]$ and $[\theta_d]$, for which we need the following computations. Define

$$\delta' : C \xrightarrow{f} C^d \to C^{(d)}$$

where the first map is given by $x \mapsto (x, x_1, \ldots, x_{d-1})$.

Lemma 6. *Let $d \geq 1$. We have the following*

(1) $\deg(\delta^*[\theta_d]) = d^2 g$
(2) $\deg(\delta'^*[\theta_d]) = g$
(3) $\deg(\delta^*[x]) = d$
(4) $\deg(\delta'^*[x]) = 1$

Proof. Recall that $\theta_d = u_d^*\Theta$, where $u_d : C^{(d)} \to J(C)$ is given by $D \mapsto \mathscr{O}(D - dx_0)$ for a fixed point $x_0 \in C$. Therefore the composition $u_d \circ \delta : C \to J(C)$ is given by $x \mapsto dx \mapsto \mathscr{O}(dx - dx_0)$, which is the map

$$C \xrightarrow{u_1} J(C) \xrightarrow{\times d} J(C) .$$

The pullback of Θ under the map $J(C) \xrightarrow{\times d} J(C)$ is Θ^{d^2} and the degree of the pullback of Θ under the map $u_1 : C \to J(C)$ is g. Hence degree of $\delta^*\theta_d = d^2 g$. This proves (1).

The composition $u_d \circ \delta' : C \to J(C)$ is given by $C \to C^{(d)} \to J(C)$

$$x \mapsto x + \sum_{i=1}^{d-1} x_i \mapsto \mathscr{O}\left(x + \sum_{i=1}^{d-1} x_i - dx_0\right)$$

which is the composition $C \xrightarrow{u_1} J(C) \xrightarrow{t_a} J(C)$, where t_a is translation by an element in $J(C)$. Hence degree of $\delta'^*\theta_d = g$. This proves (2).

For a line bundle L on C, we will denote by $L^{\boxtimes d}$ to be the unique line bundle on $C^{(d)}$, whose pullback under the quotient map $\pi : C^d \to C^{(d)}$ is $\bigotimes_{i=1}^d p_i^* L$. Recall that by [15, § 2], we have that $[x] = [\mathscr{O}(x)^{\boxtimes d}]$ for a point $x \in C$. By definition under the map $\pi : C^d \to C^{(d)}$ the pullback of $\mathscr{O}(x)^{\boxtimes d}$ is $\bigotimes_{i=1}^d p_i^*\mathscr{O}(x)$. Now $\delta : C \hookrightarrow C^{(d)}$ is the composition $C \to C^d \to C^{(d)}$

$$x \mapsto (x, .., x) \mapsto dx .$$

Hence we get that the pullback of $\mathscr{O}(x)^{\boxtimes d}$ to δ is $\mathscr{O}(dx)$. Therefore degree of $\delta^*[x] = d$. This proves(3).

We know δ' is the composition $C \to C^d \to C^{(d)}$

$$x \mapsto (x, x_1, .., x_{d-1}) \mapsto x + x_1 + \ldots + x_{d-1} .$$

Hence we get that $\delta'^*[x] = \mathscr{O}(x)$. Therefore degree of $\delta'^*[x] = 1$. This proves (4). □

Lemma 7. *Let $g, d \geq 1$. Let $\mu_0 := \frac{d+g-1}{dg}$. Then*

$$\begin{aligned}[L_0] &= dg[x] - [\theta_d] \\ &= \left(dg - d - g + 1\right) . [x] + [\Delta_d/2] \\ &= \left(\frac{1}{\mu_0} - 1\right)[\theta_d] + \frac{1}{\mu_0}[\Delta_d/2] .\end{aligned}$$

Proof. Let $[L_0] = a[\theta_d] + b[x]$. We need two equations to solve for a and b. The first equation is $\delta^*[L_0] = 0$. Recall

$$\delta' : C \xrightarrow{f} C^d \to C^{(d)}$$

where the first map is given by $x \mapsto (x, x_1, \ldots, x_d)$. Hence

$$\delta'^*[L_0] = f^*\phi^*\left(\otimes p_i^*\Theta\right).$$

Now the composition

$$C \xrightarrow{f} C^d \xrightarrow{\phi} J(C)^{\binom{d}{2}}$$

is given by $x \mapsto (x - x_1, x - x_2, \ldots, x - x_{d-1}, x_i - x_j)_{i<j}$. Hence

$$\deg\left(\delta'^*[L_0]\right) = \sum_{i=1}^{d-1} \deg(\theta_1) = (d-1)g.$$

This will be our second equation.

We use these two equations and the preceding computations to compute a and b.

$$\begin{aligned}0 &= \deg\left(\delta^*[L_0]\right)\\ &= a.\deg\left(\delta^*[\theta_d]\right) + b.\deg\left(\delta^*[x]\right)\\ &= ad^2g + bd.\end{aligned}$$

Therefore

$$b = -adg.$$

Now using the second equation we get

$$\begin{aligned}(d-1)g &= \deg\left(\delta'^*[L_0]\right)\\ &= a.\deg\left(\delta'^*[\theta_d]\right) + b.\deg\left(\delta'^*[x]\right)\\ &= ag + b\\ &= ag - adg = ag(1-d).\end{aligned}$$

Therefore

$$a = -1, \qquad b = dg.$$

Hence we get $[L_0] = dg[x] - [\theta_d]$. For the other two equalities, we use the relation

$$[\theta_d] = \left(d + g - 1\right)[x] - [\Delta_d/2]$$

between $[x], [\Delta_d/2]$ and $[\theta_d]$ [15, Lemma 2.1]. □

3. Picard group and Neron–Severi group of $\mathcal{Q}$

Let E be a locally free sheaf over C. Throughout this section $\mathcal{Q}$ will denote the Quot scheme $\mathcal{Q}(E,d)$ which parametrizes torsion quotients of E of degree d. In this section we compute the Picard group of $\mathcal{Q}$, and the vector spaces $N^1(\mathcal{Q})$ and $N_1(\mathcal{Q})$.

Lemma 8. *Let S be a scheme over k. Let F be a coherent sheaf over $C \times S$ which is S-flat and for all $s \in S$, $F|_{C\times s}$ is a torsion sheaf over C of degree d. Let $p_S : C \times S \to S$ be the projection. Then*

(i) *$p_{S*}(F)$ is locally free of rank d and $\forall\ s \in S$ the natural map $p_{S*}(F)|_s \to H^0(C, F|_{C\times s})$ is an isomorphism.*

(ii) *Assume that we are given a morphism $\phi : T \to S$. We have the following diagram:*

$$\begin{array}{ccc} C\times T & \xrightarrow{id\times\phi} & C\times S \\ \big\downarrow{\scriptstyle p_T} & & \big\downarrow{\scriptstyle p_S} \\ T & \xrightarrow{\phi} & S \end{array}$$

Then the natural morphism

$$\phi^* p_{S*}(F) \to (p_T)_*(id\times\phi)^* F$$

is an isomorphism.

Proof. Since $F|_{C\times s}$ is a torsion sheaf for all $s \in S$, we have $H^1(C, F|_{C\times s}) = 0$. By [9, Chapter III, Theorem 12.11 (a)] we get $R^1 p_{S*}(F) = 0$. Using [9, Chapter III, Theorem 12.11 (b)] (ii) with $i = 1$ we get that the morphism $p_{S*}(F)|_s \to H^0(C, F|_{C\times s})$ is surjective. Again using the same with $i = 0$ we get that $p_{S*}(F)$ is locally free of rank d and the map $p_{S*}(F)|_s \to H^0(C, F|_{C\times s})$ is an isomorphism.

Since F is S-flat it follows that $(id\times\phi)^*F$ is T-flat. Applying the above we see $\phi^* p_{S*}(F)$ and $(p_T)_*(id\times\phi)^*F$ are locally free of rank d. For each $t \in T$ we have the commutative diagram:

$$\begin{array}{ccc} \phi^* p_{S*}(F)|_t = p_{S*}(F)|_{\phi(t)} & \longrightarrow & (p_T)_*(id\times\phi)^*F|_t \\ \big\downarrow & & \big\downarrow \\ H^0\left(C, F\big|_{C\times\phi(t)}\right) & =\!=\!=\!= & H^0\left(C, (id\times\phi^*)F\big|_{C\times t}\right) \end{array}$$

By the first part we get that the vertical arrows are isomorphisms. Hence we get that the first row of the diagram is an isomorphism. Therefore

$$\phi^* p_{S*}(F) \to (p_T)_*(id\times\phi)^* F$$

is a surjective morphism of vector bundles of same rank and hence an isomorphism. □

We define a line bundle on $\mathcal{Q}$. Let us denote the projections $C\times\mathcal{Q}$ to C and $\mathcal{Q}$ by p_C and p_Q respectively. Then we have the universal quotient $p_C^* E \to \mathcal{B}_{\mathcal{Q}}$ over $C\times\mathcal{Q}$. By Lemma 8, $p_{\mathcal{Q}*}(\mathcal{B}_{\mathcal{Q}})$ is a vector bundle of rank d.

Definition 9. *Denote the line bundle* $\det(p_{\mathcal{Q}*}(\mathcal{B}_{\mathcal{Q}}))$ *by* $\mathcal{O}_{\mathcal{Q}}(1)$.

Denote the d^{th} symmetric product of C by $C^{(d)}$. Recall the Hilbert–Chow map $\Phi : \mathcal{Q} \to C^{(d)}$ which sends $[E \to B]$ to $\sum l(B_p)p$, where $l(B_p)$ is the length of the $\mathcal{O}_{C,p}$-module B_p. Therefore, we have the pullback $\Phi^* : \mathrm{Pic}(C^{(d)}) \to \mathrm{Pic}(\mathcal{Q})$ which is in fact an inclusion. To see this, recall that the fibres of Φ are projective integral varieties [8, Corollary 6.6] and Φ is flat [8, Corollary 6.3]. Hence $\Phi_*(\mathcal{O}_{\mathcal{Q}}) = \mathcal{O}_{C^{(d)}}$. Now by projection formula $\Phi_*\Phi^*L \cong L$ for all $L \in \mathrm{Pic}(C^{(d)})$ and the statement follows.

The big diagonal is the image of the map $C\times C^{(d-2)} \to C^{(d)}$ given by $(x, A) \mapsto 2x + A$. Let us denote the big diagonal in $C^{(d)}$ by Δ. Let $U_C := C^{(d)} \setminus \Delta$ and $\mathcal{U} := \Phi^{-1}(U_C)$. Then $\mathcal{U} \subset \mathcal{Q}$.

Lemma 10. *For any line bundle $\mathcal{L} \in \mathrm{Pic}(\mathcal{Q})$, $\exists$ an unique $n \in \mathbb{Z}$ such that $(\mathcal{L}\otimes\mathcal{O}_{\mathcal{Q}}(-n))|_{\Phi^{-1}(p)} \cong \mathcal{O}_{\Phi^{-1}(p)}$ for all $p \in U_C$.*

Proof. Let $\pi : \mathbb{P}(E) \to C$ be the projective bundle associated to E and let $\mathcal{O}_{\mathbb{P}(E)}(1)$ be the universal line bundle over $\mathbb{P}(E)$. Let $Z = \mathbb{P}(E)^d$. Let $p_i : Z \to \mathbb{P}(E)$ be the i^{th} projection. Let $\pi_d : Z \to C^d$ be the product map. The symmetric group S_d acts on Z and the map π_d is equivariant for this action. Let $\psi : C^d \to C^{(d)}$ be the quotient map. Define $U_Z := (\psi\circ\pi_d)^{-1}(U)$.

Let $c \in C$ be a closed point and let k_c denote the skyscraper sheaf supported at c. A closed point of $\mathbb{P}(E)$ which maps to $c \in C$ corresponds to a quotient $E \to E_c \to k_c$. Recall that we have a map [7, Theorem 2.2 (a)]

$$\widetilde{\psi} : U_Z \to \mathcal{U}$$

which sends a closed point

$$\left(E_{c_i} \to k_{c_i}\right)_{i=1}^{d} \in U_Z$$

to the quotient

$$E \to \bigoplus_i E_{c_i} \to \bigoplus_i k_{c_i} \in \mathscr{U}\,.$$

So we have a commutative diagram:

$$\begin{tikzcd} U_Z \arrow[r, "\tilde{\psi}"] \arrow[d, "\pi_d"] & \mathscr{U} \arrow[d, "\Phi"] \\ \psi^{-1}(U_C) \arrow[r, "\psi"] & U_C \end{tikzcd}$$

Moreover, if $\underline{c} = (c_1, \ldots, c_d) \in \psi^{-1}(U_C)$, then by [8, Lemma 6.5] $\widetilde{\psi}$ induces an isomorphism

$$\prod \mathbb{P}(E_{c_i}) = \pi_d^{-1}(\underline{c}) \xrightarrow{\sim} \Phi^{-1}\left(\psi(\underline{c})\right).$$

Applying Lemma 8 by taking $T = U_Z$, $S = \mathscr{U}$ and $\phi = \widetilde{\psi}$ and the definition of the map $\widetilde{\psi}$ (see the proof of [7, Theorem 2.2 (a)]) we see that

$$\widetilde{\psi}^* \mathscr{O}_{\mathscr{Q}}(1) = \bigotimes_{i=1}^{d} p_i^* \mathscr{O}_{\mathbb{P}(E)}(1)|_{U_Z}\,.$$

Hence it is enough to show that $\exists\ n \in \mathbb{Z}$ such that $\forall\ \underline{c} \in \psi^{-1}(U_C)$

$$\widetilde{\psi}^* \mathscr{L}|_{\pi_d^{-1}(\underline{c})} \cong \bigotimes_{i=1}^{d} p_i^* \mathscr{O}(n)|_{\pi_d^{-1}(\underline{c})}\,.$$

For $\underline{c} \in \psi^{-1}(U_C)$ define $n_i(\underline{c}) \in \mathbb{Z}$ using the equation

$$\widetilde{\psi}^* \mathscr{L}|_{\pi_d^{-1}(\underline{c})} = \bigotimes_{i=1}^{d} p_i^* \mathscr{O}_{\mathbb{P}(E_{c_i})}\left(n_i(\underline{c})\right).$$

We may view the n_i as functions $n_i : \psi^{-1}(U_C) \to \mathbb{Z}$. Since the line bundle $\widetilde{\psi}^* \mathscr{L}$ is invariant under the action of the group S_d, it follows that

$$n_{\sigma(i)}(\underline{c}) = n_i\left(\sigma(\underline{c})\right). \tag{3}$$

Here $\sigma(\underline{c}) := (c_{\sigma(1)}, \ldots, c_{\sigma(d)})$. Hence it suffices to show that n_1 is a constant function.

Let $c_2, \ldots, c_d$ be distinct points in C. Define $V := C \setminus \{c_2, .., c_d\}$ and a map

$$i : V \hookrightarrow \psi^{-1}(U_C) \qquad\qquad i(c) := (c, c_2, .., c_d)\,.$$

Then $\pi_d^{-1}(V)$ is equal to $\mathbb{P}(E|_V) \times \mathbb{P}(E_{c_2}) \times \ldots \times \mathbb{P}(E_{c_d})$. The restriction of $\widetilde{\psi}^* \mathscr{L}$ to $\mathbb{P}(E|_V) \times \mathbb{P}(E_{c_2}) \times \ldots \times \mathbb{P}(E_{c_d})$ is isomorphic to

$$\pi^* M \otimes p_1^* \mathscr{O}_{\mathbb{P}(E|_V)}(a_1) \otimes p_2^* \mathscr{O}_{\mathbb{P}(E_{c_2})}(a_2) \ldots \otimes p_d^* \mathscr{O}_{\mathbb{P}(E_{c_d})}(a_d)\,,$$

where M is a line bundle on V. Further restricting to $(c, c_2, \ldots, c_d)$ and $(c', c_2, \ldots, c_d)$, where $c, c' \in V$, we see that

$$n_i(c, c_2, .., c_d) = n_i(c', c_2, \ldots, c_d) \qquad\qquad \forall\ i\,. \tag{4}$$

This proves that for distinct points $c, c', c_2, \ldots, c_d \in C$ we have

$$n_i(c, c_2, .., c_d) = n_i(c', c_2, \ldots, c_d) \qquad\qquad \forall\ i\,. \tag{5}$$

Choose $2d$ distinct points $c_1,\dots,c_d,c'_1,\dots,c'_d$ in C. Then using equations (4) and (5) we get

$$\begin{aligned} n_1(c_1,c_2,\dots,c_d) &= n_1\left(c'_1,c_2,\dots,c_d\right)\\ &= n_2\left(c_2,c'_1,\dots,c_d\right)\\ &= n_2\left(c'_2,c'_1,c_3,\dots,c_d\right)\\ &= n_1\left(c'_1,c'_2,c_3,\dots,c_d\right)\\ &= \dots\\ &= n_1\left(c'_1,c'_2,\dots,c'_d\right). \end{aligned}$$

Finally, for any two points $\underline{c},\underline{c}'\in\psi^{-1}(U_C)$ choose a third point $\underline{c}''$ such that the coordinates of $\underline{c}''$ are distinct from those of $\underline{c}$ and $\underline{c}'$. Then we see that $n_1(\underline{c})=n_1(\underline{c}'')=n_1(\underline{c}')$. This proves that n_1 is the constant function. Therefore, $\psi^*\mathscr{L}|_{\pi_d^{-1}(\underline{c})}$ is of the form $\bigotimes p_i^*\mathscr{O}_{\mathbb{P}(E_{c_i})}(n)$, $\forall\,\underline{c}\in\psi^{-1}(U_C)$. The uniqueness of n is obvious. ☐

Theorem 11. $\operatorname{Pic}(\mathscr{Q})=\Phi^*\operatorname{Pic}(C^{(d)})\bigoplus\mathbb{Z}[\mathscr{O}_{\mathscr{Q}}(1)]$.

Proof. Let $\mathscr{L}\in\operatorname{Pic}(\mathscr{Q})$. By [8, Corollary 6.3] and [8, Corollary 6.4] the morphism Φ is flat and fibres of Φ are integral. Then by [13, Lemma 2.1.2] and Lemma 10 we get that $\mathscr{L}\otimes\mathscr{O}_{\mathscr{Q}}(-n)=\Phi^*\mathscr{M}$ for some $\mathscr{M}\in\operatorname{Pic}(C^{(d)})$. Hence $\mathscr{L}=\Phi^*\mathscr{M}\otimes\mathscr{O}_{\mathscr{Q}}(n)$. The uniqueness of such an expression follows from the statement on uniqueness in Lemma 10. ☐

For a projective variety X over k recall that $N^1(X)$ (respectively, $N_1(X)$) is the vector space of $\mathbb{R}$-divisors (respectively, 1-cycles) modulo numerical equivalences [12, § 1.4]. It is known that $N^1(X)$ and $N_1(X)$ are finite dimensional and the intersection product defines a non-degenerate pairing

$$N^1(X)\times N_1(X)\to\mathbb{R}\qquad\qquad \big([\beta],[\gamma]\big)\mapsto[\beta]\cdot[\gamma].$$

We will compute $N^1(\mathscr{Q})$ and $N_1(\mathscr{Q})$. Let $\underline{c}\in U_C\subset C^{(d)}$. As we saw in the proof of Theorem 11,

$$\Phi^{-1}(\underline{c})\cong\prod\mathbb{P}\left(E_{c_i}\right).$$

Let $\mathbb{P}^1\hookrightarrow\mathbb{P}(E_{c_1})$ be a line and let $v_i\in\mathbb{P}(E_{c_i})$ for $i\geq 2$. Then we have an embedding:

$$\mathbb{P}^1\cong\mathbb{P}^1\times v_2\times\dots\times v_d\hookrightarrow\mathbb{P}\left(E_{c_1}\right)\times\prod_{i\geq 2}\mathbb{P}(E|_{c_i})=\Phi^{-1}(\underline{c})\subset\mathscr{Q}.\tag{6}$$

Definition 12. *Let us denote the class of this curve in $N_1(\mathscr{Q})$ by $[l]$.*

Corollary 13. $N^1(\mathscr{Q})=\Phi^*N^1(C^{(d)})\bigoplus\mathbb{R}[\mathscr{O}_{\mathscr{Q}}(1)]$.

Proof. Since Φ is surjective, $N^1(C^{(d)})\to N^1(\mathscr{Q})$ is an inclusion [12, Example 1.4.4]. Note that $\mathscr{O}_{\mathscr{Q}}(1)\neq 0$ in $N^1(\mathscr{Q})$ since $[\mathscr{O}_{\mathscr{Q}}(1)]\cdot[l]=1$. Hence $\mathscr{O}_{\mathscr{Q}}(1)\neq 0$ in $N^1(\mathscr{Q})$. This also shows that $\mathscr{O}_{\mathscr{Q}}(1)\notin\Phi^*N^1(C^{(d)})$.

By Theorem 11, we know that any $N^1(\mathscr{Q})$ is generated by $\Phi^*N^1(C^{(d)})$ and $[\mathscr{O}_{\mathscr{Q}}(1)]$. The only thing left is to show that

$$\Phi^*N^1\left(C^{(d)}\right)\cap\mathbb{R}\,[\mathscr{O}_{\mathscr{Q}}(1)]=0.$$

For $a\in\mathbb{R}$ if $a[\mathscr{O}_{\mathscr{Q}}(1)]\in N^1(C^{(d)})$, then $a[\mathscr{O}_{\mathscr{Q}}(1)]\cdot[l]=a=0$. Hence the result follows. ☐

Hence, it follows from Corollary 13 that

Proposition 14. *If $g=1$ or C is very general with $g\geq 2$, then* $\dim_{\mathbb{R}}N^1(\mathscr{Q})=3$.

Proof. We already saw that $N^1(C^{(d)})$ is of dimension 2. The Proposition follows. ☐

To compute $N_1(\mathscr{Q})$ we first construct a section of $\Phi : \mathscr{Q} \to C^{(d)}$. Over $C \times C^{(d)}$ we have the universal divisor Σ which gives us the universal quotient $\mathscr{O}_{C\times C^{(d)}} \to \mathscr{O}_\Sigma$. Choose a surjection $E \to L$ over C, where L is a line bundle on C. This induces a surjection $E \otimes \mathscr{O}_{C\times C^{(d)}} \to L \otimes \mathscr{O}_{C\times C^{(d)}}$. Then the composition

$$E \otimes \mathscr{O}_{C\times C^{(d)}} \to L \otimes \mathscr{O}_{C\times C^{(d)}} \to L \otimes \mathscr{O}_\Sigma$$

gives us a morphism

$$\eta : C^{(d)} \to \mathscr{Q} \tag{7}$$

which is easily seen to be a section of Φ.

Corollary 15. *$N_1(\mathscr{Q}) = N_1(C^{(d)}) \oplus \mathbb{R}[l]$ where $N_1(C^{(d)}) \hookrightarrow N_1(\mathscr{Q})$ is the morphism given by the pushforward η_*.*

Proof. Since $\Phi \circ \eta = id_{C^{(d)}}$ we have that η_* is an injection. Also since $[\mathscr{O}_\mathscr{Q}(1)] \cdot [l] = 1$, we have $[l] \neq 0$. We claim that $[l] \notin N_1(C^{(d)})$. If not, assume that $[l] = \eta_*[\gamma]$ for $[\gamma] \in N^1(C^{(d)})$. Then for every $\beta \in N^1(C^{(d)})$ we have

$$[l] \cdot \Phi^*\beta = \Phi_*([l]) \cdot \beta = 0 = \gamma \cdot \beta.$$

This proves that $\gamma = 0$.

Let $\gamma \in N_1(\mathscr{Q})$. Then we claim that

$$\gamma = \eta_*\Phi_*\gamma + \big([\mathscr{O}_\mathscr{Q}(1)] \cdot (\gamma - \eta_*\Phi_*\gamma)\big)\,[l]\,.$$

This can be seen as follows. It is enough to show that $\forall\ D \in N^1(\mathscr{Q})$,

$$[D] \cdot \gamma = [D] \cdot \big(\eta_*\Phi_*\gamma\big) + \big([\mathscr{O}_\mathscr{Q}(1)] \cdot \gamma\big)\,[D] \cdot [l]\,.$$

By Corollary 13, it is enough to consider the case when $D = \Phi^*D'$ where $D' \in N^1(C^{(d)})$ or $D = \mathscr{O}_\mathscr{Q}(1)$. In the first case the statement follows from projection formula and the second case is by definition. This completes the proof of the Corollary 15. □

Let $p_C : C \times \mathscr{Q} \to \mathscr{Q}$ and $p_\mathscr{Q} : C \times \mathscr{Q} \to C$ be the projections. Let $\mathscr{B}_\mathscr{Q}$ denote the universal quotient on $C \times \mathscr{Q}$. For a vector bundle F over C, we define

$$B_{F,\mathscr{Q}} := \det\big(p_{\mathscr{Q}*}\big(\mathscr{B}_\mathscr{Q} \otimes p_C^*F\big)\big)\,.$$

Lemma 16. *Suppose we are given a map $f : T \to \mathscr{Q}$. Let $(id \times f)^*\mathscr{B}_\mathscr{Q} = \mathscr{B}_T$. Let $p_T : C \times T \to T$ and $p_{1,T} : C \times T \to C$ be the projections.*

$$\begin{tikzcd} C \times T \arrow[r, "id\times f"] \arrow[d, "p_T"] & C \times \mathscr{Q} \arrow[d, "p_\mathscr{Q}"] \\ T \arrow[r, "f"] & \mathscr{Q} \end{tikzcd}$$

(i) *$f^*p_{\mathscr{Q}*}(\mathscr{B}_\mathscr{Q} \otimes p_C^*F) \to p_{T*}(\mathscr{B}_T \otimes p_{1,T}^*F)$ is an isomorphism.*
(ii) *For a vector bundle F on C define $B_{F,T} := \det(p_{T*}(\mathscr{B}_T \otimes p_{1,T}^*F))$. Then $f^*B_{F,\mathscr{Q}} = B_{F,T}$.*

Proof. For (i) take $\mathscr{B}_\mathscr{Q} \otimes p_C^*F$ and use Lemma 8. The assertion (ii) follows from (i) by applying determinant to the isomorphism

$$f^*p_{\mathscr{Q}*}\big(\mathscr{B}_\mathscr{Q} \otimes p_C^*F\big) \xrightarrow{\sim} p_{T*}\big(\mathscr{B}_T \otimes p_{1,T}^*F\big)\,.$$ □

Recall the definition of η from equation (7), this is a section of Φ. For a line bundle L on C we have a line bundle $\mathscr{G}_{d,L}$ over $C^{(d)}$ (see [15, page 8] for notation).

Lemma 17. *Let η be defined by a quotient $E \to M \to 0$. Then*

$$\eta^*B_{L,\mathscr{Q}} \cong \mathscr{G}_{d,L\otimes M}\,.$$

Proof. We have the diagram:

$$\begin{array}{ccc} C\times C^{(d)} & \xrightarrow{id_C\times\eta} & C\times\mathscr{Q} \\ \downarrow & & \downarrow \\ C^{(d)} & \xrightarrow{\eta} & \mathscr{Q} \end{array}$$

Recall that by definition of η, the pullback of the universal quotient on $C\times\mathscr{Q}$ to $C\times C^{(d)}$ via the section $(id_C\times\eta)$ is the quotient

$$E\otimes\mathscr{O}_{C\times C^{(d)}}\to L\otimes\mathscr{O}_{C\times C^{(d)}}\to L\otimes\mathscr{O}_\Sigma$$

Hence by Lemma 16, we have

$$\eta^* B_{L,\mathscr{Q}}\cong\mathscr{G}_{d,L\otimes M}.$$ □

Proposition 18. *For any two line bundles L, L' over C*

$$B_{L,\mathscr{Q}}\otimes B^{-1}_{L',\mathscr{Q}}=\Phi^*\left(\left(L\otimes L'^{-1}\right)^{\boxtimes d}\right).$$

Proof. First we show that $B_{L,\mathscr{Q}}\otimes B^{-1}_{L',\mathscr{Q}}\in\Phi^*\operatorname{Pic}(C^{(d)})$. Since any line bundle over $\mathscr{Q}$ is of the form $\mathscr{O}_{\mathscr{Q}}(a)\otimes\phi^*\mathscr{L}$, where $\mathscr{L}\in\operatorname{Pic}(C^{(d)})$, it is enough to show that both $B_{L,\mathscr{Q}}$ and $B_{L',\mathscr{Q}}$ have the same $\mathscr{O}_{\mathscr{Q}}(1)^{\text{th}}$ coeffcient.

To compute the coefficient of this component of any line bundle over $\mathscr{Q}$, we can do the following. Fix d distinct points $c_1,\dots,c_d\in C$. These define a point $\underline{c}\in C^{(d)}$. As we saw in the proof of Theorem 11,

$$\Phi^{-1}\left(\underline{c}\right)\cong\prod_{i=1}^d\mathbb{P}\left(E_{c_i}\right).$$

Let $v_i\in\mathbb{P}(E_{c_i})$ for $i\geq 2$. Then we have an embedding:

$$f:\mathbb{P}(E_{c_1})\times v_2\times\ldots\times v_d\hookrightarrow\mathbb{P}\left(E_{c_1}\right)\times\prod_{i\geq 2}\mathbb{P}(E_{c_i})=\Phi^{-1}(\underline{c}).$$

Then the $\mathscr{O}_{\mathscr{Q}}(1)^{\text{th}}$ coefficient of a line bundle $\mathscr{M}$ over $\mathscr{Q}$ is the degree of $f^*\mathscr{M}$ with respect to $\mathscr{O}_{\mathbb{P}(E_{c_1})}(1)$. Let $Y=\mathbb{P}(E_{c_1})$. Using Lemma 16, $f^*B_{L,\mathscr{Q}}=\det(p_{Y*}(\mathscr{B}_Y\otimes p^*_{1,Y}L))$.

The $v_j\in\mathbb{P}(E_{c_j})$ correspond to quotients $v_j:E\to E_{c_j}\to k_{c_j}$, for $2\leq j\leq d$. Over $C\times Y$ we have the inclusions $i_j:Y\cong c_j\times Y\hookrightarrow C\times Y$ for every $1\leq j\leq d$. We have a map

$$p^*_{1,Y}E\to\bigoplus_{j=1}^d i_{j*}\left(p^*_{1,Y}E|_{c_j\times Y}\right).$$

The bundle $p^*_{1,Y}E|_{c_j\times Y}$ is just the trivial bundle on Y, and using v_j we can get quotients $p^*_{1,Y}E|_{c_j\times Y}\to\mathscr{O}_Y$ for $2\leq j\leq d$. For $j=1$ we have the quotient $p^*_{1,Y}E|_{c_1\times Y}\to i_{1*}(\mathscr{O}_Y(1))$. Since the $c_j\times Y$ are disjoint we can put these together to get a quotient on $C\times Y$

$$p^*_{1,Y}E\to\left(\bigoplus_{j=2}^d i_{j*}\mathscr{O}_Y\right)\bigoplus i_{1*}\mathscr{O}_Y(1).$$

By definition, the sheaf $\mathscr{B}_Y$ is the sheaf in the RHS. Then

$$\begin{aligned}\mathscr{B}_Y\otimes p^*_{1,Y}L&=\left(\bigoplus_{j=2}^d i_{j*}\mathscr{O}_Y\right)\otimes p^*_{1,Y}L\bigoplus i_{1*}\mathscr{O}_Y(1)\otimes p^*_{1,Y}L\\&=\left(\bigoplus_{j=2}^d i_{j*}\mathscr{O}_Y\right)\bigoplus i_{1*}\mathscr{O}_Y(1)\\&=\mathscr{B}_Y.\end{aligned}$$

Thus, using the remark in the preceding para, we get that the $\mathscr{O}_{\mathscr{Q}}(1)^{\text{th}}$ coefficient of $B_{L,\mathscr{Q}}$ is the same as that of $B_{L',\mathscr{Q}}$. Hence $B_{L,\mathscr{Q}}\otimes B^{-1}_{L',\mathscr{Q}}=\Phi^*\mathscr{L}$.

Recall the section η of Φ from equation (7), constructed using some line bundle quotient $E \to M$. Then $\eta^*(B_{L,\mathcal{Q}} \otimes B^{-1}_{L',\mathcal{Q}}) = s^*\Phi^*\mathcal{L} = \mathcal{L}$. Now using Lemma 17, we get that $\eta^* B_{L,\mathcal{Q}} = \mathcal{G}_{d,L\otimes M}$.

By Göttsche's theorem ([15, page 9]) we get that $\eta^* B_{L,\mathcal{Q}} = \mathcal{G}_{d,L\otimes M} = (L\otimes M)^{\boxtimes d} \otimes \mathcal{O}(-\Delta_d/2)$. Therefore, we get

$$\mathcal{L} = \eta^*\left(B_{L,\mathcal{Q}} \otimes B^{-1}_{L',\mathcal{Q}}\right) = \left(L\otimes L'^{-1}\right)^{\boxtimes d}.$$

This completes the proof of the Proposition 18. □

Corollary 19. $[B_{L,\mathcal{Q}}] = [\mathcal{O}_{\mathcal{Q}}(1)] + \deg(L)[x]$ *in* $N^1(\mathcal{Q})$.

4. Upper bound on NEF cone

Let V be a vector space of dimension n. From now, unless mentioned otherwise, the notation $\mathcal{Q}$ will be reserved for the space $\mathcal{Q}(V\otimes\mathcal{O}_C, d)$. Sometimes we will also denote this space by $\mathcal{Q}(n,d)$ when we want to emphasize n and d.

Notation

For the rest of this article, except in section 6, the genus of the curve C will be $g(C)\geq 1$. If $g(C)\geq 2$ then we will also assume that C is very general.

Our aim is to compute the NEF cone of $\mathcal{Q}$. Since this cone is dual to the cone of effective curves, it follows that if we take effective curves $C_1, C_2, \ldots, C_r$, take the cone generated by these in $N_1(\mathcal{Q})$, and take the dual cone T in $N^1(\mathcal{Q})$, then Nef($\mathcal{Q}$) is contained in T. This gives us an upper bound on Nef($\mathcal{Q}$). We already know two curves in $\mathcal{Q}$. The first being a line in the fiber of $\Phi: \mathcal{Q} \to C^{(d)}$, see Definition 12, which was denoted $[l]$. Recall the section η of Φ from equation (7), taking L to be the trivial bundle. The second curve is $\eta_*([l'])$, where $[l']$ is from Definition 2. Now we will construct a third curve in $\mathcal{Q}$.

Define a morphism

$$\widetilde{\delta}: C \to \mathcal{Q} \tag{8}$$

as follows. Let $p_1, p_2 : C\times C \to C$ be the first and second projections respectively. Let $i: C \to C\times C$ be the diagonal. Fix a surjection $k^n \to k^d$ of vector spaces. Then define the quotient over $C\times C$

$$\mathcal{O}^n_{C\times C} \to \mathcal{O}^d_{C\times C} \to i_* i^* \mathcal{O}^d_{C\times C}.$$

This induces a morphism $\widetilde{\delta}: C \to \mathcal{Q}$ which sends $c \mapsto [\mathcal{O}^n_C \to k^d_c \to 0]$. We will abuse notation and denote the class $[\widetilde{\delta}_*(C)] \in N_1(\mathcal{Q})$ by $[\widetilde{\delta}]$.

We now give an upper bound for the NEF cone when $n \geq d \geq \mathrm{gon}(C)$.

Proposition 20. *Consider the Quot scheme $\mathcal{Q} = \mathcal{Q}(n,d)$. Assume $n\geq d\geq \mathrm{gon}(C)$. Let $\mu_0 := \frac{d+g-1}{dg}$. Then*

$$\mathrm{Nef}(\mathcal{Q}) \subset \mathbb{R}_{\geq 0}\left([\mathcal{O}_{\mathcal{Q}}(1)] + \mu_0[L_0]\right) + \mathbb{R}_{\geq 0}[\theta_d] + \mathbb{R}_{\geq 0}[L_0].$$

Proof. We claim that the cone dual to $\langle [l], \eta_*([l']), [\widetilde{\delta}]\rangle$ is precisely

$$\left\langle \left([\mathcal{O}_{\mathcal{Q}}(1)] + \mu_0[L_0]\right), [L_0], [\theta_d]\right\rangle.$$

We have the following equalities:

(1) $([\mathcal{O}_{\mathcal{Q}}(1)] + \mu_0[L_0])\cdot [l] = 1$. This is clear.

(2) $([\mathcal{O}_{\mathcal{Q}}(1)] + \mu_0[L_0])\cdot \eta_*[l'] = 0$. By projection formula and Lemma 17, we get that

$$\left([\mathcal{O}_{\mathcal{Q}}(1)] + \mu_0[L_0]\right)\cdot\left[\eta_* l'\right] = \left([-\Delta_d/2] + \mu_0[L_0]\right)\cdot[l'].$$

By Lemma 7 we get that $[-\Delta_d/2] + \mu_0[L_0] = (1-\mu_0)[\theta_d]$. But as we saw earlier, $[\theta_d]\cdot[l'] = 0$.

(3) $([\mathcal{O}_{\mathcal{Q}}(1)]+\mu_0[L_0])\cdot[\widetilde{\delta}]=0$. By Lemma 8, it is easy to see that $[\mathcal{O}_{\mathcal{Q}}(1)]\cdot[\widetilde{\delta}]=0$. By projection formula, we get

$$\left([\mathcal{O}_{\mathcal{Q}}(1)]+\mu_0[L_0]\right)\cdot\left[\widetilde{\delta}\right]=\left[\mu_0 L_0\right]\cdot\left[\Phi_*\widetilde{\delta}\right]=\left[\mu_0 L_0\right]\cdot[\delta]=0\,.$$

(4) $[\theta_d]\cdot[l]=[L_0]\cdot[l]=0$ follows using the projection formula.

Now the claim follows from Proposition 5. As explained before, since $\mathrm{Nef}(\mathcal{Q})$ is contained in the dual to the cone $\langle[l],\eta_*([l']),[\widetilde{\delta}]\rangle$, the proposition follows. □

When the genus $g=1$, we have the following improvement of Proposition 20.

Proposition 21. *Let C be a smooth projective curve of genus $g=1$. Consider the Quot scheme $\mathcal{Q}=\mathcal{Q}(n,d)$. Assume $d\geq \mathrm{gon}(C)=2$. Then*

$$\mathrm{Nef}(\mathcal{Q})\subset\mathbb{R}_{\geq 0}([\mathcal{O}_{\mathcal{Q}}(1)]+[L_0])+\mathbb{R}_{\geq 0}[\theta_d]+\mathbb{R}_{\geq 0}[L_0]\,.$$

Proof. We claim that the cone dual to $\langle[l],\eta_*([l']),\eta_*[\delta]\rangle$ is precisely

$$\left\langle([\mathcal{O}_{\mathcal{Q}}(1)]+[L_0]),[L_0],[\theta_d]\right\rangle.$$

Let us check that $[([\mathcal{O}_{\mathcal{Q}}(1)]+[L_0])]\cdot\eta_*[\delta]=0$. Since $[L_0]\cdot[\delta]=0$ it is clear that it suffices to check that $[\mathcal{O}_{\mathcal{Q}}(1)]\cdot\eta_*[\delta]=0$. Applying the definition of the map $\eta\circ\delta:C\to\mathcal{Q}$ we see that $[\mathcal{O}_{\mathcal{Q}}(1)]\cdot\eta_*[\delta]=\deg(p_{2*}(\mathcal{O}/\mathcal{I}^d))$, where $\mathcal{I}$ is the ideal sheaf of the diagonal in $E\times E$. Since $\mathcal{I}/\mathcal{I}^2$ is trivial and $\mathcal{I}^j/\mathcal{I}^{j+1}=(\mathcal{I}/\mathcal{I}^2)^{\otimes j}$, it follows that $\deg(p_{2*}(\mathcal{O}/\mathcal{I}^d))=0$. The rest of the proof is the same as that of Proposition 20. □

5. Lower bound on NEF cone

In this section we obtain a lower bound for $\mathrm{Nef}(\mathcal{Q})$ ($\mathcal{Q}=\mathcal{Q}(n,d)$).

Lemma 22. *Let $f:D\to\mathcal{Q}$ be a morphism, where D is a smooth projective curve. Fix a point $q\in f(D)$ and an effective divisor A on C containing the scheme theoretic support of $\mathcal{B}_q$. If there is a line bundle L on C such that $H^0(L)\to H^0(L|_A)$ is surjective then $[B_{L,\mathcal{Q}}]\cdot[D]\geq 0$.*

Proof. Consider the map

$$p_{\mathcal{Q}*}\left(p_C^*(V\otimes\mathcal{O}_C)\otimes p_C^*L\right)\to p_{\mathcal{Q}*}\left(\mathcal{B}_{\mathcal{Q}}\otimes p_C^*L\right)$$

on $\mathcal{Q}$. We claim that this map is surjective at the point q. In view of Lemma 8 when we restrict this map to q, it becomes equal to the map

$$H^0(V\otimes L)\to H^0\left(\mathcal{B}_q\otimes L\right).$$

The map $V\otimes L\to\mathcal{B}_q\otimes L$ on C factors as

$$V\otimes L\to V\otimes L|_A\to\mathcal{B}_q\otimes L.$$

Taking global sections we see that the map $H^0(V\otimes L)\to H^0(\mathcal{B}_q\otimes L)$ factors as

$$H^0(V\otimes L)\to H^0(V\otimes L|_A)\to H^0\left(\mathcal{B}_q\otimes L\right).$$

The second arrow is surjective since these are coherent sheaves on a zero dimensional scheme. The first arrow is simply

$$V\otimes H^0(L)\to V\otimes H^0(L|_A).$$

Since $H^0(L)\to H^0(L|_A)$ is surjective by our choice of L, it follows that $H^0(V\otimes L)\to H^0(\mathcal{B}_q\otimes L)$ is surjective, and so it follows that $p_{\mathcal{Q}*}(V\otimes p_C^*L)\to p_{\mathcal{Q}*}(\mathcal{B}_{\mathcal{Q}}\otimes p_C^*L)$ is surjective at the point q.

The rank of the vector bundle $p_{\mathcal{Q}*}(\mathcal{B}_{\mathcal{Q}}\otimes p_C^*L)$ on $\mathcal{Q}$ is d. Taking the dth exterior of $p_{\mathcal{Q}*}(V\otimes p_C^*L)\to p_{\mathcal{Q}*}(\mathcal{B}_{\mathcal{Q}}\otimes p_C^*L)$ we get a map

$$\bigwedge^d\left(V\otimes H^0(L)\right)\to B_{L,\mathcal{Q}}\,.$$

This map is nonzero and that can be seen by looking at the restriction to the point q. This shows that there is a global section of $B_{L,\mathscr{Q}}$ whose restriction to q does not vanish. It follows that $[B_{L,\mathscr{Q}}]\cdot[D]\geq 0$.This completes the proof of the Lemma 22. □

Lemma 23. *Let A be an effective divisor on C of degree d. Then there is a line bundle L of degree $d+g-1$ such that the natural map*

$$H^0(L)\to H^0(L|_A)$$

is surjective.

Proof. It suffices to find a line bundle of degree $d+g-1$ such that $H^1(L\otimes\mathscr{O}_C(-A))=0$. By Serre duality this is same as saying that $H^0(L^\vee\otimes K_C\otimes\mathscr{O}_C(A))=0$. The degree of $L^\vee\otimes K_C\otimes\mathscr{O}_C(A)$ is $g-1$. Thus, fixing A we may choose a general L such that $L^\vee\otimes K_C\otimes\mathscr{O}_C(A)$ line bundle has no global sections. □

Definition 24. *Define $U\subset\mathscr{Q}$ to be the set of quotients of the form*

$$\mathscr{O}_C^n\to\frac{\mathscr{O}_C}{\prod_{i=1}^{r}\mathfrak{m}_{C,c_i}^{d_i}}\cong\bigoplus\frac{\mathscr{O}_{C,c_i}}{\mathfrak{m}_{C,c_i}^{d_i}}\qquad c_i\neq c_j\,.$$

We now prove a lemma, which is implicitly contained [8, Section 5]. Let $\Sigma\subset C\times C^{(d)}$ denote the closed sub-scheme which is the universal divisor. In the following Lemma we work more generally with $\mathscr{Q}(E,d)$.

Lemma 25. *Let E be a locally free sheaf of rank r on C. Let $\mathscr{Q}=\mathscr{Q}(E,d)$ denote the Quot scheme of torsion quotients of length d. The universal quotient $\mathscr{B}_{\mathscr{Q}}$ is supported on $\Phi^*\Sigma\subset C\times\mathscr{Q}$. The set U is open in $\mathscr{Q}$. On $C\times U$ the sheaf $\mathscr{B}_{\mathscr{Q}}$ is a line bundle supported on the scheme $\Phi^*\Sigma\cap(C\times U)$.*

Proof. Let A denote the kernel of the universal quotient on $C\times\mathscr{Q}$

$$0\to A\xrightarrow{h}p_C^*E\to\mathscr{B}_{\mathscr{Q}}\to 0\,.$$

The map Φ is defined taking the determinant of h, that is, using the quotient

$$0\to\det(A)\xrightarrow{\det(h)}p_C^*\det(E)\to\mathscr{F}\to 0\,.$$

If $\mathscr{I}_\Sigma$ denotes the ideal sheaf of Σ then this shows that

$$\Phi^*\mathscr{I}_\Sigma=\det(A)\otimes p_C^*\det(E)^{-1}\,.$$

Let $0\to E'\xrightarrow{h}E$ be locally free sheaves of the same rank on a scheme Y. Let $\mathscr{I}$ denote the ideal sheaf determined by $\det(h)$. Then it is easy to see that $\mathscr{I}E\subset h(E')\subset E$. Applying this we get that $(\Phi^*\mathscr{I}_\Sigma)p_C^*E\subset A$. This proves that $\mathscr{B}$ is supported on $\Phi^*\Sigma$. Let us denote by $Z:=\Phi^*\Sigma\subset C\times\mathscr{Q}$. Consider the closed subset $Z_2\subset Z$ defined as follows

$$Z_2:=\left\{z=(c,q)\in Z\,\middle|\,\mathrm{rank}_k(\mathscr{B}_{\mathscr{Q}}\otimes k(z))\geq 2\right\}.$$

Then the image of Z_2 in $\mathscr{Q}$ is closed and U is precisely the complement of Z_2. This proves that U is open in $\mathscr{Q}$.

Let R be a local ring with maximal ideal $\mathfrak{m}$ and let $R\to S$ be a finite map. Let M be a finite S module, which is flat over R and such that $M/\mathfrak{m}M\cong S/\mathfrak{m}S$. Then it follows easily that $M\cong S$.

Let $q\in U\subset\mathscr{Q}$ be a point. The sheaf $\mathscr{B}_{\mathscr{Q}}$ is a coherent sheaf supported on Z, the map $Z\to\mathscr{Q}$ is finite, the fiber

$$\mathscr{B}_q=\bigoplus\frac{\mathscr{O}_{C,c_i}}{\mathfrak{m}_{C,c_i}^{d_i}}\cong\mathscr{O}_\Sigma|_q\cong\mathscr{O}_Z|_q\,.$$

From the preceding remark it follows that $\mathscr{B}_{\mathscr{Q}}$ is a line bundle over $Z\cap(C\times U)$. □

Lemma 26. *Consider the Quot scheme $\mathcal{Q} = \mathcal{Q}(n,d)$. Let D be a smooth projective curve and let $D \to \mathcal{Q}$ be a morphism such that its image intersects U. Then* $([\mathcal{O}_{\mathcal{Q}}(1)] + [\Delta_d/2]) \cdot [D] \geq 0$.

Proof. Denote by $\mathcal{B}_D$ the pullback of the universal quotient over $C \times \mathcal{Q}$ to $C \times D$. Denote by $i_D : \Gamma \hookrightarrow C \times D$ the pullback of the universal subscheme $\Sigma \hookrightarrow C \times C^{(d)}$ to $C \times D$. Then $\mathcal{B}_D$ is supported on Γ.

Let Γ_i be the irreducible components of Γ. Since $\Gamma \to D$ is flat each Γ_i dominates D. Let $f : \Gamma \to D$ denote the projection. There is an open subset $U_1 \subset D$ such that

$$f^{-1}(U_1) = \bigsqcup_i \Gamma_i \cap f^{-1}(U_1)$$

and $\mathcal{B}_D$ restricted to $f^{-1}(U_1)$ is a line bundle. Note that by $\Gamma_i \cap f^{-1}(U_1)$ we mean this open subscheme of Γ. Fix a closed point $x_i \in \Gamma_i \cap f^{-1}(U_1)$. Consider the quotient

$$V \otimes \mathcal{O}_{C\times D} \to \mathcal{B}_D$$

and restrict it to the point x_i. We get a quotient

$$V \to \mathcal{B}_D \otimes k(x_i) \to 0.$$

If we pick a general line in V, then it surjects onto $\mathcal{B}_D \otimes k(x_i)$. Thus, for the general element $s \in V$, $s \otimes \mathcal{O}_{C\times D}$ surjects onto $\mathcal{B}_D \otimes k(x_i)$. This map factors through $\mathcal{O}_\Gamma$, and we get an exact sequence

$$0 \to \mathcal{O}_\Gamma \to \mathcal{B}_D \to F \to 0$$

where F is supported on a 0 dimensional scheme. Then we have

$$0 \to f_*\mathcal{O}_\Gamma \to f_*\mathcal{B}_D \to f_*F \to 0.$$

Since f_*F is again supported on finitely many points, hence we have

$$\deg\left(f_*\mathcal{B}_D\right) - \deg\left(f_*\mathcal{O}_\Gamma\right) \geq 0$$

By Lemma 8, $\deg(f_*\mathcal{B}_D) = [\mathcal{O}_{\mathcal{Q}}(1)] \cdot [D]$ and by [15, § 3] we have

$$\deg\left(f_*\mathcal{O}_\Gamma\right) = [\mathcal{O}(-\Delta_d/2)] \cdot [D].$$

Hence the result follows. □

Corollary 27. *If the image of $f : D \to \mathcal{Q}$ intersects U, then* $([\mathcal{O}_{\mathcal{Q}}(1)] + \mu_0[L_0]) \cdot [D] \geq 0$.

Proof. If its image intersects U, then by Lemma 26,

$$([\mathcal{O}_{\mathcal{Q}}(1)] + [\Delta_d/2]) \cdot [D] \geq 0.$$

By Lemma 7,

$$[\Delta_d/2] = \mu_0[L_0] - \left(1-\mu_0\right)[\theta_d].$$

Since θ_d is nef, we have that

$$\left([\mathcal{O}_{\mathcal{Q}}(1)] + \mu_0[L_0]\right) \cdot [D] \geq 0.$$ □

Lemma 28. *Consider the Quot scheme $\mathcal{Q} = \mathcal{Q}(n,d)$. Let D be a smooth projective curve and let $f : D \to (\mathcal{Q} \setminus U) \subset \mathcal{Q}$ be a morphism. Then* $([\mathcal{O}_{\mathcal{Q}}(1)] + (d+g-2)[x]) \cdot [D] \geq 0$.

Proof. Fix a point $q \in f(D)$. Let A be the scheme theoretic support of the quotient $\mathcal{B}_q$ on C. Let $\deg(A) = d'$. Since $q \notin \mathcal{U}$, we have $d' < d$. By Lemma 23 we have a line bundle L of degree $d' + g - 1$ such that $H^0(L) \to H^0(L|_A)$ is surjective. By Lemma 22 and Corollary 19 we get that $[B_{L,\mathcal{Q}}] \cdot [D] = ([\mathcal{O}_{\mathcal{Q}}(1)] + (d' + g - 1)[x]) \cdot [D] \geq 0$. Since $[x]$ is nef on $\mathcal{Q}$ and $d' \leq d-1$ we get that $([\mathcal{O}_{\mathcal{Q}}(1)] + (d+g-2)[x]) \cdot [D] \geq 0$. □

Proposition 29. *Consider the Quot scheme $\mathcal{Q} = \mathcal{Q}(n,d)$. Let $\mu_0 = \frac{d+g-1}{dg}$. Then the class $\kappa_1 := [\mathcal{O}_{\mathcal{Q}}(1)] + \mu_0[L_0] + \frac{d+g-2}{dg}[\theta_d]$ is nef.*

Proof. Let $D \to \mathcal{Q}$ is a morphism, where D is a smooth projective curve. If the image of this morphism intersects U then by Lemma 26 we have $([\mathcal{O}_{\mathcal{Q}}(1)]+[\Delta_d/2])\cdot[D]\geq 0$. By Lemma 7 we have $[\Delta_d/2]=\mu_0[L_0]-(1-\mu_0)[\theta_d]$. Hence we get

$$\big([\mathcal{O}_{\mathcal{Q}}(1)]+\mu_0[L_0]\big)\cdot[D]\geq \big(1-\mu_0\big)[\theta_d]\cdot[D]\geq 0\,.$$

Since $[\theta_d]$ is nef, we get

$$\big([\mathcal{O}_{\mathcal{Q}}(1)]+\mu_0[L_0]\big)\cdot[D]+\frac{d+g-2}{dg}[\theta_d]\cdot[D]\geq 0\,.$$

Now assume $D\to\mathcal{Q}$ does not intersect U. Then by Lemma 28 we get

$$\big([\mathcal{O}_{\mathcal{Q}}(1)]+(d+g-2)[x]\big)\cdot[D]\geq 0\,.$$

By Lemma 7 we have $[x]=\frac{1}{dg}[L_0]+\frac{1}{dg}[\theta_d]$. Therefore

$$\begin{aligned}(d+g-2)[x]=&\frac{d+g-2}{dg}[L_0]+\frac{d+g-2}{dg}[\theta_d]\\ =&\mu_0[L_0]-\frac{1}{dg}[L_0]+\frac{d+g-2}{dg}[\theta_d]\,.\end{aligned}$$

Since L_0 is nef we get that

$$\left([\mathcal{O}_{\mathcal{Q}}(1)]+\mu_0[L_0]+\frac{d+g-2}{dg}[\theta_d]\right)\cdot[D]\geq 0\,.$$ □

Lemma 30. *Let L be a line bundle on C of degree $d+g-1$. If $d\geq \operatorname{gon}(C)$ then the line bundle $B_{L,\mathcal{Q}}$ is not ample. Moreover, for any $t\in[0,1]$ the class $t[B_{L,\mathcal{Q}}]+(1-t)[\theta_d]$ is nef but not ample.*

Proof. We saw in the last para of the proof of Proposition 18 that $\eta^*B_{L,\mathcal{Q}}=L^{\boxtimes d}\otimes\mathcal{O}(-\Delta_d/2)$. Its class in the nef cone is $(d+g-1)[x]-[\Delta_d/2]$. It follows from Lemma 7 that this is equal to $[\theta_d]$. Since $d\geq\operatorname{gon}(C)$ we have θ_d is not ample on $C^{(d)}$. That $t[B_{L,\mathcal{Q}}]+(1-t)[\theta_d]$ is nef is clear since both $[B_{L,\mathcal{Q}}]$ and $[\theta_d]$ are nef. This is not ample since η^* of this class is $[\theta_d]$ on $C^{(d)}$, which is not ample. □

Proposition 31. *Consider the Quot scheme $\mathcal{Q}=\mathcal{Q}(n,d)$. Then the class $[\mathcal{O}_{\mathcal{Q}}(1)]+(d+g-1)[x]\in N^1(\mathcal{Q})$ is nef.*

Proof. It is easily checked that the class $[\mathcal{O}_{\mathcal{Q}}(1)]+(d+g-1)[x]$ can be written as a positive linear combination of $[\theta_d]$ and the class in Proposition 29. □

We may slightly improve Proposition 31 in a special case using the results in [15]. For this we first recall the main results in [15, § 4]. Let C be a very general curve of genus $g(C)=2k$. Since the gonality is given by $\lfloor\frac{g+3}{2}\rfloor$, in this case it is $k+1$. Let L_i' denote the finitely many g^1_{k+1}'s on C and define $L_i=K_C-L_i'$. Then $\deg(L_i)=3(k-1)$. It is proved in [15, Proposition 3.6, Theorem 4.1] that $\mathcal{G}_{k,L_i}$ is nef but not ample.

Proposition 32. *Let C be a very general curve of genus $g(C)=2k$. Consider the Quot scheme $\mathcal{Q}=\mathcal{Q}(n,k)$. The line bundle $B_{L,\mathcal{Q}}$ is nef when $\deg(L)\geq 3(k-1)$. When $\deg(L)=3(k-1)$ the class $t[B_{L,\mathcal{Q}}]+(1-t)[\mathcal{G}_{k,L}]$ is nef but not ample for any $t\in[0,1]$.*

We remark that this is an improvement since Proposition 31 only shows that $B_{L,\mathcal{Q}}$ is nef when $\deg(L)\geq 3k-1$.

Proof. It follows from Proposition 18 that the class of $B_{L,\mathcal{Q}}$ in $N^1(\mathcal{Q})$ is $[\mathcal{O}_{\mathcal{Q}}(1)]+\deg(L)[x]$, since $B_{\mathcal{O}_C,\mathcal{Q}}=\mathcal{O}_{\mathcal{Q}}(1)$. Notice that this class only depends on the degree of L. Since the sum of nef line bundles is nef, it suffices to show that $[B_{L,\mathcal{Q}}]=[\mathcal{O}_{\mathcal{Q}}(1)]+\deg(L)[x]$ is nef when $\deg(L)=3(k-1)$.

The set $V(\sigma_{L_i})$ is defined in equation [15, equation (18)]. Then (A) in [15, Theorem 4.1] says that for every $A\in C^{(k)}$ there is an L_i such that $H^0(C,L_i)\to H^0(C,L_i|_A)$ is surjective.

Let $f : D \to \mathcal{Q}$ be morphism, where D is a smooth projective curve. Fix a point $q \in f(D)$. Let A be the divisor corresponding to $\Phi(q)$, then A is an effective divisor of degree k. For this A, choose a line bundle L_i such that

$$H^0(C, L_i) \to H^0(C, L_i|_A)$$

is surjective. The scheme theoretic support of $\mathscr{B}_q$ is contained in A. It follows from Lemma 22 that

$$f^* B_{L_i,\mathcal{Q}} = f^*([\mathscr{O}_{\mathcal{Q}}(1)] + 3(k-1)[x]) \geq 0.$$

It follows that $B_{L,\mathcal{Q}}$ is nef.

Note that

$$\begin{aligned}\eta^* B_{L,\mathcal{Q}} &= \eta^*[\mathscr{O}_{\mathcal{Q}}(1)] + \deg(L)\eta^*[x] \\ &= [\mathscr{O}(-\Delta_k/2)] + 3(k-1)[x] \\ &= \left[\mathscr{G}_{k,L}\right].\end{aligned}$$

Thus, when $t \in [0,1]$ the pullback along η of $t[B_{L,\mathcal{Q}}] + (1-t)[\mathscr{G}_{k,L}]$ is $[\mathscr{G}_{k,L}]$, which is not ample. □

6. The genus 0 case

Throughout this section we will work with $C = \mathbb{P}^1$. Let us first compute the nef cone of $\mathcal{Q}(n,d)$.

Note that we have $C^{(d)} \cong \mathbb{P}^d$. Hence $N^1(C^{(d)}) = \mathbb{R}[\mathscr{O}_{\mathbb{P}^d}(1)]$. By Corollary 13 it follows that $N^1(\mathcal{Q})$ is two dimensional. Hence, it suffices to find a line bundle on $\mathcal{Q}$ which is different from the pullback of $\mathscr{O}_{\mathbb{P}^d}(1)$ and which is nef but not ample. The following result is proved in [16, Theorem 6.2], but we include it for the benefit of the reader.

Proposition 33.

$$\begin{aligned}\mathrm{Nef}(\mathcal{Q}(n,d)) =&\mathbb{R}_{\geq 0}\left[B_{\mathscr{O}(d-1),\mathcal{Q}}\right] + \mathbb{R}_{\geq 0}\left[\mathscr{O}_{\mathbb{P}^d}(1)\right] \\ =&\mathbb{R}_{\geq 0}\left([\mathscr{O}_{\mathcal{Q}}(1)] + (d-1)\left[\mathscr{O}_{\mathbb{P}^d}(1)\right]\right) + \mathbb{R}_{\geq 0}\left[\mathscr{O}_{\mathbb{P}^d}(1)\right].\end{aligned}$$

Proof. Let $W := H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(d))$. There is a natural isomorphism $\mathbb{P}W^* \xrightarrow{\sim} C^{(d)}$. The universal subscheme $\Sigma \subset \mathbb{P}^1 \times \mathbb{P}W^*$ is given by the tautological section

$$p_2^*\mathscr{O}_{\mathbb{P}W^*}(-1) \to p_2^* W = p_1^* W \to p_1^*\mathscr{O}_{\mathbb{P}^1}(d).$$

By Lemma 22 and Lemma 23 we get that $B_{\mathscr{O}(d-1),\mathcal{Q}}$ is nef. To show $B_{\mathscr{O}(d-1),\mathcal{Q}}$ is not ample, consider a section $\eta : C^{(d)} \to \mathcal{Q}$ constructed as in (7) with L the trivial bundle. Let p_i denote the two projections from $\mathbb{P}^1 \times \mathbb{P}W^*$. By definition and Lemma 16 it follows that $\eta^* B_{\mathscr{O}(d-1),\mathcal{Q}} = \det(p_{2*}(\mathscr{O}_\Sigma \otimes p_1^*\mathscr{O}_{\mathbb{P}^1}(d-1)))$. Tensoring the exact sequence

$$0 \to p_1^*\mathscr{O}_{\mathbb{P}^1}(-d) \otimes p_2^*\mathscr{O}_{\mathbb{P}W^*}(-1) \to \mathscr{O}_{\mathbb{P}^1 \times \mathbb{P}W^*} \to \mathscr{O}_\Sigma \to 0$$

with $p_1^*\mathscr{O}_{\mathbb{P}^1}(d-1)$ and applying p_{2*} it easily follows that $p_{2*}(\mathscr{O}_\Sigma \otimes p_1^*\mathscr{O}_{\mathbb{P}^1}(d-1))$ is the trivial bundle and so $\eta^* B_{\mathscr{O}(d-1),\mathcal{Q}}$ is trivial. This proves that $B_{\mathscr{O}(d-1),\mathcal{Q}}$ is nef but not ample.

By restricting to a fiber of Φ and using Corollary 19 we see that $[B_{\mathscr{O}(d-1),\mathcal{Q}}]$ is linearly independent from $[\mathscr{O}_{\mathbb{P}^d}(1)]$. This completes the proof of the first equality. The second equality will follow from the first equality once we show that

$$\left[B_{\mathscr{O}(d-1),\mathcal{Q}}\right] = [\mathscr{O}_{\mathcal{Q}}(1)] + (d-1)\left[\mathscr{O}_{\mathscr{P}^d}(1)\right].$$

By Corollary 19, we have that $[B_{\mathscr{O}(d-1),\mathcal{Q}}] = [\mathscr{O}_{\mathcal{Q}}(1)] + (d-1)[x]$. Now recall that given $x \in \mathbb{P}^1$, $[x]$ is the class of the divisor in $C^{(d)}$ whose underlying set consists of effective divisors of degree d containing x (see (4)). Hence, $[x]$ is the class of the hyperplane section

$$\mathbb{P}\left(H^0\left(\mathbb{P}^1, \mathscr{O}(d) \otimes \mathscr{O}(-x)\right)^*\right) \subset \mathbb{P}\left(H^0\left(\mathbb{P}^1, \mathscr{O}(d)\right)^*\right) = C^{(d)}.$$

Therefore $[x] = [\mathscr{O}_{\mathbb{P}^1}(1)]$ and this completes the proof of the second equality. □

Theorem 34. *Let $C=\mathbb{P}^1$. Let $E=\bigoplus_{i=1}^{k}\mathcal{O}(a_i)$ with $a_i \le a_j$ for $i<j$. Let $d\ge 1$. Let $L=\mathcal{O}(-a_1+d-1)$. Then*

$$\begin{aligned}\mathrm{Nef}(\mathcal{Q}(E,d)) =&\mathbb{R}_{\ge 0}\left[B_{L,\mathcal{Q}(E,d)}\right]+\mathbb{R}_{\ge 0}\left[\mathcal{O}_{\mathbb{P}^d}(1)\right]\\ =&\mathbb{R}_{\ge 0}\left(\left[\mathcal{O}_{\mathcal{Q}(E,d)}(1)\right]+(-a_1+d-1)\left[\mathcal{O}_{\mathbb{P}^d}(1)\right]\right)+\mathbb{R}_{\ge 0}\left[\mathcal{O}_{\mathbb{P}^d}(1)\right].\end{aligned}$$

Proof. By Corollary 13 we get that $N^1(\mathcal{Q}(E,d))$ is 2-dimensional. Hence it is enough to give two line bundles which are nef but not ample. Clearly $\Phi^*_{\mathcal{Q}(E,d)}\mathcal{O}_{\mathbb{P}^d}(1)$ is nef but not ample. So it is enough to show that $B_{L,\mathcal{Q}(E,d)}$ is nef but not ample.

Since $a_j-a_1\ge 0\ \forall\ j\ge 1$, we get that $E(-a_1)$ is globally generated. Let $V:=H^0(C,E(-a_1))$ and let dim $V=n$. Then we have a surjection $V\otimes\mathcal{O}_C\to E(-a_1)$. Then gives us a surjection

$$V\otimes\mathcal{O}_C\to p_C^*E(-a_1)\to\mathcal{B}_{\mathcal{Q}(E,d)}\otimes p_C^*\mathcal{O}_C(-a_1)\to 0.$$

This defines a map $f:\mathcal{Q}(E,d)\to\mathcal{Q}(n,d)$. By Lemma 16 we get that

$$f^*B_{\mathcal{O}(d-1),\mathcal{Q}(n,d)}=B_{L,\mathcal{Q}(E,d)}=\det\left(p_{\mathcal{Q}(E,d)*}\left(\mathcal{B}_{\mathcal{Q}(E,d)}\otimes p_C^*L\right)\right).$$

Since $B_{\mathcal{O}(d-1),\mathcal{Q}(n,d)}$ is nef we get that $B_{L,\mathcal{Q}(E,d)}$ is nef. We next show that the $B_{L,\mathcal{Q}(E,d)}$ is not ample. Consider the section $\eta_{\mathcal{Q}(E,d)}$ of $\Phi_{\mathcal{Q}(E,d)}:\mathcal{Q}(E,d)\to C^{(d)}$ defined by the quotient $p_C^*E\to p_C^*\mathcal{O}(a_1)\otimes\mathcal{O}_\Sigma$ on $C\times C^{(d)}$ (see (7)). Then $f\circ\eta_{\mathcal{Q}(E,d)}$ is a section of $\Phi:\mathcal{Q}(n,d)\to C^{(d)}$ defined by a quotient $\mathcal{O}_C^n\to\mathcal{O}_\Sigma\to 0$ on $C\times C^{(d)}$. Therefore $\eta^*_{\mathcal{Q}(E,d)}B_{L,\mathcal{Q}(E,d)}=\eta^*B_{\mathcal{O}(d-1),\mathcal{Q}(n,d)}$. As $\eta^*B_{\mathcal{O}(d-1),\mathcal{Q}(n,d)}$ is not ample, we get that $B_{L,\mathcal{Q}(E,d)}$ is not ample. The second equality follows again from the fact that $[x]=[\mathcal{O}_{\mathbb{P}^d}(1)]$. □

7. Some cases of equality

Now we are back to the assumption that the genus of the curve satisfies $g(C)\ge 1$ and if $g(C)\ge 2$ then we also assume that C is very general.

Definition 35. *Let $U'\subset\mathcal{Q}$ be the open set consisting of quotients $\mathcal{O}_C^n\to B\to 0$ such that the induced map $H^0(C,\mathcal{O}_C^n)\to H^0(C,B)$ is surjective.*

Lemma 36. *Consider the Quot scheme $\mathcal{Q}=\mathcal{Q}(n,d)$. Let D be a smooth projective curve and let $D\to\mathcal{Q}$ be a morphism such that its image intersects U'. Then $[\mathcal{O}_{\mathcal{Q}}(1)]\cdot[D]\ge 0$.*

Proof. We continue with the notations of Lemma 26. Let $p_D:C\times D\to D$ be the projection. Then applying $(p_D)_*$ to the quotient $\mathcal{O}^n_{C\times\mathcal{Q}}\to\mathcal{B}_D$ we get that the morphism

$$(p_D)_*\mathcal{O}^n_{C\times D}=\mathcal{O}_D^n\to(p_D)_*\mathcal{B}_D$$

is generically surjective by our assumption and Lemma 8. Hence we get that

$$[\mathcal{O}_{\mathcal{Q}}(1)]\cdot[D]=\deg\left((p_D)_*\mathcal{B}_D\right)\ge 0.$$ □

One extremal ray in Nef$(C^{(2)})$ is given by L_0. Let other extremal ray of Nef$(C^{(2)})$ be given by

$$\alpha_t=(t+1)x-\Delta_2/2, \tag{9}$$

(see [12, page 75]). Then using Lemma 7, we get that

$$\Delta_2/2=\frac{t+1}{g+t}L_0-\frac{g-1}{g+t}\alpha_t. \tag{10}$$

Theorem 37. *Let $d=2$. Consider the Quot scheme $\mathcal{Q}=\mathcal{Q}(n,2)$. Then*

$$\mathrm{Nef}(\mathcal{Q})=\mathbb{R}_{\ge 0}\left([\mathcal{O}_{\mathcal{Q}}(1)]+\frac{t+1}{g+t}[L_0]\right)+\mathbb{R}_{\ge 0}[L_0]+\mathbb{R}_{\ge 0}[\alpha_t].$$

Proof. We first prove that $[\mathscr{O}_{\mathscr{Q}}(1)] + \frac{t+1}{g+t}[L_0]$ is nef. Since $d = 2$, then there are only three types of quotients:

(1) $\mathscr{O}_C^n \to \frac{\mathscr{O}_{C,c_1}}{\mathfrak{m}_{C,c_1}} \oplus \dfrac{\mathscr{O}_{C,c_2}}{\mathfrak{m}_{C,c_2}}$ with $c_1 \neq c_2$,

(2) $\mathscr{O}_C^n \to \frac{\mathscr{O}_{C,c_1}}{\mathfrak{m}_{C,c_1}^2}$,

(3) $\mathscr{O}_C^n \to \frac{\mathscr{O}_{C,c}}{\mathfrak{m}_{C,c}} \oplus \dfrac{\mathscr{O}_{C,c}}{\mathfrak{m}_{C,c}}$.

The first two quotients are in U while the third one is in U', that is, we get $U \cup U' = \mathscr{Q}$. Now let D be a smooth projective curve and $D \to \mathscr{Q}$ be a morphism. If its image intersects U, then by Corollary 27, $([\mathscr{O}_{\mathscr{Q}}(1)] + \Delta_2/2) \cdot [D] \geq 0$. Using (10) and the fact that α_t is nef, we get that $([\mathscr{O}_{\mathscr{Q}}(1)] + \frac{t+1}{g+t}[L_0]) \cdot [D] \geq 0$. If D does not intersect U then $D \subset U'$. Hence by Lemma 36, we have

$$[\mathscr{O}_{\mathscr{Q}}(1)] \cdot [D] \geq 0.$$

Since $[L_0]$ is nef we have that

$$\left([\mathscr{O}_{\mathscr{Q}}(1)] + \frac{t+1}{g+t}[L_0]\right) \cdot [D] \geq 0.$$

Also $([\mathscr{O}_{\mathscr{Q}}(1)] + \frac{t+1}{g+t}[L_0]) \cdot [\widetilde{\delta}] = 0$. Hence any convex linear combination of $[\mathscr{O}_{\mathscr{Q}}(1)] + \frac{t+1}{g+t}[L_0]$ and $[L_0]$ is nef but not ample. By (10) $\eta^*([\mathscr{O}_{\mathscr{Q}}(1)] + \frac{t+1}{g+t}[L_0]) = \frac{g-1}{g+t}\alpha_t$. Hence any convex linear combination of $[\mathscr{O}_{\mathscr{Q}}(1)] + \frac{t+1}{g+t}[L_0]$ and $[\alpha_t]$ is not ample. Hence the result follows. □

Precise values for t depending on g are known when

(1) When $g = 1, t = 1$.
(2) When $g = 2, t = 2$.
(3) When $g = 3, t = 9/5$.
(4) When g is a perfect square $t = \sqrt{g}$, see [11, Theorem 2].
(5) In [5, Proposition 3.2], when $g \geq 9$, assuming the Nagata conjecture, they prove that $t = \sqrt{g}$.

Thus, in all these cases using Theorem 37 we get the Nef cone of $\mathscr{Q}(n, 2)$.

7.1. *Criterion for nefness*

In the remainder of this section, we will need to work with $C^{(d)}$ for different values of d. The line bundles L_0 on $C^{(d)}$ will therefore be denoted by $L_0^{(d)}$ when we want to emphasize the d. Similarly, we will denote $\mu_0^{(d)} = \frac{d+g-1}{dg}$. Let $\mathscr{P}_{(d)}^{\leq n}$ be the set of all partitions $(d_1, d_2, \ldots, d_k)$ of d of length at most n. Given an element $\mathbf{d} \in \mathscr{P}_{(d)}^{\leq n}$ define

$$C^{(\mathbf{d})} := C^{(d_1)} \times C^{(d_2)} \times \ldots \times C^{(d_k)}$$

and if $p_i : C^{(\mathbf{d})} \to C^{(d_i)}$ is the i^{th} projection we define a class

$$[\mathscr{O}(-\Delta_{\mathbf{d}}/2)] := \left[\sum p_i^* \mathscr{O}\left(-\Delta_{d_i}/2\right)\right] \in N^1\left(C^{(\mathbf{d})}\right).$$

Note that we have a natural addition

$$\pi_{\mathbf{d}} : C^{(\mathbf{d})} \to C^{(d)}.$$

For a partition $\mathbf{d} \in \mathscr{P}_d^{\leq n}$ define a morphism

$$\eta_{\mathbf{d}} : C^{(\mathbf{d})} \to \mathscr{Q}$$

as follows. For any $l \geq 1$, we define the universal subscheme of $C^{(l)}$ over $C \times C^{(l)}$ by Σ_l. Then over $C \times C^{(\mathbf{d})}$ we have the subschemes $(id \times p_i)^* \Sigma_{d_i}$. We have a quotient

$$q_{\mathbf{d}} : \mathcal{O}^n_{C \times C^{(d)}} \to \bigoplus_i \mathcal{O}_{(id \times p_{i,\mathbf{d}})^* \Sigma_{d_i}}$$

defined by taking direct sum of morphisms $\mathcal{O}_{C \times C^{(d)}} \to \mathcal{O}_{(id \times p_{i,\mathbf{d}})^* \Sigma_{d_i}}$. Then $q_{\mathbf{d}}$ defines a map $C^{(\mathbf{d})} \to \mathcal{Q}$. By Lemma 16, we have

$$\left[\eta^*_{\mathbf{d}} \mathcal{O}_{\mathcal{Q}}(1)\right] = [\mathcal{O}(-\Delta_{\mathbf{d}}/2)]\,. \tag{11}$$

Lemma 38. *Let D be a smooth projective curve. Let $D \to \mathcal{Q}$ be a morphism. Then there exists a partition $\mathbf{d} \in \mathcal{P}^{\leq n}_{(d)}$ such that the composition $D \to \mathcal{Q} \to C^{(d)}$ factors as $D \to C^{(\mathbf{d})} \to C^{(d)}$ and $[\mathcal{O}_{\mathcal{Q}}(1)] \cdot [D] \geq [\mathcal{O}(-\Delta_{\mathbf{d}}/2)] \cdot [D]$.*

Proof. We will proceed by induction on d. When $d = 1$ the statement is obvious.

Let us denote the pullback of the universal quotient on $C \times \mathcal{Q}$ to $C \times D$ by $\mathcal{B}_D$ and let $f : C \times D \to D$ be the natural projection. Consider a section such that the composite $\mathcal{O}_{C \times D} \to \mathcal{O}^n_{C \times D} \to \mathcal{B}_D$ is non-zero and let $\mathcal{F}$ denote the cokernel of the composite map. We have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_{C\times D} \arrow[r] \arrow[d] & \mathcal{O}^n_{C\times D} \arrow[r] \arrow[d] & \mathcal{O}^{n-1}_{C\times D} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{O}_{\Gamma'} \arrow[r] & \mathcal{B}_D \arrow[r] & \mathcal{F} \arrow[r] & 0 \end{tikzcd} \tag{12}$$

Let $T_0(\mathcal{F}) \subset \mathcal{F}$ denote the maximal subsheaf of dimension 0, see [10, Definition 1.1.4]. Define $\mathcal{F}' := \mathcal{F}/T_0(\mathcal{F})$. Now, either $\mathcal{F}' = 0$ or $\mathcal{F}'$ is torsion free over D, and hence, flat over D. In the first case, it follows that D meets the open set U in Lemma 26. Then we take $\mathbf{d} = (d)$ and the statement follows from Lemma 26. So we assume $\mathcal{F}'$ is flat over D and let d' be the degree of $\mathcal{F}'|_{C \times x}$, for $x \in D$. So $0 < d' < d$. By (12) we have

$$\deg\, f_* \mathcal{B}_D = \deg\, f_* \mathcal{O}_{\Gamma'} + \deg\, f_* \mathcal{F}\,.$$

Since $T_0(\mathcal{F})$ is supported on finitely many points, we have $\deg\, \mathcal{F} \geq \deg\, \mathcal{F}'$. In other words, we have

$$\deg\, f_* \mathcal{B}_D \geq \deg\, f_* \mathcal{O}_{\Gamma'} + f_* \mathcal{F}'\,. \tag{13}$$

Now Γ' defines a morphism $D \to C^{(d-d')}$ and note that

$$\deg\, f_* \mathcal{O}_{\Gamma'} = [\mathcal{O}(-\Delta_{d-d'}/2)] \cdot [D]\,.$$

The quotient $\mathcal{O}^{n-1}_{C \times D} \to \mathcal{F}' \to 0$ defines a map $D \to \mathcal{Q}(n-1, d')$. By induction hypothesis, we get that there exists a partition $\mathbf{d}' \in \mathcal{P}^{\leq n-1}_{d'}$ such that the composition $D \to \mathcal{Q}(n-1, d') \to C^{(d')}$ factors as $D \to C^{(\mathbf{d}')} \to C^{(d')}$ and

$$\left[\mathcal{O}_{\mathcal{Q}(n-1,d')}(1)\right] \cdot [D] \geq [\mathcal{O}(-\Delta_{\mathbf{d}'}/2)] \cdot [D]\,.$$

Since $\deg\, f_* \mathcal{F}' = [\mathcal{O}_{\mathcal{Q}(n-1,d')}(1)] \cdot [D]$ we have that $\deg\, f_* \mathcal{F}' \geq [\mathcal{O}(-\Delta_{\mathbf{d}'}/2)] \cdot [D]$. From (13) we get that

$$[\mathcal{O}_{\mathcal{Q}}(1)] \cdot [D] \geq [\mathcal{O}(-\Delta_{d-d'}/2)] \cdot D + [\mathcal{O}(-\Delta_{\mathbf{d}'}/2)] \cdot [D]\,.$$

Now we define $\mathbf{d} := (d - d', \mathbf{d}')$ and the statement follows from the above inequality. □

Theorem 39. *Let $\beta \in N^1(C^{(d)})$. Then the class $[\mathcal{O}_{\mathcal{Q}}(1)] + \beta \in N^1(\mathcal{Q})$ is nef iff the class $[\mathcal{O}(-\Delta_{\mathbf{d}}/2)] + \pi^*_{\mathbf{d}}\beta \in N^1(C^{(\mathbf{d})})$ is nef for all $\mathbf{d} \in \mathcal{P}^{\leq n}_{d}$.*

Proof. From (11) it is clear that if $[\mathscr{O}_{\mathscr{Q}}(1)]+\beta$ is nef, then $\eta_{\mathbf{d}}^*([\mathscr{O}_{\mathscr{Q}}(1)]+\beta)=[\mathscr{O}(-\Delta_{\mathbf{d}}/2)]+\pi_{\mathbf{d}}^*\beta$ is nef.

For the converse, we assume $[\mathscr{O}(-\Delta_{\mathbf{d}}/2)]+\pi_{\mathbf{d}}^*\beta$ is nef for all $\mathbf{d}\in\mathscr{P}_d^{\leq n}$. Let D be a smooth projective curve and $D\to\mathscr{Q}$ be a morphism. By Lemma 38 we have that there exists $\mathbf{d}\in\mathscr{P}_d^{\leq n}$ such that $D\to C^{(d)}$ factors as $D\to C^{(\mathbf{d})}\to C^{(d)}$ and

$$[\mathscr{O}_{\mathscr{Q}}(1)]\cdot[D]\geq[\mathscr{O}(-\Delta_{\mathbf{d}}/2)]\cdot[D].$$

Now by assumption we have that

$$[\mathscr{O}(-\Delta_{\mathbf{d}}/2)]\cdot[D]\geq-\beta\cdot[D].$$

Therefore we get

$$[\mathscr{O}_{\mathscr{Q}}(1)]\cdot[D]\geq-\beta\cdot[D].$$

Hence we get that the class $[\mathscr{O}_{\mathscr{Q}}(1)]+\beta$ is nef. □

Lemma 40. *Suppose we are given a map $D\to C^{(\mathbf{d})}\xrightarrow{\pi_{\mathbf{d}}}C^{(d)}$. Then we have*

$$\left[L_0^{(d)}\right]\cdot[D]\geq\sum_i\left[L_0^{(d_i)}\right]\cdot[D].$$

Proof. By $[L_0^{d_i}]\cdot[D]$ we mean the degree of the pullback of $[L_0^{(d_i)}]$ along $D\to C^{(\mathbf{d})}\xrightarrow{p_i}C^{(d_i)}$. The lemma follows easily from the definition of $L_0^{(d)}$ and is left to the reader. □

Proposition 41. *Let $n\geq1$, $g\geq1$ and $\mathscr{Q}=\mathscr{Q}(n,d)$. Then the class $\kappa_2:=[\mathscr{O}_{\mathscr{Q}}(1)]+\frac{g+1}{2g}[L_0^{(d)}]\in N^1(\mathscr{Q})$ is nef. As a consequence we get that*

$$\mathrm{Nef}(\mathscr{Q})\supset\mathbb{R}_{\geq0}\kappa_1+\mathbb{R}_{\geq0}\kappa_2+\mathbb{R}_{\geq0}[\theta_d]+\mathbb{R}_{\geq0}[L_0^{(d)}].$$

Proof. Recall $\mu_0^{(2)}=\frac{g+1}{2g}$. By Theorem 39 it suffices to show that for all $\mathbf{d}\in\mathscr{P}_{(d)}^{\leq n}$ we have $[\mathscr{O}(-\Delta_{\mathbf{d}}/2)]+\mu_0^{(2)}\pi_{\mathbf{d}}^*[L_0^{(d)}]$ is nef. Using Lemma 7, $[L_0^{(1)}]=0$ and Lemma 40 we get

$$\begin{aligned}&\left([\mathscr{O}(-\Delta_{\mathbf{d}}/2)]+\mu_0^{(2)}\pi_{\mathbf{d}}^*\left[L_0^{(d)}\right]\right)\cdot[D]\\&=\left(\sum_i\left(1-\mu_0^{(d_i)}\right)[\theta_{d_i}]-\mu_0^{(d_i)}\left[L_0^{d_i}\right]\right)\cdot[D]+\mu_0^{(2)}\left[L_0^{(d)}\right]\cdot[D]\geq\sum_i\left(\mu_0^{(2)}-\mu_0^{(d_i)}\right)\left[L_0^{d_i}\right]\cdot[D].\end{aligned}$$

This proves that κ_2 is nef. That κ_1 is nef is proved in Proposition 29. This completes the proof of the theorem. □

Corollary 42. *Let $n\geq d$. Then the class $[\mathscr{O}_{\mathscr{Q}}(1)]+\mu_0^{(2)}[L_0^{(d)}]\in N^1(\mathscr{Q})$ is nef but not ample.*

Proof. By Proposition 41 we have that $[\mathscr{O}_{\mathscr{Q}}(1)]+\mu_0^{(2)}[L_0^{(d)}]$ is nef. Now recall that when $n\geq d$ we have the curve $\widetilde{\delta}\hookrightarrow\mathscr{Q}$ (8). From the definition of $\widetilde{\delta}$ and Lemma 16 we have $[\mathscr{O}_{\mathscr{Q}}(1)]\cdot[\widetilde{\delta}]=0$. Also $\Phi_*\widetilde{\delta}=\delta$. Hence $[L_0^{(d)}]\cdot[\widetilde{\delta}]=[L_0^{(d)}]\cdot[\delta]=0$. From this we get $[\mathscr{O}_{\mathscr{Q}}(1)]+\mu_0^{(2)}[L_0^{(d)}]\cdot[\widetilde{\delta}]=0$ and hence $[\mathscr{O}_{\mathscr{Q}}(1)]+\mu_0^{(2)}L_0^{(d)}$ is not ample. □

As a corollary we get the following result. When $g=1$ note that $\mu_0^{(2)}=1$.

Theorem 43. *Let $g=1$, $n\geq1$ and $\mathscr{Q}=\mathscr{Q}(n,d)$. Then the class $[\mathscr{O}_{\mathscr{Q}}(1)]+[\Delta_d/2]\in N^1(\mathscr{Q})$ is nef. Moreover,*

$$\mathrm{Nef}(\mathscr{Q})=\mathbb{R}_{\geq0}([\mathscr{O}_{\mathscr{Q}}(1)]+[\Delta_d/2])+\mathbb{R}_{\geq0}[\theta_d]+\mathbb{R}_{\geq0}[\Delta_d/2].$$

8. Curves over the small diagonal

Throughout this section the genus of the curve C will be $g(C) \geq 2$ and C is a very general curve. Recall that $\Phi : \mathscr{Q} \to C^{(d)}$ is the Hilbert–Chow map.

Proposition 44. *Let $f : D \to \mathscr{Q}(n,d)$ be such that $\Phi \circ f$ factors through the small diagonal. Then $[\mathscr{O}_{\mathscr{Q}}(1)] \cdot [D] \geq 0$.*

Proof. Since $\Phi \circ f$ factors through the small diagonal, there is a map $g : D \to C$ such that if $\Gamma := \Gamma_g$ denotes the graph of g in $C \times D$, and $\mathscr{O}^n_{C\times D} \to \mathscr{B}_D$ is the quotient on $C \times D$, then $\mathscr{B}_D$ is supported on $\mathscr{O}_{C\times D}/\mathscr{I}(\Gamma)^d$. Denote $\mathscr{I} := \mathscr{I}(\Gamma)$. Then $\mathscr{B}_D/\mathscr{I}\mathscr{B}_D$ is a globally generated sheaf on D and so its determinant has degree ≥ 0. Now consider the sheaf

$$\mathscr{I}^i\mathscr{B}_D/\mathscr{I}^{i+1}\mathscr{B}_D \cong \left(\mathscr{I}/\mathscr{I}^2\right)^{\otimes i} \otimes \mathscr{B}_D/\mathscr{I}\mathscr{B}_D.$$

Using adjunction it is easily seen that $\mathscr{I}/\mathscr{I}^2 \cong g^*\omega_C$. Since $\det(\mathscr{B}_D/\mathscr{I}\mathscr{B}_D)$ has degree ≥ 0, it follows that $\det(\mathscr{I}^i\mathscr{B}_D/\mathscr{I}^{i+1}\mathscr{B}_D)$ has degree ≥ 0. From the filtration

$$\mathscr{B}_D \supset \mathscr{I}\mathscr{B}_D \supset \mathscr{I}^2\mathscr{B}_D \supset \ldots \supset \mathscr{I}^d\mathscr{B}_D = 0$$

we easily conclude that $[\mathscr{O}_{\mathscr{Q}}(1)] \cdot [D] \geq 0$. □

Lemma 45. *Let $D \to C^{(d)}$ be a morphism. Then we can find a cover $\widetilde{D} \to D$ such that the composite $\widetilde{D} \to D \to C^{(d)}$ factors through C^d.*

Proof. Let D_1 be a component of $D \times_{C^{(d)}} C^d$ which dominates D. Take $\widetilde{D}$ to be a resolution of D_1. □

Corollary 46. *Let $D \to \mathscr{Q}$ be a morphism. Replacing D by a cover $\widetilde{D}$ we may assume that the map $\widetilde{D} \to D \to \mathscr{Q} \to C^{(d)}$ factors through C^d.*

In view of the above, given a map $D \to Q$ we may assume that the composite $D \to \mathscr{Q} \to C^{(d)}$ factors through C^d. Let each component be given by a map $f_i : D \to C$. Denote by $i_D : \Gamma \hookrightarrow C \times D$ the pullback of the universal subscheme $\Sigma \hookrightarrow C \times C^{(d)}$ to $C \times D$. The ideal sheaf of Γ is the product $\mathscr{I}(\Gamma_{f_i})$, the ideal sheaves of the graphs $\Gamma_{f_i} \subset C \times D$. Moreover, $\mathscr{B}_D$ is supported on Γ. Let $g_1, g_2, \ldots, g_r$ be the distinct maps in the set $\{f_1, f_2, \ldots, f_d\}$ and assume that g_i occurs d_i many times. Then we have $\mathscr{I}(\Gamma) = \prod_{i=1}^r \mathscr{I}(\Gamma_{g_i})^{d_i}$. There is a natural map

$$\psi : \mathscr{B}_D \to \bigoplus \mathscr{B}_D/\mathscr{I}(\Gamma_{g_i})^{d_i}\mathscr{B}_D.$$

Lemma 47. *Let $f : D \to \mathscr{Q}$ be such that $\Phi \circ f$ factors through $C^d \to C^{(d)}$. If ψ is an isomorphism then $[\mathscr{O}_{\mathscr{Q}}(1)] \cdot [D] \geq 0$.*

Proof. Since $\mathscr{B}_D$ is a quotient of $\mathscr{O}^n_{C\times D}$ it follows that each $\mathscr{B}_D/\mathscr{I}(\Gamma_{g_i})^{d_i}\mathscr{B}_D$ is a quotient of $\mathscr{O}^n_{C\times D}$. Thus, each $\mathscr{B}_D/\mathscr{I}(\Gamma_{g_i})^{d_i}\mathscr{B}_D$ defines a map $D \to \mathscr{Q}(n, d'_i)$ such that the image under the map $\Phi : \mathscr{Q}(n, d'_i) \to C^{(d'_i)}$ is the small diagonal. By Proposition 44 it follows that degree of $\det(p_{D*}(\mathscr{B}_D/\mathscr{I}(\Gamma_{g_i})^{d_i}\mathscr{B}_D))$ is ≥ 0. Since ψ is an isomorphism it follows that degree of $\det(p_{D*}(\mathscr{B}_D))$ is ≥ 0. □

We can use the above method to prove a result similar to Theorem 37 when $d = 3$.

Corollary 48. *Let $d = 3$. Consider the Quot scheme $\mathscr{Q} = \mathscr{Q}(n,3)$. Let $\mu_0^{(3)} = \frac{g+2}{3g}$. Then $[\mathscr{O}_{\mathscr{Q}}(1)] + \mu_0^{(3)}[L_0^{(3)}]$ is nef.*

Proof. If $d = 3$ there are only these types of quotients:

(1) $\mathscr{O}^n_C \to \mathscr{O}_C/\mathfrak{m}_{C,c_1}\mathfrak{m}_{C,c_2}\mathfrak{m}_{C,c_3}$,

(2) $\mathscr{O}_C^n \to \mathscr{O}_{C,c_1}/\mathfrak{m}_{C,c_1} \oplus \mathscr{O}_C/\mathfrak{m}_{C,c_1}\mathfrak{m}_{C,c_2}$,

(3) $\mathscr{O}_C^n \to \frac{\mathscr{O}_{C,c}}{\mathfrak{m}_{C,c}} \oplus \frac{\mathscr{O}_{C,c}}{\mathfrak{m}_{C,c}} \oplus \frac{\mathscr{O}_{C,c}}{\mathfrak{m}_{C,c}}$.

Let $f : D \to \mathscr{Q}$ be a map. If D contains a quotient of type (1) or (3) then D meets U or U'(see Definition 24 and Definition 35). Thus, in these cases $([\mathscr{O}_{\mathscr{Q}}(1)] + \mu_0^{(3)}[L_0^{(3)}]) \cdot [D] \geq 0$ by Corollary 27 and Lemma 36.

Now consider the case when all points in the image of D are of type (2). After replacing D by a cover, using Corollary 46, we may assume that the map $D \to \mathscr{Q}$ factors through C^3. Since the images of points of D represent quotients of type (2), we may assume that the map from $D \to C^3$ looks like $d \mapsto (g_1(d), g_1(d), g_2(d))$. Now consider a general section $\mathscr{O}_{C\times D} \to \mathscr{B}_D$. Arguing as in the proof of Lemma 26 we get a diagram as in equation (12), such that $\mathscr{O}_{\Gamma'}$ defines a map $D \to C^{(2)}$ and $\mathscr{F}' = \mathscr{F}/T_0(\mathscr{F})$ is a line bundle on D which is globally generated. Hence

$$\begin{aligned}[\mathscr{O}_{\mathscr{Q}}(1)] \cdot [D] &\geq [\mathscr{O}(-\Delta_2/2)] \cdot [D] + \left[c_1\left(p_{D*}(\mathscr{F})\right)\right] \cdot [D] \\ &\geq -\mu_0^{(2)}\left[L_0^{(2)}\right] \cdot [D].\end{aligned}$$

One easily checks using the definition of L_0 that in this case $[L_0^{(3)}] \cdot [D] = 2[L_0^{(2)}] \cdot [D]$. Thus,

$$\left([\mathscr{O}_{\mathscr{Q}}(1)] + \mu_0^{(3)}\left[L_0^{(3)}\right]\right) \cdot [D] \geq \left(2\mu_0^{(3)} - \mu_0^{(2)}\right)\left[L_0^{(2)}\right] \cdot [D] \geq 0.$$

This completes the proof of the Corollary 48. □

Combining this with Proposition 20 we get the following result.

Theorem 49. *Let C be a very general curve of genus $2 \leq g(C) \leq 4$. Let $n \geq 3$ and let $\mathscr{Q} = \mathscr{Q}(n,3)$. Let $\mu_0 = \dfrac{g+2}{3g}$ Then*

$$\mathrm{Nef}(\mathscr{Q}) = \mathbb{R}_{\geq 0}\left([\mathscr{O}_{\mathscr{Q}}(1)] + \mu_0\left[L_0^{(3)}\right]\right) + \mathbb{R}_{\geq 0}[\theta_d] + \mathbb{R}_{\geq 0}\left[L_0^{(3)}\right].$$

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