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Nef cones of some Quot schemes on a Smooth Projective Curve

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Abstract. Let \( C \) be a smooth projective curve over \( \mathbb{C} \). Let \( n, d \geq 1 \). Let \( \mathcal{Q} \) be the Quot scheme parameterizing torsion quotients of the vector bundle \( \mathcal{O}_n^d \) of degree \( d \). In this article we study the nef cone of \( \mathcal{Q} \). We give a complete description of the nef cone in the case of elliptic curves. We compute it in the case when \( d = 2 \) and \( C \) very general, in terms of the nef cone of the second symmetric product of \( C \). In the case when \( n \geq d \) and \( C \) very general, we give upper and lower bounds for the Nef cone. In general, we give a necessary and sufficient criterion for a divisor on \( \mathcal{Q} \) to be nef.

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1. Introduction

Throughout this article we assume that the base field to be \( \mathbb{C} \). Let \( X \) be a smooth projective variety and let \( N^1(X) \) be the \( \mathbb{R} \)-vector space of \( \mathbb{R} \)-divisors modulo numerical equivalence. It is known that \( N^1(X) \) is a finite dimensional vector space. The closed cone \( \text{Nef}(X) \subset N^1(X) \) is the cone of all \( \mathbb{R} \)-divisors whose intersection product with any curve in \( X \) is non-negative. It has been an interesting problem to compute \( \text{Nef}(X) \). For example, when \( X = \mathbb{P}(E) \) where \( E \) is a semistable vector bundle over a smooth projective curve, Miyaoka computed the \( \text{Nef}(X) \) in [14]. In [4], \( \text{Nef}(X) \) was computed in the case when \( X \) is the Grassmann bundle associated to a vector bundle \( E \) on a smooth projective curve \( C \), in terms of the Harder Narasimhan filtration of \( E \). Let \( C^{(d)} \) denote the \( d \)th symmetric product. In [15], the author computed the \( \text{Nef}(C^{(d)}) \) in the case when \( C \) is a very general curve of even genus and \( d = \text{gon}(C) - 1 \). In [11] \( \text{Nef}(C^{(2)}) \) is computed in the case when \( C \) is very general and \( g \) is a perfect square. In [5] \( \text{Nef}(C^{(2)}) \) was computed assuming the Nagata conjecture. We refer the reader to [12, Section 1.5] for more such examples and details.

The reader is referred to [6] for the definition and details on Quot schemes. Let \( E \) be a vector bundle over a smooth projective curve \( C \). Fix a polynomial \( P \in \mathbb{Q}[t] \). Let \( \mathcal{Q}(E, P) \) denote the Quot
scheme parametrizing quotients of $E$ with Hilbert polynomial $P$. In [16], when $C = \mathbb{P}^1$, the quot scheme $\mathcal{Q}(\mathcal{O}_C^r, P)$ is studied as a natural compactification of the set of all maps from $C$ to some Grassmannians of a fixed degree. In this article we will consider the case when $P = d$ a constant, that is, when $\mathcal{Q}(E, d)$ parametrizes torsion quotients of $E$ of degree $d$. For notational convenience, we will denote $\mathcal{Q}(E, d)$ by $\mathcal{Q}$, when there is no possibility of confusion. It is known that $\mathcal{Q}$ is a smooth projective variety. Many properties of $\mathcal{Q}$ have been studied. In [1], the Betti cohomologies of $\mathcal{Q}(\mathcal{O}_C^r, d)$ are computed, $\mathcal{Q}(\mathcal{O}_C^r, d)$ has been interpreted as the space of higher rank divisors of rank $n$, and an analogue of the Abel–Jacobi map was constructed. In [2] the automorphism group scheme of $\mathcal{Q}(\mathcal{O}_C^r, d)$ was computed in the case when the genus of $C$ satisfies $g(C) > 1$ and a Torelli theorem for these Quot schemes was proved. In [3] the Brauer group of $\mathcal{Q}(\mathcal{O}_C^r, d)$ is computed. In [7], the automorphism group scheme of $\mathcal{Q}(E, d)$ was computed in the case when either $\text{rk } E \geq 3$ or $E$ is semistable and genus of $C$ satisfies $g(C) > 1$. In [8], the S-fundamental group scheme of $\mathcal{Q}(E, d)$ was computed.

In this article, we address the question of computing $\text{Nef}(\mathcal{Q})$. Recall that we have a Hilbert–Chow map $\Phi : \mathcal{Q} \to C^{(d)}$ (this map is explained after Definition 9). A precise definition can be found, for example, in [8]. For notational convenience, for a divisor $D \in N^1(C^{(d)})$ we will denote its pullback $\Phi^* D \in N^1(\mathcal{Q})$ by $D$, when there is no possibility of confusion. The line bundle $\mathcal{O}_{\mathcal{Q}}(1)$ is defined in Definition 9. In Section 2 we recall the results we need on $\text{Nef}(\mathcal{Q})$. In Section 3 we compute $\text{Pic}(\mathcal{Q})$.

**Theorem (Theorem 11).** $\text{Pic}(\mathcal{Q}) = \Phi^* \text{Pic}(C^{(d)}) \oplus \mathbb{Z}[\mathcal{O}_{\mathcal{Q}}(1)]$.

As a corollary (Corollary 13) we get that $N^1(\mathcal{Q}) \cong N^1(C^{(d)}) \oplus \mathbb{R}[\mathcal{O}_{\mathcal{Q}}(1)]$. The computation of $N^1(\mathcal{Q})$ can also be found in [3]. As a result, when $C \cong \mathbb{P}^1$, since $C^{(d)} \cong \mathbb{P}_d^1$, we have that the $N^1(\mathcal{Q})$ is 2-dimensional and we prove that its nef cone is given as follows.

**Theorem (Theorem 34).** Let $C = \mathbb{P}^1$. Let $E = \bigoplus_{i=1}^k \mathcal{O}(a_i)$ with $a_i \leq a_j$ for $i < j$. Let $d \geq 1$. Then \[ \text{Nef}(\mathcal{Q}(E, d)) = \mathbb{R}_{\geq 0} \left( [\mathcal{O}_{\mathcal{Q}}(E, d)^{(1)}(1)] + (-a_1 + d - 1) [\mathcal{O}_{\mathcal{Q}}(1)] \right) + \mathbb{R}_{\geq 0} [\mathcal{O}_{\mathcal{Q}}(1)] . \]

Note that this theorem was already known in the case when $E = V \otimes \mathcal{O}_C$, for a vector space $V$ over $k$ ([16, Theorem 6.2]).

For the rest of the introduction, we will assume $E = V \otimes \mathcal{O}_C$ with $\dim_k V = n$ and denote by $\mathcal{Q} = \mathcal{Q}(n, d)$ the Quot scheme $\mathcal{Q}(E, d)$. Let us consider the case $g = 1$. In this case, $N^1(\mathcal{Q})$ is three-dimensional (see Proposition 14), and we prove that its nef cone is given as follows (see Definition 4 for notations).

**Theorem (Theorem 43).** Let $g = 1$, $n \geq 1$ and $\mathcal{Q} = \mathcal{Q}(n, d)$. Then the class $[\mathcal{O}_{\mathcal{Q}}(1)] + [\Delta_d/2] \in N^1(\mathcal{Q})$ is nef. Moreover, \[ \text{Nef}(\mathcal{Q}) = \mathbb{R}_{\geq 0} ([\mathcal{O}_{\mathcal{Q}}(1)] + [\Delta_d/2]) + \mathbb{R}_{\geq 0} [\mathcal{O}_{\mathcal{Q}}(1)] . \]

From now on assume that $g \geq 2$ and $C$ is very general. See Definition 9 for the definition of $t$ and $\alpha_t$. When $d = 2$ we have the following result.

**Theorem (Theorem 37).** Let $g \geq 2$ and $C$ be very general. Let $d = 2$. Consider the Quot scheme $\mathcal{Q} = \mathcal{Q}(n, 2)$. Then \[ \text{Nef}(\mathcal{Q}) = \mathbb{R}_{\geq 0} \left( [\mathcal{O}_{\mathcal{Q}}(1)] + \frac{t + 1}{g + t} [L_0] \right) + \mathbb{R}_{\geq 0} [L_0] + \mathbb{R}_{\geq 0} [\alpha_t] . \]

Precise values of $t$ are known for small genus. When $g \geq 9$ it is conjectured that $t = \sqrt{g}$. This is known when $g$ is a perfect square. The precise statements have been mentioned after Theorem 37.
In general (without any assumptions on \( n \) and \( d \)), we give a criterion for certain line bundle on \( \mathcal{O} \) to be nef in terms of its pullback along certain natural maps from products \( \prod_i C^{(d)} \), see Subsection 7.1 for notation.

**Theorem (Theorem 39).** Let \( \beta \in N^1(C^{(d)}) \). Then the class \([\mathcal{O}_2(1)] + \beta \in N^1(\mathcal{O})\) is nef iff the class \([\mathcal{O}(-\Delta_d/2)] + \pi^* \beta \in N^1(C^{(d)})\) is nef for all \( \mathbf{d} \in \mathcal{O}^{\leq n} \).

Using the above we show that certain classes are in \( \text{Nef}(\mathcal{O}) \). Define

\[
\kappa_1 := [\mathcal{O}_2(1)] + \mu_0[\bar{L}_0] + \frac{d + g - 2}{d g} \left[ \theta_d \right] \quad \kappa_2 := [\mathcal{O}_2(1)] + \frac{g + 1}{2 g} [L_0] \in N^1(\mathcal{O}) \, .
\]

**Proposition (Proposition 41).** Let \( g \geq 1 \), \( n \geq 1 \) and \( \mathcal{O} = \mathcal{O}(n, d) \). Then

\[
\text{Nef}(\mathcal{O}) \supseteq \mathbb{R}_{\geq 0} \kappa_1 + \mathbb{R}_{\geq 0} \kappa_2 + \mathbb{R}_{\geq 0} \theta_d + \mathbb{R}_{\geq 0} [L_0] \, .
\]

Now consider the case when \( n \geq d \geq \text{gon}(C) \). Then \( \text{Nef}(C^{(d)}) \) is generated by \( \theta_d \) and \( L_0 \) (see Definitions 1 and 4). In this case we give the following upper bound for the nef cone in Proposition 20. Let \( \mu_0 := \frac{d + g - 1}{d g} \). Then

\[
\text{Nef}(\mathcal{O}) \subset \mathbb{R}_{\geq 0} \left( [\mathcal{O}_2(1)] + \mu_0[\bar{L}_0] \right) + \mathbb{R}_{\geq 0} \theta_d + \mathbb{R}_{\geq 0} [L_0] \, .
\]

When \( d \geq \text{gon}(C) \), in Lemma 30 we show that any convex linear combination of the \( \kappa_1 \) and \( \theta_d \) is nef but not ample. In particular, any such class lies on the boundary of \( \text{Nef}(\mathcal{O}) \). Similarly, in Corollary 42 we show when \( n \geq d \), any convex linear combination of the class \( \kappa_2 \) and \( L_0^{(d)} \) is nef but not ample. So any such class lies on the boundary of \( \text{Nef}(\mathcal{O}) \).

In terms of the above diagram, we have that when \( n \geq d \geq \text{gon}(C) \)

\[
\left\langle OD, OE, OC, OB \right\rangle \subset \text{Nef}(\mathcal{O}) \subset \left\langle OA, OC, OB \right\rangle \, .
\]

We do not know if the inclusion in the right is an equality when \( n \geq d \geq \text{gon}(C) \). This is same as saying that \([\mathcal{O}_2(1)] + \mu_0[\bar{L}_0] \) is nef when \( n \geq d \geq \text{gon}(C) \). In Section 8 we give a sufficient condition for when the pullback of \([\mathcal{O}_2(1)] + \mu_0[\bar{L}_0] \) along a map \( D \to \mathcal{O} \) is nef. However, when \( d = 3 \) we have the following result.
**Theorem (Theorem 49).** Let $C$ be a very general curve of genus $2 \leq g(C) \leq 4$. Let $n \geq 3$ and let $\mathcal{Z} = \mathcal{Z}(n, 3)$. Let $\mu_0 = \frac{g^2 + 2}{3g}$ Then

$$\text{Nef}(\mathcal{Z}) = \mathbb{R}_{\geq 0} \left[ (\theta_{\mathcal{Z}}(1)) + \mu_0 [L_0] \right] + \mathbb{R}_{\geq 0} [\theta_d] + \mathbb{R}_{\geq 0} [L_0].$$

Some of the results above can be improved in the case when $g = 2k$ using the results in [15]. (See Proposition 32.)

2. Nef cone of $C^{(d)}$

We follow [15, § 2] for this section. Assume that either $C$ is an elliptic curve or is a very general curve of genus $g \geq 2$. Then it is known that the Neron–Severi space is 2-dimensional. So in this case, to compute the nef cone, it is enough to give two classes in $\text{N}_1(C)$ which are nef but not ample.

For any smooth projective curve and $d \geq 2$ (not just a very general curve) there is a natural line bundle $L_0$ on $C^{(d)}$ which is nef but not ample. This line bundle is constructed in the following manner. Consider the map

$$\phi : C^d \to J(C)^{[d]},$$

$$(x_i) \mapsto (x_i - x_j)_{i < j}.$$

Let $p_{ij}$ denote the projections from $J(C)^{[d]}$. Since $\phi$ is not finite, as it contracts the diagonal, the line bundle $\phi^* (\phi p_{ij}^* \Theta)$ is nef but not ample. This line bundle is invariant under the action of $S_d$ on $C^d$. This follows from the fact that $\Theta$ in $J(C)$ is invariant under the involution $L \mapsto L^{-1}$.

**Definition 1.** $\phi^* (\phi p_{ij}^* \Theta)$ descends to a line bundle $L_0$ on $C^{(d)}$.

Since $\phi$ contracts the small diagonal $\delta : C \to C^{(d)}$, we have $\delta^*[L_0] = 0$. Hence $L_0$ is nef but not ample [15, Lemma 2.2]. Therefore, in the case when $C$ is very general, computing the nef cone of $C^{(d)}$ boils down to finding another class which is nef but not ample.

In the case when $d \geq \text{gon}(C) =: e$, [15, Lemma 2.3] we can easily construct another line bundle which is nef but not ample: Then we have a map $g_e : C \to \mathbb{P}^1$ of degree $e$. This induces a closed immersion $\mathbb{P}^1 \to C^{(e)}$ with $v \mapsto [(g_e)^{-1} (v)] \in C^{(e)}$. This in turn gives a closed immersion $\mathbb{P}^1 \to C^{(d)}$ with $v \mapsto [(g_e)^{-1} (v) + (d-e)x]$ for some point $x \in C$.

**Definition 2.** Denote the class of this $\mathbb{P}^1$ in $\text{N}_1(C^{(d)})$ by $[l']$.

The composition $\mathbb{P}^1 \to C^{(d)} \xrightarrow{u_d} J(C)$ is constant, since there can be no non-constant maps from $\mathbb{P}^1 \to J(C)$. Hence $u_d : C^{(d)} \to J(C)$ is not finite and we get that $u_d^* \Theta$ is nef but not ample.

**Definition 3.** Define $\theta_d := u_d^* \Theta$.

Recall that over $C^{(d)}$ we have natural divisors [15, § 2]:

**Definition 4.** Define

1. $\theta_d$
2. the big diagonal $\Delta_d \to C^{(d)}$
3. If $i_{d-1} : C^{(d-1)} \to C^{(d)}$ is the map given by $D \mapsto D + x$ for a point $x \in C$, then the image $i_{d-1}^*(C^{(d-1)})$. This divisor will be denoted $[x]$.

It is known that when $g = 1$ or $C$ is very general of $g \geq 2$, then $\text{N}_1(C^{(d)})$ is of dimension 2 and any two of the above three forms a basis.

By abuse of notation, let us denote the class ($\delta$ is the small diagonal) $[\delta_+(C)] \in \text{N}_1(C^{(d)})$ by $\delta$. We summarise the above discussion in the following theorem.
Proposition 5 ([15, Proposition 2.4]). When \( d \geq \text{gon}(C) \), we have:
\begin{enumerate}
\item \( \text{Nef}(C^{(d)}) = \mathbb{R}_{\geq 0}[L_0] \oplus \mathbb{R}_{\geq 0}[\theta_d] \),
\item \( \overline{\text{NE}}(C^{(d)}) = \mathbb{R}_{\geq 0}[l'] \oplus \mathbb{R}_{\geq 0}[\delta] \).
\end{enumerate}

The above bases are dual to each other.

We will need to write \([L_0]\) in terms of \([x]\) and \([\theta_d]\), for which we need the following computations. Define

\[ \delta^t : C \xrightarrow{\mu} C^d \to C^{(d)} \]

where the first map is given by \( x \mapsto (x, x_1, \ldots, x_{d-1}) \).

**Lemma 6.** Let \( d \geq 1 \). We have the following
\begin{enumerate}
\item \( \deg(\delta^t[\theta_d]) = d^2 g \)
\item \( \deg(\delta^t[\theta_d]) = g \)
\item \( \deg(\delta^t[x]) = d \)
\item \( \deg(\delta^t[x]) = 1 \)
\end{enumerate}

**Proof.** Recall that \( \theta_d = u_d^* \Theta \), where \( u_d : C^{(d)} \to J(C) \) is given by \( D \mapsto \Theta(D - dx_0) \) for a fixed point \( x_0 \in C \). Therefore the composition \( u_d \circ \delta : C \to J(C) \) is given by \( x \mapsto dx \mapsto \Theta(dx - dx_0) \), which is the map

\[ C \xrightarrow{u_1} J(C) \xrightarrow{x} J(C). \]

The pullback of \( \Theta \) under the map \( J(C) \xrightarrow{x} J(C) \) is \( \Theta^d^2 \) and the degree of the pullback of \( \Theta \) under the map \( u_1 : C \to J(C) \) is \( g \). Hence degree of \( \delta^t[\theta_d] = d^2 g \). This proves (1).

The composition \( u_d \circ \delta^t : C \to J(C) \) is given by \( C \to C^{(d)} \to J(C) \)

\[ x \mapsto x + \sum_{i=1}^{d-1} x_i \mapsto \Theta \left( x + \sum_{i=1}^{d-1} x_i - dx_0 \right) \]

which is the composition \( C \xrightarrow{u_1} J(C) \xrightarrow{t_x} J(C) \), where \( t_x \) is translation by an element in \( J(C) \). Hence degree of \( \delta^t[\theta_d] = g \). This proves (2).

For a line bundle \( L \) on \( C \), we will denote by \( L^{\boxtimes d} \) to be the unique line bundle on \( C^{(d)} \), whose pullback under the quotient map \( \pi : C^d \to C^{(d)} \) is \( \boxtimes_{i=1}^d p_i^* L \). Recall that by [15, § 2], we have that \([x] = [\Theta(x)^{\boxtimes d}] \) for a point \( x \in C \). By definition under the map \( \pi : C^d \to C^{(d)} \) the pullback of \( \Theta(x)^{\boxtimes d} \) is \( \boxtimes_{i=1}^d p_i^* \Theta(x) \). Now \( \delta : C \to C^{(d)} \) is the composition \( C \to C^d \to C^{(d)} \)

\[ x \mapsto (x, \ldots, x) \mapsto dx. \]

Hence we get that the pullback of \( \Theta(x)^{\boxtimes d} \) to \( \delta \) is \( \Theta(dx) \). Therefore degree of \( \delta^* [x] = d \). This proves (3).

We know \( \delta^t \) is the composition \( C \to C^d \to C^{(d)} \)

\[ x \mapsto (x, x_1, \ldots, x_{d-1}) \mapsto x + x_1 + \ldots + x_{d-1}. \]

Hence we get that \( \delta^t[x] = \Theta(x) \). Therefore degree of \( \delta^t[x] = 1 \). This proves (4).

**Lemma 7.** Let \( g, d \geq 1 \). Let \( \mu_0 := \frac{d g - 1}{d g} \). Then

\[ [L_0] = d g [x] - [\theta_d] = (d g - d - g + 1) \cdot [x] + [\Delta_d / 2] \]

\[ = \left( \frac{1}{\mu_0} - 1 \right) [\theta_d] + \frac{1}{\mu_0} [\Delta_d / 2]. \]
Proof. Let \([L_0] = a[\theta_d] + b[x]\). We need two equations to solve for \(a\) and \(b\). The first equation is \(\delta^* [L_0] = 0\). Recall

\[
\delta' : C \xrightarrow{f} C^d \to C^{(d)}
\]

where the first map is given by \(x \mapsto (x, x_1, \ldots, x_d)\). Hence

\[
\delta'^* [L_0] = f^* \phi^* (\otimes p_i^* \Theta).
\]

Now the composition

\[
C \xrightarrow{f} C^d \xrightarrow{\phi} J(C)^{(d)}
\]

is given by \(x \mapsto (x - x_1, x - x_2, \ldots, x - x_{d-1}, x_i - x_j)_{i < j}\). Hence

\[
\deg (\delta'^* [L_0]) = \sum_{i=1}^{d-1} \deg (\theta_1) = (d - 1) g.
\]

This will be our second equation.

We use these two equations and the preceding computations to compute \(a\) and \(b\).

\[
0 = \deg (\delta^* [L_0])
= a \cdot \deg (\delta^* [\theta_d]) + b \cdot \deg (\delta^* [x])
= ad^2 g + bd.
\]

Therefore

\[
b = -ad g.
\]

Now using the second equation we get

\[
(d - 1) g = \deg (\delta'^* [L_0])
= a \cdot \deg (\delta'^* [\theta_d]) + b \cdot \deg (\delta'^* [x])
= ag + b
= ag - ad g = ag(1 - d).
\]

Therefore

\[
a = -1, \quad b = dg.
\]

Hence we get \([L_0] = dg[x] - [\theta_d]\). For the other two equalities, we use the relation

\[
[\theta_d] = (d + g - 1) [x] - [\Delta_d / 2]
\]

between \([x]\), \([\Delta_d / 2]\) and \([\theta_d]\) [15, Lemma 2.1].

3. Picard group and Neron–Severi group of \(\mathcal{D}\)

Let \(E\) be a locally free sheaf over \(C\). Throughout this section \(\mathcal{D}\) will denote the Quot scheme \(\mathcal{D}(E, d)\) which parametrizes torsion quotients of \(E\) of degree \(d\). In this section we compute the Picard group of \(\mathcal{D}\), and the vector spaces \(N^1(\mathcal{D})\) and \(N_1(\mathcal{D})\).

Lemma 8. Let \(S\) be a scheme over \(k\). Let \(F\) be a coherent sheaf over \(C \times S\) which is \(S\)-flat and for all \(s \in S\), \(F|_{C \times s}\) is a torsion sheaf over \(C\) of degree \(d\). Let \(p_S : C \times S \to S\) be the projection. Then

(i) \(p_S^*(F)\) is locally free of rank \(d\) and \(\forall s \in S\) the natural map \(p_S^*(F)|_s \to H^0(C, F|_{C \times s})\) is an isomorphism.
Lemma 10. For any line bundle denote the big diagonal in Let

\[ \Phi \sim= \... \]

Then the natural morphism \( \phi^* p_{S*}(F) \to (p_T)_*(id \times \phi)^* F \) is an isomorphism.

Proof. Since \( F|_{C \times s} \) is a torsion sheaf for all \( s \in S \), we have \( H^1(C, F|_{C \times s}) = 0 \). By [9, Chapter III, Theorem 12.11 (a)] we get \( R^1 p_{S*}(F) = 0 \). Using [9, Chapter III, Theorem 12.11 (b)] (ii) with \( i = 1 \) we get that the morphism \( p_{S*}(F)|_{i} \to H^0(C, F|_{C \times s}) \) is surjective. Again using the same with \( i = 0 \) we get that \( p_{S*}(F) \) is locally free of rank \( d \) and the map \( p_{S*}(F)|_{i} \to H^0(C, F|_{C \times s}) \) is an isomorphism.

Since \( F \) is \( S \)-flat it follows that \( (id \times \phi)^* F \) is \( T \)-flat. Applying the above we see \( \phi^* p_{S*}(F) \) and \( (p_T)_*(id \times \phi)^* F \) are locally free of rank \( d \). For each \( t \in T \) we have the commutative diagram:

\[
\begin{array}{ccc}
\phi^* p_{S*}(F)|_{t} = p_{S*}(F)|_{\phi(t)} & \to & (p_T)_*(id \times \phi)^* F|_{t} \\
\downarrow & & \downarrow \\
H^0(C, F|_{C \times \phi(t)}) & \to & H^0(C, (id \times \phi)^* F|_{C \times t})
\end{array}
\]

By the first part we get that the vertical arrows are isomorphisms. Hence we get that the first row of the diagram is an isomorphism. Therefore

\[
\phi^* p_{S*}(F) \to (p_T)_*(id \times \phi)^* F
\]

is a surjective morphism of vector bundles of same rank and hence an isomorphism.

We define a line bundle on \( \mathcal{O} \). Let us denote the projections \( C \times \mathcal{O} \) to \( C \) and \( \mathcal{O} \) by \( p_C \) and \( p_Q \) respectively. Then we have the universal quotient \( p_C^* E \to \mathcal{O}_\mathcal{O} \) over \( C \times \mathcal{O} \). By Lemma 8, \( p_{\mathcal{O}}* (\mathcal{O}_\mathcal{O}) \) is a vector bundle of rank \( d \).

Definition 9. Denote the line bundle \( det(p_{\mathcal{O}}* (\mathcal{O}_\mathcal{O})) \) by \( \mathcal{O}_\mathcal{O}(1) \).

Denote the \( d \)-th symmetric product of \( C \) by \( C^{(d)} \). Recall the Hilbert–Chow map \( \Phi : \mathcal{O} \to C^{(d)} \) which sends \( [E \to B] \) to \( \sum l(B)p \), where \( l(B)p \) is the length of the \( \mathcal{O}_C \cdot p \)-module \( B \). Therefore, we have the pullback \( \Phi^* : Pic(C^{(d)}) \to Pic(\mathcal{O}) \) which is in fact an inclusion. To see this, recall that the fibres of \( \Phi \) are projective integral varieties [8, Corollary 6.6] and \( \Phi \) is flat [8, Corollary 6.3]. Hence \( \Phi_*(\mathcal{O}_\mathcal{O}) = \mathcal{O}_C^{(d)} \). Now by projection formula \( \Phi_* \Phi^* L \cong L \) for all \( L \in Pic(C^{(d)}) \) and the statement follows.

The big diagonal is the image of the map \( C \times C^{(d-2)} \to C^{(d)} \) given by \( (x, A) \to 2x + A \). Let us denote the big diagonal in \( C^{(d)} \) by \( \Delta \). Let \( U_C := C^{(d)} \setminus \Delta \) and \( \mathcal{U} := \Phi^{-1}(U_C) \). Then \( \mathcal{U} \subset \mathcal{O} \).

Lemma 10. For any line bundle \( \mathcal{L} \in Pic(\mathcal{O}) \), \( \exists \) an unique \( n \in \mathbb{Z} \) such that \( (\mathcal{L} \otimes \mathcal{O}_\mathcal{O}(-n))|_{\Phi^{-1}(p)} \equiv \mathcal{O}_\Phi^{-1}(p) \) for all \( p \in U_C \).

Proof. Let \( \pi : \mathbb{P}(E) \to C \) be the projective bundle associated to \( E \) and let \( \mathcal{O}_{\mathbb{P}(E)}(1) \) be the universal line bundle over \( \mathbb{P}(E) \). Let \( Z = \mathbb{P}(E)^d \). Let \( p_1 : Z \to \mathbb{P}(E) \) be the \( i \)-th projection. Let \( \pi_d : Z \to C^d \) be the product map. The symmetric group \( S_d \) acts on \( Z \) and the map \( \pi_d \) is equivariant for this action. Let \( \psi : C^d \to C^{(d)} \) be the quotient map. Define \( U_Z := (\psi \circ \pi_d)^{-1}(U) \).

Let \( c \in C \) be a closed point and let \( k_c \) denote the skyscraper sheaf supported at \( c \). A closed point of \( \mathbb{P}(E) \) which maps to \( c \in C \) corresponds to a quotient \( E \to E_c \to k_c \). Recall that we have a map [7, Theorem 2.2(a)]

\[
\bar{\psi} : U_Z \to \mathcal{U}
\]
which sends a closed point
\[(E_{c_i} \rightarrow k_{c_i})_{i=1}^d \in U_Z\]
to the quotient
\[E \rightarrow \bigoplus_i E_{c_i} \rightarrow \bigoplus_i k_{c_i} \in \mathcal{U}.
\]
So we have a commutative diagram:
\[
\begin{array}{ccc}
U_Z & \xrightarrow{\psi} & \mathcal{U} \\
\downarrow{\pi_d} & & \downarrow{\Phi} \\
\psi^{-1}(U_C) & \xrightarrow{} & U_C
\end{array}
\]
Moreover, if \(\underline{c} = (c_1, \ldots, c_d) \in \psi^{-1}(U_C)\), then by [8, Lemma 6.5] \(\overline{\psi}\) induces an isomorphism
\[
\prod \mathbb{P}(E_{c_i}) = \pi_d^{-1}(\underline{c}) \xrightarrow{\sim} \Phi^{-1}(\psi(\underline{c})).
\]
Applying Lemma 8 by taking \(T = U_Z\), \(S = \mathcal{U}\) and \(\phi = \overline{\psi}\) and the definition of the map \(\overline{\psi}\) (see the proof of [7, Theorem 2.2(a)]) we see that
\[
\overline{\psi}^* \mathcal{O}_{\mathcal{U}}(1) = \bigotimes_{i=1}^d p_i^* \mathcal{O}_{\mathbb{P}(E_i)}(1)|_{U_Z}.
\]
Hence it is enough to show that \(\exists \ n \in \mathbb{Z}\) such that \(\forall \ \underline{c} \in \psi^{-1}(U_C)\)
\[
\overline{\psi}^* \mathcal{L}|_{\pi_d^{-1}(\underline{c})} \cong \bigotimes_{i=1}^d p_i^* \mathcal{O}(n)|_{\pi_d^{-1}(\underline{c})}.
\]
For \(\underline{c} \in \psi^{-1}(U_C)\) define \(n_i(\underline{c}) \in \mathbb{Z}\) using the equation
\[
\overline{\psi}^* \mathcal{L}|_{\pi_d^{-1}(\underline{c})} = \bigotimes_{i=1}^d p_i^* \mathcal{O}(n_i)|_{\pi_d^{-1}(\underline{c})}.
\]
We may view the \(n_i\) as functions \(n_i : \psi^{-1}(U_C) \rightarrow \mathbb{Z}\). Since the line bundle \(\overline{\psi}^* \mathcal{L}\) is invariant under the action of the group \(S_d\), it follows that
\[
n_{\sigma(i)}(\underline{c}) = n_i(\sigma(\underline{c})). \quad \text{(3)}
\]
Here \(\sigma(\underline{c}) := (c_{\sigma(1)}, \ldots, c_{\sigma(d)})\). Hence it suffices to show that \(n_1\) is a constant function.

Let \(c_2, \ldots, c_d\) be distinct points in \(C\). Define \(V := C \setminus \{c_2, \ldots, c_d\}\) and a map
\[
i : V \rightarrow \psi^{-1}(U_C) \quad i(c) := (c, c_2, \ldots, c_d).
\]
Then \(\pi_d^{-1}(V)\) is equal to \(\mathbb{P}(E_{|V}) \times \mathbb{P}(E_{c_2}) \times \ldots \times \mathbb{P}(E_{c_d})\). The restriction of \(\overline{\psi}^* \mathcal{L}\) to \(\mathbb{P}(E_{|V}) \times \mathbb{P}(E_{c_2}) \times \ldots \times \mathbb{P}(E_{c_d})\) is isomorphic to
\[
\pi^* M \otimes p_1^* \mathcal{O}_{\mathbb{P}(E_{|V})}(a_1) \otimes p_2^* \mathcal{O}_{\mathbb{P}(E_{c_2})}(a_2) \ldots \otimes p_d^* \mathcal{O}_{\mathbb{P}(E_{c_d})}(a_d),
\]
where \(M\) is a line bundle on \(V\). Further restricting to \((c, c_2, \ldots, c_d)\) and \((c', c_2, \ldots, c_d)\), where \(c, c' \in V\), we see that
\[
n_i(c, c_2, \ldots, c_d) = n_i(c', c_2, \ldots, c_d) \quad \forall \ i. \quad \text{(4)}
\]
This proves that for distinct points \(c, c', c_2, \ldots, c_d \in C\) we have
\[
n_i(c, c_2, \ldots, c_d) = n_i(c', c_2, \ldots, c_d) \quad \forall \ i. \quad \text{(5)}
\]
Choose $2d$ distinct points $c_1, \ldots, c_d, c'_1, \ldots, c'_d$ in $C$. Then using equations (4) and (5) we get

$$\begin{align*}
n_1 (c_1, c_2, \ldots, c_d) &= n_1 (c'_1, c_2, \ldots, c_d) \\
&= n_2 (c_2, c'_1, \ldots, c_d) \\
&= n_2 (c'_2, c'_1, \ldots, c_d) \\
&= n_1 (c'_1, c'_2, \ldots, c_d) \\
&= \ldots \\
&= n_1 (c'_1, c'_2, \ldots, c'_d).
\end{align*}$$

Finally, for any two points $c, c' \in \psi^{-1}(U_C)$ choose a third point $c''$ such that the coordinates of $c''$ are distinct from those of $c$ and $c'$. Then we see that $n_1 (c) = n_1 (c'') = n_1 (c')$. This proves that $n_1$ is the constant function. Therefore, $\psi^* \mathcal{L} |_{\pi^{-1}(c)}$ is of the form $\bigotimes p_i^* \mathcal{O}_{P(L_i)} (n)$, $\forall c \in \psi^{-1}(U_C)$. The uniqueness of $n$ is obvious.

**Theorem 11.** \(\text{Pic}(\mathcal{O}) = \Phi^* \text{Pic}(C^{(d)}) \oplus \mathbb{Z}[\mathcal{O}_\mathcal{Z}(1)].\)

**Proof.** Let $\mathcal{L} \in \text{Pic}(\mathcal{O})$. By [8, Corollary 6.3] and [8, Corollary 6.4] the morphism $\Phi$ is flat and fibres of $\Phi$ are integral. Then by [13, Lemma 2.1.2] and Lemma 10 we get that $\mathcal{L} \otimes \mathcal{O}_\mathcal{Z} (-n) = \Phi^* \mathcal{M}$ for some $\mathcal{M} \in \text{Pic}(C^{(d)})$. Hence $\mathcal{L} = \Phi^* \mathcal{M} \otimes \mathcal{O}_\mathcal{Z}(n)$. The uniqueness of such an expression follows from the statement on uniqueness in Lemma 10.

For a projective variety $X$ over $k$ recall that $N^1 (X)$ (respectively, $N_1 (X)$) is the vector space of $\mathbb{R}$-divisors (respectively, 1-cycles) modulo numerical equivalence [12, §1.4]. It is known that $N^1 (X)$ and $N_1 (X)$ are finite dimensional and the intersection product defines a non-degenerate pairing

$$N^1 (X) \times N_1 (X) \to \mathbb{R} \quad ([\beta], [\gamma]) \mapsto [\beta] \cdot [\gamma].$$

We will compute $N^1 (\mathcal{O})$ and $N_1 (\mathcal{O})$. Let $\mathcal{O} \in U_C \subset C^{(d)}$. As we saw in the proof of Theorem 11,

$$\Phi^{-1} (\mathcal{O}) \cong \prod_{i \geq 2} p(E_{c_i}).$$

Let $\mathbb{P}^1 \hookrightarrow P(E_{c_i})$ be a line and let $v_i \in P(E_{c_i})$ for $i \geq 2$. Then we have an embedding:

$$\mathbb{P}^1 \cong \mathbb{P}^1 \times v_2 \times \ldots \times v_d \hookrightarrow P(E_{c_i}) \times \prod_{i \geq 2} P(E_{c_i}) = \Phi^{-1} (\mathcal{O}) \subset \mathcal{O}.$$  \hspace{1cm} (6)

**Definition 12.** Let us denote the class of this curve in $N_1 (\mathcal{O})$ by $[I]$.

**Corollary 13.** \(N^1 (\mathcal{O}) = \Phi^* N^1 (C^{(d)}) \oplus \mathbb{R}[\mathcal{O}_\mathcal{Z}(1)].\)

**Proof.** Since $\Phi$ is surjective, $N^1 (C^{(d)}) \to N^1 (\mathcal{O})$ is an inclusion [12, Example 1.4.4]. Note that $\mathcal{O}_\mathcal{Z}(1) \neq 0$ in $N^1 (\mathcal{O})$ since $[\mathcal{O}_\mathcal{Z}(1)] \cdot [I] = 1$. Hence $\mathcal{O}_\mathcal{Z}(1) \neq 0$ in $N^1 (\mathcal{O})$. This also shows that $\mathcal{O}_\mathcal{Z}(1) \in \Phi^* N^1 (C^{(d)})$.

By Theorem 11, we know that any $N^1 (\mathcal{O})$ is generated by $\Phi^* N^1 (C^{(d)})$ and $[\mathcal{O}_\mathcal{Z}(1)]$. The only thing left is to show that

$$\Phi^* N^1 \left( C^{(d)} \right) \cap \mathbb{R} \left[ \mathcal{O}_\mathcal{Z}(1) \right] = 0.$$

For $a \in \mathbb{R}$ if $a[\mathcal{O}_\mathcal{Z}(1)] \in N^1 (C^{(d)})$, then $a[\mathcal{O}_\mathcal{Z}(1)] \cdot [I] = a = 0$. Hence the result follows.

Hence, it follows from Corollary 13 that

**Proposition 14.** If $g = 1$ or $C$ is very general with $g \geq 2$, then $\dim_{\mathbb{R}} N^1 (\mathcal{O}) = 3$.

**Proof.** We already saw that $N^1 (C^{(d)})$ is of dimension 2. The Proposition follows.
To compute $N_1(\mathcal{D})$ we first construct a section of $\Phi : \mathcal{D} \to C^{(d)}$. Over $C \times C^{(d)}$ we have the universal divisor $\Sigma$ which gives us the universal quotient $\mathcal{O}_{C \times C^{(d)}} \to \mathcal{O}_\Sigma$. Choose a surjection $E \to L$ over $C$, where $L$ is a line bundle on $C$. This induces a surjection $E \otimes \mathcal{O}_{C \times C^{(d)}} \to L \otimes \mathcal{O}_{C \times C^{(d)}}$. Then the composition

$$E \otimes \mathcal{O}_{C \times C^{(d)}} \to L \otimes \mathcal{O}_{C \times C^{(d)}} \to L \otimes \mathcal{O}_\Sigma$$

gives us a morphism

$$\eta : C^{(d)} \to \mathcal{D}$$

which is easily seen to be a section of $\Phi$.

**Corollary 15.** $N_1(\mathcal{D}) = N_1(C^{(d)}) \oplus \mathcal{R}(l)$ where $N_1(C^{(d)}) \hookrightarrow N_1(\mathcal{D})$ is the morphism given by the pushforward $\eta_*$.  

**Proof.** Since $\Phi \circ \eta = id_{C^{(d)}}$ we have that $\eta_*$ is an injection. Also since $[\mathcal{O}_\mathcal{D}(1)] \cdot [l] = 1$, we have $[l] \neq 0$. We claim that $[l] \notin N_1(C^{(d)})$. If not, assume that $[l] = \eta_*[\gamma]$ for $[\gamma] \in N_1(C^{(d)})$. Then for every $\beta \in N^1(C^{(d)})$ we have

$$[l] \cdot \Phi^* \beta = \Phi_*([l]) \cdot \beta = 0 = \gamma \cdot \beta.$$  

This proves that $\gamma = 0$.

Let $\gamma \in N_1(\mathcal{D})$. Then we claim that

$$\gamma = \eta_* \Phi_* \gamma + ([\mathcal{O}_\mathcal{D}(1)] \cdot [\gamma - \eta_* \Phi_* [\gamma]]) [l].$$

This can be seen as follows. It is enough to show that $\forall \ D \in N^1(\mathcal{D}),$

$$[D] \cdot \gamma = [D] \cdot ([\eta_* \Phi_* \gamma] + ([\mathcal{O}_\mathcal{D}(1)] \cdot [\gamma]) [D] \cdot [l].$$

By Corollary 13, it is enough to consider the case when $D = \Phi^* D'$ where $D' \in N^1(C^{(d)})$ or $D = \mathcal{O}_\mathcal{D}(1)$. In the first case the statement follows from projection formula and the second case is by definition. This completes the proof of the Corollary 15.  

Let $p_C : C \times \mathcal{D} \to \mathcal{D}$ and $p_\mathcal{D} : C \times \mathcal{D} \to C$ be the projections. Let $\mathcal{R}_\mathcal{D}$ denote the universal quotient on $C \times \mathcal{D}$. For a vector bundle $F$ over $C$, we define

$$B_{F,\mathcal{D}} := \det(p_{\mathcal{D}}_* (\mathcal{R}_\mathcal{D} \otimes p_C^* F)).$$

**Lemma 16.** Suppose we are given a map $f : T \to \mathcal{D}$. Let $(id \times f)^* \mathcal{R}_\mathcal{D} = \mathcal{R}_T$. Let $p_T : C \times T \to T$ and $p_{1,T} : C \times T \to C$ be the projections. 

$$
\begin{array}{cccc}
C \times T & \xrightarrow{id \times f} & C \times \mathcal{D} \\
\downarrow p_T & & \downarrow p_\mathcal{D} \\
T & \xrightarrow{f} & \mathcal{D}
\end{array}
$$

(i) $f^* p_{\mathcal{D}}_* (\mathcal{R}_\mathcal{D} \otimes p_C^* F) \to p_T^* (\mathcal{R}_T \otimes p_{1,T}^* F)$ is an isomorphism.

(ii) For a vector bundle $F$ on $C$ define $B_{F,T} := \det(p_T^* (\mathcal{R}_T \otimes p_{1,T}^* F))$. Then $f^* B_{F,\mathcal{D}} = B_{F,T}.$

**Proof.** For (i) take $\mathcal{R}_\mathcal{D} \otimes p_C^* F$ and use Lemma 8. The assertion (ii) follows from (i) by applying determinant to the isomorphism

$$f^* p_{\mathcal{D}}_* (\mathcal{R}_\mathcal{D} \otimes p_C^* F) \xrightarrow{\sim} p_T^* (\mathcal{R}_T \otimes p_{1,T}^* F).$$

Recall the definition of $\eta$ from equation (7), this is a section of $\Phi$. For a line bundle $L$ on $C$ we have a line bundle $\mathcal{O}_{d,L}$ over $C^{(d)}$ (see [15, page 8] for notation).

**Lemma 17.** Let $\eta$ be defined by a quotient $E \to M \to 0$. Then

$$\eta^* B_{L,\mathcal{D}} \cong \mathcal{O}_{d,L \otimes M}.$$
Thus, using the remark in the preceding para, we get that the same as that of $\mathcal{B}$ by definition, the sheaf $\mathcal{B}$ then the following. Fix $d$. For any two line bundles $L$ and $L'$ over $C$

\[ B_{L, \mathcal{O}} \otimes B_{L', \mathcal{O}}^{-1} = \Phi^* \left( (L \otimes L'^{-1}) \boxtimes d \right). \]

**Proposition 18.** For any two line bundles $L, L'$ over $C$

\[ B_{L, \mathcal{O}} \otimes B_{L', \mathcal{O}}^{-1} = \Phi^* \left( (L \otimes L'^{-1}) \boxtimes d \right). \]

**Proof.** First we show that $B_{L, \mathcal{O}} \otimes B_{L', \mathcal{O}}^{-1} \in \Phi^* \operatorname{Pic}(C^{(d)})$. Since any line bundle over $\mathcal{O}$ is of the form $\mathcal{O}(a) \otimes \phi^* \mathcal{L}$, where $\mathcal{L} \in \operatorname{Pic}(C^{(d)})$, it is enough to show that both $B_{L, \mathcal{O}}$ and $B_{L', \mathcal{O}}$ have the same $\mathcal{O}(1)$ coefficient.

To compute the coefficient of this component of any line bundle over $\mathcal{O}$, we can do the following. Fix $d$ distinct points $c_1, \ldots, c_d \in C$. These define a point $c \in C^{(d)}$. As we saw in the proof of Theorem 11,

\[ \Phi^{-1}(c) = \prod_{i=1}^{d} \mathbb{P}(E_{c_i}). \]

Let $v_i \in \mathbb{P}(E_{c_i})$ for $i \geq 2$. Then we have an embedding:

\[ f : \prod_{i=1}^{d} \mathbb{P}(E_{c_i}) \times \prod_{i \geq 2} \mathbb{P}(E_{c_i}) \to \prod_{i \geq 2} \mathbb{P}(E_{c_i}) = \Phi^{-1}(c). \]

Then the $\mathcal{O}(1)$ coefficient of a line bundle $\mathcal{M}$ over $\mathcal{O}$ is the degree of $f^* \mathcal{M}$ with respect to $\mathcal{O}$. Let $Y = \mathbb{P}(E_{c_1})$. Using Lemma 16, $f^* B_{L, \mathcal{O}} = \det(p_{Y*} (\mathcal{B}_Y \otimes p_{1, Y}^* L))$.

The $v_j \in \mathbb{P}(E_{c_j})$ correspond to quotients $p_j : E \to E_{c_j} \to k_{c_j}$, for $2 \leq j \leq d$. Over $C \times Y$ we have the inclusions $i_j : Y \equiv c_j \times Y \to C \times Y$ for every $1 \leq j \leq d$. We have a map

\[ p_{1, Y}^* E \to \bigoplus_{j=1}^{d} i_{j*} \left( p_{1, Y}^* E|_{c_j \times Y} \right). \]

The bundle $p_{1, Y}^* E|_{c_j \times Y}$ is just the trivial bundle on $Y$, and using $v_j$ we can get quotients $p_{1, Y}^* E|_{c_j \times Y} \to \mathcal{O}_Y$ for $2 \leq j \leq d$. For $j = 1$ we have the quotient $p_{1, Y}^* E|_{c_1 \times Y} \to i_{1*} (\mathcal{O}_Y(1))$. Since the $c_j \times Y$ are disjoint we can put these together to get a quotient on $C \times Y$

\[ p_{1, Y}^* E \to \left( \bigoplus_{j=2}^{d} \bigoplus_{j=2}^{d} i_{j*} \mathcal{O}_Y \right) \bigoplus i_{1*} \mathcal{O}_Y(1). \]

By definition, the sheaf $\mathcal{B}_Y$ is the sheaf in the RHS. Then

\[ \mathcal{B}_Y \otimes p_{1, Y}^* L = \left( \bigoplus_{j=2}^{d} i_{j*} \mathcal{O}_Y \right) \otimes p_{1, Y}^* L \bigoplus i_{1*} \mathcal{O}_Y(1) \otimes p_{1, Y}^* L \]

\[ = \left( \bigoplus_{j=2}^{d} i_{j*} \mathcal{O}_Y \right) \bigoplus i_{1*} \mathcal{O}_Y(1) \]

\[ = \mathcal{B}_Y. \]

Thus, using the remark in the preceding para, we get that the $\mathcal{O}(1)$ coefficient of $B_{L, \mathcal{O}}$ is the same as that of $B_{L', \mathcal{O}}$. Hence $B_{L, \mathcal{O}} \otimes B_{L', \mathcal{O}}^{-1} = \Phi^* \mathcal{L}$. 

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Recall the section $\eta$ of $\Phi$ from equation (7), constructed using some line bundle quotient $E \to M$. Then $\eta^*(B_{L,0} \otimes B_{L,0}^{-1}) = s^* \Phi^* \mathcal{L} = \mathcal{L}$. Now using Lemma 17, we get that $\eta^* B_{L,0} = \mathcal{Q}_{d, L \circ M}$.

By Göttsche’s theorem ( [15, page 9]) we get that $\eta^* B_{L,0} = \mathcal{Q}_{d, L \circ M} = (L \otimes M)^{\otimes d} \otimes \mathcal{O}(-\Delta_d/2)$. Therefore, we get

$$\mathcal{L} = \eta^* \left( B_{L,0} \otimes B_{L,0}^{-1} \right) = (L \otimes L^{-1})^{\otimes d}.$$ 

This completes the proof of the Proposition 18. \hfill \Box

**Corollary 19.** $[B_{L,0}] = [\mathcal{Q}_{d,0}] + \deg(L)[x]$ in $N^1(\mathcal{D})$.

### 4. Upper bound on NEF cone

Let $V$ be a vector space of dimension $n$. From now, unless mentioned otherwise, the notation $\mathcal{D}$ will be reserved for the space $\mathcal{D}(V \otimes \mathcal{O}_C, d)$. Sometimes we will also denote this space by $\mathcal{D}(n, d)$ when we want to emphasize $n$ and $d$.

**Notation**

For the rest of this article, except in section 6, the genus of the curve $C$ will be $g(C) \geq 1$. If $g(C) \geq 2$ then we will also assume that $C$ is very general.

Our aim is to compute the NEF cone of $\mathcal{D}$. Since this cone is dual to the cone of effective curves, it follows that if we take effective curves $C_1, C_2, \ldots, C_r$, take the cone generated by these in $N_1(\mathcal{D})$, and take the dual cone $T$ in $N^1(\mathcal{D})$, then Nef($\mathcal{D}$) is contained in $T$. This gives us an upper bound on Nef($\mathcal{D}$). We already know two curves in $\mathcal{D}$. The first being a line in the fiber of $\Phi : \mathcal{D} \to C^{(d)}$, see Definition 12, which was denoted $[l]$. Recall the section $\eta$ of $\Phi$ from equation (7), taking $L$ to be the trivial bundle. The second curve is $\eta_*([l'])$, where $[l']$ is from Definition 2. Now we will construct a third curve in $\mathcal{D}$.

Define a morphism

$$\dd : C \to \mathcal{D}$$

as follows. Let $p_1, p_2 : C \times C \to C$ be the first and second projections respectively. Let $i : C \to C \times C$ be the diagonal. Fix a surjection $k^n \to k^d$ of vector spaces. Then define the quotient over $C \times C$

$$\Theta^n_{C \times C} \to \Theta^d_{C \times C} \to i_* i^* \Theta^d_{C \times C}.$$ 

This induces a morphism $\dd : C \to \mathcal{D}$ which sends $c \to [\Theta^n_C \to k^d \to 0]$. We will abuse notation and denote the class $[\dd_*, (C)] \in N_1(\mathcal{D})$ by $[\dd]$.

We now give an upper bound for the NEF cone when $n \geq d \geq \text{gon}(C)$.

**Proposition 20.** Consider the Quot scheme $\mathcal{D} = \mathcal{D}(n, d)$. Assume $n \geq d \geq \text{gon}(C)$. Let $\mu_0 := \frac{d+g-1}{dg}$. Then

$$\text{Nef}(\mathcal{D}) \subset \mathcal{R}_{\geq 0} \left[ (\Theta^n_{\mathcal{D}}(1)) + \mu_0 [L_0] \right] + \mathcal{R}_{\geq 0} [\theta_d] + \mathcal{R}_{\geq 0} [L_0].$$

**Proof.** We claim that the cone dual to $(\eta_*([l'])), [\dd])$ is precisely

$$\{ ([\Theta^n_{\mathcal{D}}(1)) + \mu_0 [L_0]), [L_0], [\theta_d] \}.$$ 

We have the following equalities:

1. $(\eta_*([l])) \cdot [l] = 1$. This is clear.
2. $(\eta_*([l])) \cdot \eta_*([l']) = 0$. By projection formula and Lemma 17, we get that

$$\left( (\Theta^n_{\mathcal{D}}(1)) + \mu_0 [L_0] \right) \cdot \eta_*([l']) = \left( \eta_*([l']) = \left( \Theta^n_{\mathcal{D}}(1)) + \mu_0 [L_0] \right) \cdot \eta_*([l']) \right.$$

By Lemma 7 we get that $\left( \Theta^n_{\mathcal{D}}(1)) + \mu_0 [L_0] \right) \cdot \eta_*([l']) = 0$. But as we saw earlier, $[\theta_d] \cdot [l'] = 0$. 

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(3) \((\mathcal{O}_{\mathcal{D}}(1) + \mu_0[L_0]) \cdot [\delta] = 0\). By Lemma 8, it is easy to see that \(\mathcal{O}_{\mathcal{D}}(1) \cdot [\delta] = 0\). By projection formula, we get

\[
(\mathcal{O}_{\mathcal{D}}(1) + \mu_0[L_0]) \cdot [\delta] = \mu_0[L_0] \cdot [\Phi_{*} \delta] = \mu_0[L_0] \cdot [\delta] = 0.
\]

(4) \([\delta_{d}] \cdot [l] = [L_0] \cdot [l] = 0\) follows using the projection formula.

Now the claim follows from Proposition 5. As explained before, since \(\text{Nef}(\mathcal{D})\) is contained in the dual to the cone \(\langle [l], \eta_{*}([l']), \tilde{\delta}\rangle\), the proposition follows.

When the genus \(g = 1\), we have the following improvement of Proposition 20.

**Proposition 21.** Let \(C\) be a smooth projective curve of genus \(g = 1\). Consider the Quot scheme \(\mathcal{D} = \mathcal{D}(n, d)\). Assume \(d \geq \text{gon}(C) = 2\). Then

\[
\text{Nef}(\mathcal{D}) \subset \mathbb{R}_{\geq 0} \left(\mathcal{O}_{\mathcal{D}}(1) + [L_0]\right) + \mathbb{R}_{\geq 0}[\delta_{d}] + \mathbb{R}_{\geq 0}[L_0].
\]

**Proof.** We claim that the cone dual to \(\langle [l], \eta_{*}([l']), \eta_{*}([\delta])\rangle\) is precisely

\[
\langle \left(\mathcal{O}_{\mathcal{D}}(1) + [L_0], [L_0], [\delta_{d}]\right) \rangle.
\]

Let us check that \(\langle \left(\mathcal{O}_{\mathcal{D}}(1) + [L_0]\right), \eta_{*}([\delta]) = 0\). Since \([L_0] \cdot [\delta] = 0\) it is clear that it suffices to check that \(\mathcal{O}_{\mathcal{D}}(1) \cdot \eta_{*}([\delta]) = 0\). Applying the definition of the map \(\eta \circ \delta : C \rightarrow \mathcal{D}\) we see that \(\mathcal{O}_{\mathcal{D}}(1) \cdot \eta_{*}([\delta]) = \text{deg}(p_{2*}(\mathcal{O} / \mathcal{I}^2))\), where \(\mathcal{I}\) is the ideal sheaf of the diagonal in \(E \times E\). Since \(\mathcal{I} \subset \mathcal{I}^2\) is trivial and \(\mathcal{I}^f \subset \mathcal{I}^f + 1 = (\mathcal{I}^f)^{\otimes 2}\), it follows that \(\text{deg}(p_{2*}(\mathcal{O} / \mathcal{I}^2)) = 0\). The rest of the proof is the same as that of Proposition 20.

5. Lower bound on NEF cone

In this section we obtain a lower bound for \(\text{Nef}(\mathcal{D}) (\mathcal{D} = \mathcal{D}(n, d))\).

**Lemma 22.** Let \(f : D \rightarrow \mathcal{D}\) be a morphism, where \(D\) is a smooth projective curve. Fix a point \(q \in f(D)\) and an effective divisor \(A\) on \(C\) containing the scheme theoretic support of \(\mathcal{D}_{q}\). If there is a line bundle \(L\) on \(C\) such that \(H^0(L) \rightarrow H^0(L|_{A})\) is surjective then \([B_{L, \mathcal{D}}] \cdot [D] \geq 0\).

**Proof.** Consider the map

\[
p_{\mathcal{D}*} \left( p_{C}^{*} (V \otimes \mathcal{O}_{C}) \otimes p_{C}^{*} L \right) \rightarrow p_{\mathcal{D}*} \left( \mathcal{B}_{\mathcal{D}} \otimes p_{C}^{*} L \right)
\]

on \(\mathcal{D}\). We claim that this map is surjective at the point \(q\). In view of Lemma 8 when we restrict this map to \(q\), it becomes equal to the map

\[
H^0 (V \otimes L) \rightarrow H^0 (\mathcal{D}_{q} \otimes L).
\]

The map \(V \otimes L \rightarrow \mathcal{B}_{q} \otimes L\) on \(C\) factors as

\[
V \otimes L \rightarrow V \otimes L|_{A} \rightarrow \mathcal{B}_{q} \otimes L.
\]

Taking global sections we see that the map \(H^0 (V \otimes L) \rightarrow H^0 (\mathcal{B}_{q} \otimes L)\) factors as

\[
H^0 (V \otimes L) \rightarrow H^0 (V \otimes L|_{A}) \rightarrow H^0 (\mathcal{B}_{q} \otimes L).
\]

The second arrow is surjective since these are coherent sheaves on a zero dimensional scheme. The first arrow is simply

\[
V \otimes H^0 (L) \rightarrow V \otimes H^0 (L|_{A}).
\]

Since \(H^0 (L) \rightarrow H^0 (L|_{A})\) is surjective by our choice of \(L\), it follows that \(H^0 (V \otimes L) \rightarrow H^0 (\mathcal{B}_{q} \otimes L)\) is surjective, and so it follows that \(p_{\mathcal{D}*} (V \otimes p_{C}^{*} L) \rightarrow p_{\mathcal{D}*} (\mathcal{D}_{2} \otimes p_{C}^{*} L)\) is surjective at the point \(q\).

The rank of the vector bundle \(p_{\mathcal{D}*} (\mathcal{D}_{2} \otimes p_{C}^{*} L)\) on \(\mathcal{D}\) is \(d\). Taking the \(d\)th exterior of \(p_{\mathcal{D}*} (V \otimes p_{C}^{*} L) \rightarrow p_{\mathcal{D}*} (\mathcal{D}_{2} \otimes p_{C}^{*} L)\) we get a map

\[
\bigwedge^{d} (V \otimes H^0 (L)) \rightarrow B_{L, \mathcal{D}}.
\]
This map is nonzero and that can be seen by looking at the restriction to the point \( q \). This shows that there is a global section of \( B_{L,2} \) whose restriction to \( q \) does not vanish. It follows that \( |B_{L,2}| \cdot |D| \geq 0 \). This completes the proof of the Lemma 22.

**Lemma 23.** Let \( A \) be an effective divisor on \( C \) of degree \( d \). Then there is a line bundle \( L \) of degree \( d + g - 1 \) such that the natural map

\[
H^0(L) \rightarrow H^0(L|_A)
\]

is surjective.

**Proof.** It suffices to find a line bundle of degree \( d + g - 1 \) such that \( H^1(L \otimes \mathcal{O}_C(-A)) = 0 \). By Serre duality this is same as saying that \( H^0(L^\vee \otimes K_C \otimes \mathcal{O}_C(A)) = 0 \). The degree of \( L^\vee \otimes K_C \otimes \mathcal{O}_C(A) \) is \( g - 1 \). Thus, fixing \( A \) we may choose a general \( L \) such that \( L^\vee \otimes K_C \otimes \mathcal{O}_C(A) \) line bundle has no global sections. \( \square \)

**Definition 24.** Define \( U \subset \mathcal{Q} \) to be the set of quotients of the form

\[
\mathcal{O}_C^n \twoheadrightarrow \frac{\mathcal{O}_C}{\prod_{i=1}^r m_{C,c_i}^{d_i}} \cong \bigoplus_{c_i \neq c_j} \mathcal{O}_{C,c_i}^{d_i}.
\]

We now prove a lemma, which is implicitly contained [8, Section 5]. Let \( \Sigma \subset C \times C^{(d)} \) denote the closed sub-scheme which is the universal divisor. In the following Lemma we work more generally with \( \mathcal{Q}(E,d) \).

**Lemma 25.** Let \( E \) be a locally free sheaf of rank \( r \) on \( C \). Let \( \mathcal{Q} = \mathcal{Q}(E,d) \) denote the Quot scheme of torsion quotients of length \( d \). The universal quotient \( \mathcal{Q}_2 \) is supported on \( \Phi^* \Sigma \subset C \times \mathcal{Q} \). The set \( U \) is open in \( \mathcal{Q} \). On \( C \times U \) the sheaf \( \mathcal{Q}_2 \) is a line bundle supported on the scheme \( \Phi^* \Sigma \cap (C \times U) \).

**Proof.** Let \( A \) denote the kernel of the universal quotient on \( C \times \mathcal{Q} \)

\[
0 \rightarrow A \xrightarrow{h} p_C^* E \rightarrow \mathcal{Q}_2 \twoheadrightarrow 0.
\]

The map \( \Phi \) is defined taking the determinant of \( h \), that is, using the quotient

\[
0 \rightarrow \det(A) \xrightarrow{\det(h)} p_C^* \det(E) \rightarrow \mathcal{F} \twoheadrightarrow 0.
\]

If \( \mathcal{F}_2 \) denotes the ideal sheaf of \( \Sigma \) then this shows that

\[
\Phi^* \mathcal{F}_2 = \det(A) \otimes p_C^* \det(E)^{-1}.
\]

Let \( 0 \rightarrow E' \xrightarrow{h} E \) be locally free sheaves of the same rank on a scheme \( Y \). Let \( \mathcal{F} \) denote the ideal sheaf determined by \( \det(h) \). Then it is easy to see that \( \mathcal{F} E \subset h(E') \subset E \). Applying this we get that \( (\Phi^* \mathcal{F}_2) p_C^* E \subset A \). This proves that \( \mathcal{Q} \) is supported on \( \Phi^* \Sigma \). Let us denote by \( Z := \Phi^* \Sigma \subset C \times \mathcal{Q} \). Consider the closed subset \( Z_2 \subset Z \) defined as follows

\[
Z_2 := \left\{ z = (c, q) \in Z \mid \text{rank}_k (\mathcal{Q}_2 \otimes k(z)) \geq 2 \right\}.
\]

Then the image of \( Z_2 \) in \( \mathcal{Q} \) is closed and \( U \) is precisely the complement of \( Z_2 \). This proves that \( U \) is open in \( \mathcal{Q} \).

Let \( R \) be a local ring with maximal ideal \( m \) and let \( R \rightarrow S \) be a finite map. Let \( M \) be a finite \( S \) module, which is flat over \( R \) and such that \( M/mM \cong S/mS \). Then it follows easily that \( M \cong S \).

Let \( q \in U \subset \mathcal{Q} \) be a point. The sheaf \( \mathcal{Q}_2 \) is a coherent sheaf supported on \( Z \), the map \( Z \twoheadrightarrow \mathcal{Q} \) is finite, the fiber

\[
\mathcal{Q}_q = \bigoplus_{c_i} \frac{\mathcal{O}_{C,c_i}}{m_{C,c_i}^{d_i}} \cong \mathcal{O}_Z|_q \cong \mathcal{O}_Z|_q.
\]

From the preceding remark it follows that \( \mathcal{Q}_2 \) is a line bundle over \( Z \cap (C \times U) \). \( \square \)
Lemma 26. Consider the Quot scheme $\mathcal{Q} = \mathcal{Q}(n, d)$. Let $D$ be a smooth projective curve and let $\mathcal{Q} \xrightarrow{f} \mathcal{Q}$ be a morphism such that its image intersects $U$. Then $([\mathcal{Q}_\mathcal{Q}(1)] + [\Delta_d/2]) \cdot [D] \geq 0$.

Proof. Denote by $\mathcal{B}_D$ the pullback of the universal quotient over $C \times \mathcal{Q}$ to $C \times D$. Denote by $i_D : \Gamma \to C \times D$ the pullback of the universal subscheme $\Sigma \to C \times C^{(d)}$ to $C \times D$. Then $\mathcal{B}_D$ is supported on $\Gamma$.

Let $\Gamma_i$ be the irreducible components of $\Gamma$. Since $\Gamma \to D$ is flat each $\Gamma_i$ dominates $D$. Let $f : \Gamma \to D$ denote the projection. There is an open subset $U_i \subset D$ such that

$$f^{-1}(U_i) = \bigsqcup_i \Gamma_i \cap f^{-1}(U_i)$$

and $\mathcal{B}_D$ restricted to $f^{-1}(U_i)$ is a line bundle. Note that by $\Gamma_i \cap f^{-1}(U_i)$ we mean this open subscheme of $\Gamma$. Fix a closed point $x_i \in \Gamma_i \cap f^{-1}(U_i)$. Consider the quotient

$$V \otimes \mathcal{O}_{C \times D} \to \mathcal{B}_D$$

and restrict it to the point $x_i$. We get a quotient

$$V \to \mathcal{B}_D \otimes k(x_i) \to 0.$$

If we pick a general line in $V$, then it surjects onto $\mathcal{B}_D \otimes k(x_i)$. Thus, for the general element $s \in V$, $s \otimes \mathcal{O}_{C \times D}$ surjects onto $\mathcal{B}_D \otimes k(x_i)$. This map factors through $\mathcal{O}_\Gamma$, and we get an exact sequence

$$0 \to \mathcal{O}_\Gamma \to \mathcal{B}_D \to F \to 0$$

where $F$ is supported on a 0 dimensional scheme. Then we have

$$0 \to f_* \mathcal{O}_\Gamma \to f_* \mathcal{B}_D \to f_* F \to 0.$$

Since $f_* F$ is again supported on finitely many points, hence we have

$$\deg(f_* \mathcal{B}_D) - \deg(f_* \mathcal{O}_\Gamma) \geq 0$$

By Lemma 8, $\deg(f_* \mathcal{B}_D) = [\mathcal{Q}_\mathcal{Q}(1)] \cdot [D]$ and by [15, § 3] we have

$$\deg(f_* \mathcal{O}_\Gamma) = [\mathcal{O}(-\Delta_d/2)] \cdot [D].$$

Hence the result follows. \qed

Corollary 27. If the image of $f : D \to \mathcal{Q}$ intersects $U$, then $([\mathcal{Q}_\mathcal{Q}(1)] + \mu_0[L_0]) \cdot [D] \geq 0$.

Proof. If its image intersects $U$, then by Lemma 26,

$$([\mathcal{Q}_\mathcal{Q}(1)] + [\Delta_d/2]) \cdot [D] \geq 0.$$ By Lemma 7, 

$$[\Delta_d/2] = \mu_0[L_0] - (1 - \mu_0)[\theta_d].$$

Since $\theta_d$ is nef, we have that

$$([\mathcal{Q}_\mathcal{Q}(1)] + \mu_0[L_0]) \cdot [D] \geq 0.$$ \qed

Lemma 28. Consider the Quot scheme $\mathcal{Q} = \mathcal{Q}(n, d)$. Let $D$ be a smooth projective curve and let $f : D \to (\mathcal{Q} \setminus U) \subset \mathcal{Q}$ be a morphism. Then $([\mathcal{Q}_\mathcal{Q}(1)] + (d + g - 2)[x]) \cdot [D] \geq 0$.

Proof. Fix a point $q \in f(D)$. Let $A$ be the scheme theoretic support of the quotient $\mathcal{Q}_q$ on $C$. Let $\deg(A) = d'$. Since $q \notin \mathcal{Q}$, we have $d' < d$. By Lemma 23 we have a line bundle $L$ of degree $d' + g - 1$ such that $H^0(L) \to H^0(L_A)$ is surjective. By Lemma 22 and Corollary 19 we get that $[B_{\mathcal{Q}, \mathcal{Q}(1)}] \cdot [D] = ([\mathcal{Q}_\mathcal{Q}(1)] + (d' + g - 1)[x]) \cdot [D] \geq 0$. Since $[x]$ is nef on $\mathcal{Q}$ and $d' \leq d - 1$ we get that $([\mathcal{Q}_\mathcal{Q}(1)] + (d + g - 2)[x]) \cdot [D] \geq 0$. \qed

Proposition 29. Consider the Quot scheme $\mathcal{Q} = \mathcal{Q}(n, d)$. Let $\mu_0 = \frac{d + g - 1}{dg}$. Then the class $\kappa_1 := [\mathcal{Q}_\mathcal{Q}(1)] + \mu_0[L_0] + \frac{d + g - 2}{dg}[\theta_d]$ is nef.
Proof. Let \( D \to \mathcal{M} \) be a morphism, where \( D \) is a smooth projective curve. If the image of this morphism intersects \( U \) then by Lemma 26 we have \( [\theta_d] \cdot [D] \geq 0 \). Now assume \( D \to \mathcal{M} \) does not intersect \( U \). Then by Lemma 28 we get
\[
([\theta_d] + (d + g - 2)[x]) \cdot [D] \geq 0.
\]
By Lemma 7 we have \( x = \frac{1}{d} \theta_d \). Therefore
\[
(d + g - 2)[x] = \frac{d + g - 2}{d} \theta_d.
\]
Since \( [\theta_d] \) is nef, we get
\[
([\theta_d] + (d + g - 2)[x]) \cdot [D] \geq 0.
\]

Lemma 30. Let \( L \) be a line bundle on \( C \) of degree \( d + g - 1 \). If \( d \geq \text{gon}(C) \) then the line bundle \( B_{L,2} \) is not ample. Moreover, for any \( t \in [0,1] \) the class \( t[B_{L,2}] + (1-t)[\theta_d] \) is nef but not ample.

Proof. We saw in the last para of the proof of Proposition 18 that \( \eta^* B_{L,2} = L^{\otimes d} \otimes \mathcal{O}(-\Delta_d/2) \). Its class in the nef cone is \( (d + g - 1)[x] - [\Delta_d/2] \). It follows from Lemma 7 that this is equal to \([\theta_d] \). Since \( d \geq \text{gon}(C) \) we have \( \theta_d \) is not ample on \( C(d) \). That \( t[B_{L,2}] + (1-t)[\theta_d] \) is nef is clear since both \([B_{L,2}] \) and \([\theta_d] \) are nef. This is not ample since \( \eta^* \) of this class is \([\theta_d] \) on \( C(d) \), which is not ample.

Proposition 31. Consider the Quot scheme \( \mathcal{Q} = \mathcal{Q}(n, d) \). Then the class \( [\mathcal{Q}] + (d + g - 1)[x] \in N^1(\mathcal{Q}) \) is nef.

Proof. It is easily checked that the class \([\mathcal{Q}] + (d + g - 1)[x] \) can be written as a positive linear combination of \([\theta_d] \) and the class in Proposition 29.

We may slightly improve Proposition 31 in a special case using the results in [15]. For this we first recall the main results in [15, § 4]. Let \( C \) be a very general curve of genus \( g(C) = 2k \). Since the gonality is given by \( \lceil \frac{g + 2}{2} \rceil \), in this case it is \( k + 1 \). Let \( L_i \) denote the finitely many \( g_{k+1}^i \)'s on \( C \) and define \( L_i = K_C - L_i \). Then \( \deg(L_i) = 3(k - 1) \). It is proved in [15, Proposition 3.6, Theorem 4.1] that \( \mathcal{Q}_{L, L_i} \) is nef but not ample.

Proposition 32. Let \( C \) be a very general curve of genus \( g(C) = 2k \). Consider the Quot scheme \( \mathcal{Q} = \mathcal{Q}(n, k) \). The line bundle \( B_{L,2} \) is nef when \( \deg(L) \geq 3(k - 1) \). When \( \deg(L) = 3(k - 1) \) the class \( t[B_{L,2}] + (1-t)[\mathcal{Q}] \) is nef but not ample for any \( t \in [0,1] \).

We remark that this is an improvement since Proposition 31 only shows that \( B_{L,2} \) is nef when \( \deg(L) \geq 3k - 1 \).

Proof. It follows from Proposition 18 that the class of \( B_{L,2} \) in \( N^1(\mathcal{Q}) \) is \([\mathcal{Q}] + \deg(L)[x] \), since \( B_{\theta,C,2} = \mathcal{Q}(1) \). Notice that this class only depends on the degree of \( L \). Since the sum of nef line bundles is nef, it suffices to show that \([B_{L,2}] = [\mathcal{Q}] + \deg(L)[x] \) is nef when \( \deg(L) = 3(k - 1) \).

The set \( V(\mathcal{Q}(L)) \) is defined in equation [15, equation (18)]. Then (A) in [15, Theorem 4.1] says that for every \( A \in C^{(k)} \) there is an \( L_i \) such that \( H^0(C, L_i) \to H^0(C, L_i|A) \) is surjective.
Let $\mathcal{D} : D \rightarrow \Delta$ be a morphism, where $D$ is a smooth projective curve. Fix a point $q \in f(D)$. Let $A$ be the divisor corresponding to $\Phi(q)$, then $A$ is an effective divisor of degree $k$. For this $A$, choose a line bundle $L_i$ such that

$$H^0(C, L_i) \rightarrow H^0(C, L_i|_A)$$

is surjective. The scheme theoretic support of $B$ is contained in $A$. It follows from Lemma 22 that

$$f^* B_{L_i, \Delta} = f^* ([\mathcal{O}_\Delta(1)] + 3(k-1)[x]) = 0.$$

It follows that $B_{L, \Delta}$ is nef.

Note that

$$\eta^* B_{L, \Delta} = \eta^* [\mathcal{O}_\Delta(1)] + \deg(L) \eta^* [x] = [\mathcal{O}(-\Delta_k/2)] + 3(k-1)[x] = [\mathcal{O}_k, L].$$

Thus, when $t \in [0, 1]$ the pullback along $\eta$ of $t[B_{L, \Delta} + (1-t)[\mathcal{O}_k, L]]$ is $[\mathcal{O}_k, L]$, which is not ample. \qed

6. The genus 0 case

Throughout this section we will work with $C = \mathbb{P}^1$. Let us first compute the nef cone of $\Delta(n, d)$.

Note that we have $C^{(d)} = \mathbb{P}^d$. Hence $N^1(C^{(d)}) = \mathbb{N}[\mathcal{O}_{pd}(1)]$. By Corollary 13 it follows that $N^1(\Delta)$ is two dimensional. Hence, it suffices to find a line bundle on $\Delta$ which is different from the pullback of $\mathcal{O}_{pd}(1)$ and which is nef but not ample. The following result is proved in [16, Theorem 6.2], but we include it for the benefit of the reader.

**Proposition 33.**

$$\text{Nef}(\Delta(n, d)) = \mathbb{R}_{\geq 0} \left[ B_{\mathcal{O}(d-1), \Delta} \right] \cup \mathbb{R}_{\geq 0} \left[ \mathcal{O}_{pd}(1) \right]$$

$$= \mathbb{R}_{\geq 0} \left[ (\mathcal{O}_\Delta(1) + (d-1) \mathcal{O}_{pd}(1)) \right] + \mathbb{R}_{\geq 0} \left[ \mathcal{O}_{pd}(1) \right].$$

**Proof.** Let $W := H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$. There is a natural isomorphism $\mathbb{P} W^* \cong C^{(d)}$. The universal subscheme $\Sigma \subset \mathbb{P}^1 \times \mathbb{P} W^*$ is given by the tautological section

$$p^*_2 \mathcal{O}_{\mathbb{P} W^*} \cong C^{(d)}$$

$$p^*_2 \mathcal{O}_{\mathbb{P} W^*} \cong C^{(d)}$$

By Lemma 22 and Lemma 23 we get that $B_{\mathcal{O}(d-1), \Delta}$ is nef. To show $B_{\mathcal{O}(d-1), \Delta}$ is not ample, consider a section $\eta : C^{(d)} \rightarrow \Delta$ constructed as in (7) with $L$ the trivial bundle. Let $p_i$ denote the two projections from $\mathbb{P}^1 \times \mathbb{P} W^*$. By definition and Lemma 16 it follows that $\eta^* B_{\mathcal{O}(d-1), \Delta} = \det(p_{2*}(\mathcal{O}_\Sigma \otimes p^*_1 \mathcal{O}_{\mathbb{P}^1}(d-1)))$. Tensoring the exact sequence

$$0 \rightarrow p^*_1 \mathcal{O}_{\mathbb{P}^1}(d-1) \otimes p^*_1 \mathcal{O}_{\mathbb{P} W^*}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P} W^*} \rightarrow \mathcal{O}_\Sigma \rightarrow 0$$

with $p^*_1 \mathcal{O}_{\mathbb{P}^1}(d-1)$ and applying $p_{2*}$ it easily follows that $p_{2*}(\mathcal{O}_\Sigma \otimes p^*_1 \mathcal{O}_{\mathbb{P}^1}(d-1))$ is the trivial bundle and so $\eta^* B_{\mathcal{O}(d-1), \Delta}$ is trivial. This proves that $B_{\mathcal{O}(d-1), \Delta}$ is nef but not ample.

By restricting to a fiber of $\Phi$ and using Corollary 19 we see that $[B_{\mathcal{O}(d-1), \Delta}]$ is linearly independent from $[\mathcal{O}_{pd}(1)]$. This completes the proof of the first equality. The second equality will follow from the first equality once we show that

$$[B_{\mathcal{O}(d-1), \Delta}] = [\mathcal{O}_\Delta(1) + (d-1) \mathcal{O}_{pd}(1)].$$

By Corollary 19, we have that $[B_{\mathcal{O}(d-1), \Delta}] = [\mathcal{O}_\Delta(1) + (d-1)[x]]$. Now recall that given $x \in \mathbb{P}^1$, $[x]$ is the class of the divisor in $C^{(d)}$ whose underlying set consists of effective divisors of degree $d$ containing $x$ (see (4)). Hence, $[x]$ is the class of the hyperplane section

$$\mathbb{P} \left\{ H^0(\mathbb{P}^1, \mathcal{O}(d) \otimes \mathcal{O}(-x))^* \right\} \subset \mathbb{P} \left\{ H^0(\mathbb{P}^1, \mathcal{O}(d))^* \right\} = C^{(d)}.$$  

Therefore $[x] = [\mathcal{O}_{pd}(1)]$ and this completes the proof of the second equality. \qed
Theorem 34. Let $C = \mathbb{P}^1$. Let $E = \bigoplus_{i=1}^{k} \mathcal{O}(a_i)$ with $a_i \leq a_j$ for $i < j$. Let $d \geq 1$. Let $L = \mathcal{O}(-a_1 + d - 1)$. Then
\[
\text{Nef}(\mathcal{Q}(E, d)) = \mathbb{R}_{\geq 0} \left[ B_{L, \mathcal{Q}(E, d)} \right] + \mathbb{R}_{\geq 0} \left[ \mathcal{O}_{pd}(1) \right] = \mathbb{R}_{\geq 0} \left[ \left( \mathcal{Q}_{pd}(1) \right) + \left( -a_1 + d - 1 \right) \left[ \mathcal{O}_{pd}(1) \right] \right] + \mathbb{R}_{\geq 0} \left[ \mathcal{O}_{pd}(1) \right].
\]

Proof. By Corollary 13 we get that $N!^1(\mathcal{Q}(E, d))$ is 2-dimensional. Hence it is enough to give two line bundles which are nef but not ample. Clearly $\Phi_{pd}(1)$ is nef but not ample. So it is enough to show that $B_{L, \mathcal{Q}(E, d)}$ is nef but not ample.

Since $a_j - a_i \geq 0, \forall j \geq 1$, we get that $E(-a_1)$ is globally generated. Let $V := H^0(C, E(-a_1))$ and let $\dim V = n$. Then we have a surjection $V \otimes \mathcal{O}_C \rightarrow E(-a_1)$. Then gives us a surjection $V \otimes \mathcal{O}_C \rightarrow p_C^*E(-a_1) \rightarrow \mathcal{Q}(E, d) \otimes p_C^*\mathcal{O}_C(-a_1) \rightarrow 0$.

This defines a map $f : \mathcal{Q}(E, d) \rightarrow \mathcal{Q}(n, d)$. By Lemma 16 we get that
\[
f^*B_{\mathcal{Q}(d-1), \mathcal{Q}(n, d)} = B_{L, \mathcal{Q}(E, d)} = \det \left( p_{pd}(E, d) \right) \left( \mathcal{Q}(E, d) \otimes p_C^*L \right).
\]

Since $B_{\mathcal{Q}(d-1), \mathcal{Q}(n, d)}$ is nef we get that $B_{L, \mathcal{Q}(E, d)}$ is nef. We next show that the $B_{L, \mathcal{Q}(E, d)}$ is not ample. Consider the section $\eta_{\mathcal{Q}(E, d)}$ of $\Phi_{pd}(1) : \mathcal{Q}(E, d) \rightarrow C^{(d)}$ defined by the quotient $p_C^*E \rightarrow p_C^*\mathcal{O}(a_1) \otimes \mathcal{O}_\Sigma$ on $C \times C^{(d)}$ (see (7)). Then $f \circ \eta_{\mathcal{Q}(E, d)}$ is a section of $\Phi : \mathcal{Q}(n, d) \rightarrow C^{(d)}$ defined by a quotient $\mathcal{O}_C^n \rightarrow \mathcal{O}_\Sigma \rightarrow 0$ on $C \times C^{(d)}$. Therefore $\eta_{\mathcal{Q}(E, d)}B_{L, \mathcal{Q}(E, d)} = \eta^*B_{\mathcal{Q}(d-1), \mathcal{Q}(n, d)}$. As $\eta^*B_{\mathcal{Q}(d-1), \mathcal{Q}(n, d)}$ is not ample, we get that $B_{L, \mathcal{Q}(E, d)}$ is not ample. The second equality follows again from the fact that $[\alpha] = [\mathcal{O}_{pd}(1)]$. 

7. Some cases of equality

Now we are back to the assumption that the genus of the curve satisfies $g(C) \geq 1$ and if $g(C) \geq 2$ then we also assume that $C$ is very general.

Definition 35. Let $U' \subset \mathcal{Q}$ be the open set consisting of quotients $\mathcal{O}_C^n \rightarrow B \rightarrow 0$ such that the induced map $H^0(C, \mathcal{O}_C^n) \rightarrow H^0(C, B)$ is surjective.

Lemma 36. Consider the Quot scheme $\mathcal{Q} = \mathcal{Q}(n, d)$. Let $D$ be a smooth projective curve and let $D \rightarrow \mathcal{Q}$ be a morphism such that its image intersects $U'$. Then $[\mathcal{Q}(1)] \cdot [D] \geq 0$.

Proof. We continue with the notations of Lemma 26. Let $p_D : C \times D \rightarrow D$ be the projection. Then applying $(p_D)_*$ to the quotient $\mathcal{O}_C^n \rightarrow \mathcal{B}_D$ we get that the morphism
\[(p_D)_*\mathcal{O}_C^n = \mathcal{O}_D^n \rightarrow (p_D)_*\mathcal{B}_D\]
is generically surjective by our assumption and Lemma 8. Hence we get that
\[|\mathcal{Q}(1)| \cdot [D] = \deg \left( (p_D)_*\mathcal{B}_D \right) \geq 0. \]

One extremal ray in Nef$(C^{(2)})$ is given by $L_0$. Let other extremal ray of Nef$(C^{(2)})$ be given by
\[\alpha_t = (t + 1)x - \Delta_2/2, \]
(see [12, page 75]). Then using Lemma 7, we get that
\[\Delta_2/2 = \frac{t + 1}{g + t}L_0 - \frac{g - 1}{g + t}\alpha_t. \]

Theorem 37. Let $d = 2$. Consider the Quot scheme $\mathcal{Q} = \mathcal{Q}(n, 2)$. Then
\[\text{Nef}(\mathcal{Q}) = \mathbb{R}_{\geq 0} \left( [\mathcal{Q}(1)] + \frac{t + 1}{g + t}[L_0] \right) + \mathbb{R}_{\geq 0}[L_0] + \mathbb{R}_{\geq 0}[\alpha_t]. \]
Proof. We first prove that \([\mathcal{O}_\mathcal{D}(1) + \frac{t+1}{g+t}L_0]\) is nef. Since \(d = 2\), then there are only three types of quotients:

1. \(\mathcal{O}_C^n \rightarrow \frac{\mathcal{O}_C \oplus \mathcal{O}_C}{m_c c_1} \) with \(c_1 \neq c_2\),
2. \(\mathcal{O}_C^n \rightarrow \frac{\mathcal{O}_C c_2}{m_c c_1}\),
3. \(\mathcal{O}_C^n \rightarrow \frac{\mathcal{O}_C c_1}{m_c c_1} \oplus \frac{\mathcal{O}_C c_2}{m_c c_2}\).

The first two quotients are in \(U\) while the third one is in \(U'\), that is, we get \(U \cup U' = \mathcal{D}\). Now let \(D\) be a smooth projective curve and \(D \to \mathcal{D}\) be a morphism. If its image intersects \(U\), then by Corollary 27, \([\mathcal{O}_\mathcal{D}(1)] + \frac{t+1}{g+t}L_0 + \Delta_2/2 \cdot [D] \geq 0\). Using (10) and the fact that \(\alpha_t\) is nef, we get that \([\mathcal{O}_\mathcal{D}(1)] + \frac{t+1}{g+t}L_0 \cdot [D] \geq 0\). If \(D\) does not intersect \(U\) then \(D \subset U'\). Hence by Lemma 36, we have

\([\mathcal{O}_\mathcal{D}(1)] \cdot [D] \geq 0\).

Since \([L_0]\) is nef we have that

\[\left(\mathcal{O}_\mathcal{D}(1) + \frac{t+1}{g+t}L_0\right) \cdot [D] \geq 0.\]

Also \([\mathcal{O}_\mathcal{D}(1)] + \frac{t+1}{g+t}L_0\) \([\delta] = 0\). Hence any convex linear combination of \([\mathcal{O}_\mathcal{D}(1)] + \frac{t+1}{g+t}L_0\) and \([L_0]\) is nef but not ample. Hence the result follows. \(\square\)

Precise values for \(t\) depending on \(g\) are known when

1. When \(g = 1, t = 1\).
2. When \(g = 2, t = 2\).
3. When \(g = 3, t = 9/5\).
4. When \(g\) is a perfect square \(t = \sqrt{g}\), see [11, Theorem 2].
5. In [5, Proposition 3.2], when \(g \geq 9\), assuming the Nagata conjecture, they prove that \(t = \sqrt{g}\).

Thus, in all these cases using Theorem 37 we get the Nef cone of \(\mathcal{D}(n, 2)\).

7.1. Criterion for nefness

In the remainder of this section, we will need to work with \(C^{(d)}\) for different values of \(d\). The line bundles \(L_0\) on \(C^{(d)}\) will therefore be denoted by \(L_0^{(d)}\) when we want to emphasize the \(d\). Similarly, we will denote \(\mu_0 = \frac{d+g-1}{dg}\). Let \(\mathcal{D}_n\) be the set of all partitions \((d_1, d_2, \ldots, d_k)\) of \(d\) of length at most \(n\). Given an element \(d \in \mathcal{D}_n\) define

\[C^{(d)} := C^{(d_1)} \times C^{(d_2)} \times \ldots \times C^{(d_k)}\]

and if \(p_1 : C^{(d)} \to C^{(d_i)}\) is the \(i^{th}\) projection we define a class

\[\mathcal{O}(-\Delta_d/2) := [\sum p_i^* \mathcal{O}(-\Delta_d/2)] \in N^1(C^{(d)})\]

Note that we have a natural addition

\[\pi_d : C^{(d)} \to C^{(d_i)}\]

For a partition \(d \in \mathcal{D}^{\leq n}\) define a morphism

\[\eta_d : C^{(d)} \to \mathcal{D}\]
as follows. For any $l \geq 1$, we define the universal subscheme of $C^{(l)}$ over $C \times C^{(l)}$ by $\Sigma_l$. Then over $C \times C^{(d)}$ we have the subschemes $(id \times p_i)^* \Sigma_d$. We have a quotient

$$q_d : \mathcal{O}_{C \times C^{(d)}} \rightarrow \bigoplus_i \mathcal{O}_{(id \times p_i,d)\times \Sigma_d}$$

defined by taking direct sum of morphisms $\mathcal{O}_{C \times C^{(d)}} \rightarrow \mathcal{O}_{(id \times p_i,d)\times \Sigma_d}$. Then $q_d$ defines a map $C^{(d)} \rightarrow \mathcal{O}$. By Lemma 16, we have

$$[\eta^*_d \mathcal{O} (1)] = [\mathcal{O} (-\Delta_d/2)].$$

**Lemma 38.** Let $D$ be a smooth projective curve. Let $D \rightarrow \mathcal{O}$ be a morphism. Then there exists a partition $\mathbf{d} \in \mathcal{P}_{\mathcal{O}}$ such that the composition $D \rightarrow \mathcal{O} \rightarrow C^{(d)}$ factors as $D \rightarrow C^{(d)} \rightarrow C^{(d)}$ and $[\mathcal{O} (1)] \cdot [D] \geq [\mathcal{O} (-\Delta_d/2)] \cdot [D]$.

**Proof.** We will proceed by induction on $d$. When $d = 1$ the statement is obvious.

Let us denote the pullback of the universal quotient on $C \times \mathcal{O}$ to $C \times D$ by $\mathcal{B}_D$ and let $f : C \times D \rightarrow D$ be the natural projection. Consider a section such that the composite $\mathcal{O}_{C \times D} \rightarrow \mathcal{O}_{C \times D} \rightarrow \mathcal{B}_D$ is non-zero and let $\mathcal{F}$ denote the cokernel of the composite map. We have a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_{C \times D} & \longrightarrow & \mathcal{O}_{C \times D}^{n-1} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\Gamma'} & \longrightarrow & \mathcal{B}_D & \longrightarrow & \mathcal{F} & \longrightarrow & 0
\end{array}
$$

Let $T_0(\mathcal{F}) \subset \mathcal{F}$ denote the maximal subsheaf of dimension 0, see [10, Definition 1.1.4]. Define $\mathcal{F}' := \mathcal{F} / T_0(\mathcal{F})$. Now, either $\mathcal{F}' = 0$ or $\mathcal{F}'$ is torsion free over $D$, and hence, flat over $D$. In the first case, it follows that $D$ meets the open set $U$ in Lemma 26. Then we take $\mathbf{d} = (d)$ and the statement follows from Lemma 26. So we assume $\mathcal{F}'$ is flat over $D$ and let $d'$ be the degree of $\mathcal{F}'|_{C \times x}$, for $x \in D$. So $0 < d' < d$. By (12) we have

$$\deg f_* \mathcal{B}_D = \deg f_* \mathcal{O}_{\Gamma'} + \deg f_* \mathcal{F}.$$ 

Since $T_0(\mathcal{F})$ is supported on finitely many points, we have $\deg \mathcal{F} \geq \deg \mathcal{F}'$. In other words, we have

$$\deg f_* \mathcal{B}_D \geq \deg f_* \mathcal{O}_{\Gamma'} + f_* \mathcal{F}'.$$ 

(13)

Now $\Gamma'$ defines a morphism $D \rightarrow C^{(d-d')}$. And note that

$$\deg f_* \mathcal{O}_{\Gamma'} = [\mathcal{O} (-\Delta_d/2)] \cdot [D].$$

The quotient $\mathcal{O}_{C \times D}^{n-1} \rightarrow \mathcal{F}' \rightarrow 0$ defines a map $D \rightarrow \mathcal{O}(n-1, d')$. By induction hypothesis, we get that there exists a partition $\mathbf{d}' \in \mathcal{P}_{\mathcal{O}}^{n-1}$ such that the composition $D \rightarrow \mathcal{O}(n-1, d') \rightarrow C^{(d')}$ factors as $D \rightarrow C^{(d')} \rightarrow C^{(d')}$. Then

$$[\mathcal{O}(n-1, d') (1)] \cdot [D] \geq [\mathcal{O} (-\Delta_{d'}/2)] \cdot [D].$$

Since $\deg f_* \mathcal{F}' = [\mathcal{O}(n-1, d') (1)] \cdot [D]$ we have that $\deg f_* \mathcal{F}' \geq [\mathcal{O} (-\Delta_{d'}/2)] \cdot [D]$. From (13) we get that

$$[\mathcal{O}(1)] \cdot [D] \geq [\mathcal{O} (-\Delta_{d-d'}/2)] \cdot D + [\mathcal{O} (-\Delta_{d'}/2)] \cdot [D].$$

Now we define $\mathbf{d} := (d-d', d')$ and the statement follows from the above inequality. □

**Theorem 39.** Let $\beta \in N^1(C^{(d)})$. Then the class $[\mathcal{O}(1)] + \beta \in N^1(\mathcal{O})$ is nef iff the class $[\mathcal{O} (-\Delta_d/2)] + \pi_d \beta \in N^1(C^{(d)})$ is nef for all $\mathbf{d} \in \mathcal{P}_{\mathcal{O}}^{n-1}$.
Moreover, \( \Phi \) have the curve \( \Phi \).  

Proof. From (11) it is clear that if \( \mathcal{O}_\mathcal{D}(1) + \beta \) is nef, then \( \eta_\mathcal{D}(\mathcal{O}_\mathcal{D}(1) + \beta) = \mathcal{O}(-\Delta_\mathcal{D}/2) + \pi_\mathcal{D} \cdot \beta \) is nef.

For the converse, we assume \( \mathcal{O}(-\Delta_\mathcal{D}/2) + \pi_\mathcal{D} \cdot \beta \) is nef for all \( \mathcal{D} \in \mathcal{D}^{\leq n} \). Let \( D \) be a smooth projective curve and \( \mathcal{D} \rightarrow \mathcal{D} \) be a morphism. By Lemma 38 we have that there exists \( \mathcal{D} \in \mathcal{D}^{\leq n} \) such that \( D \rightarrow C^{(d)} \) factors as \( D \rightarrow C^{(d)} \rightarrow C^{(d)} \) and 

\[
\mathcal{O}_\mathcal{D}(1) \cdot [D] \geq \mathcal{O}(-\Delta_\mathcal{D}/2) \cdot [D].
\]

Now by assumption we have that 

\[
\mathcal{O}(-\Delta_\mathcal{D}/2) \cdot [D] \geq -\beta \cdot [D].
\]

Therefore we get 

\[
\mathcal{O}_\mathcal{D}(1) \cdot [D] \geq -\beta \cdot [D].
\]

Hence we get that the class \( \mathcal{O}_\mathcal{D}(1) + \beta \) is nef. \( \square \)

Lemma 40. Suppose we are given a map \( D \rightarrow C^{(d)} \rightarrow C^{(d)} \). Then we have 

\[
\mathcal{L}_0^{(d)} \cdot [D] \geq \sum \mathcal{L}_0^{(d)} \cdot [D].
\]

Proof. By \( \mathcal{L}_0^{(d)} \cdot [D] \) we mean the degree of the pullback of \( \mathcal{L}_0^{(d)} \) along \( D \rightarrow C^{(d)} \rightarrow C^{(d)} \). The lemma follows easily from the definition of \( \mathcal{L}_0^{(d)} \) and is left to the reader. \( \square \)

Proposition 41. Let \( n \geq 1 \), \( g \geq 1 \) and \( \mathcal{D} = \mathcal{O}(n,d) \). Then the class \( \kappa_2 := \mathcal{O}_\mathcal{D}(1) + \frac{g+1}{2g} \mathcal{L}_0^{(d)} \in \mathcal{N}^{1}(\mathcal{D}) \) is nef. As a consequence we get that 

\[
\text{Nef}(\mathcal{D}) \supset \mathbb{R}_{\geq 0} \kappa_1 + \mathbb{R}_{\geq 0} \kappa_2 + \mathbb{R}_{\geq 0} \{\theta_d\} + \mathbb{R}_{\geq 0} \{\mathcal{L}_0^{(d)}\}.
\]

Proof. Recall \( \mu_0^{(2)} = \frac{g+1}{2g} \). By Theorem 39 it suffices to show that for all \( \mathcal{D} \in \mathcal{D}^{\leq n} \) we have \( \mathcal{O}(-\Delta_\mathcal{D}/2) + \mu_0^{(2)} \pi_\mathcal{D}^{(d)} \mathcal{L}_0^{(d)} \) is nef. Using Lemma 7, \( \mathcal{L}_0^{(1)} = 0 \) and Lemma 40 we get

\[
\left( \mathcal{O}(-\Delta_\mathcal{D}/2) + \mu_0^{(2)} \pi_\mathcal{D}^{(d)} \mathcal{L}_0^{(d)} \right) \cdot [D] = \left( \sum_i \left( 1 - \mu_i^{(d)} \right) [\theta_d] - \mu_i^{(d)} [\mathcal{L}_0^{(d)}] \right) \cdot [D] + \mu_0^{(2)} [\mathcal{L}_0^{(d)}] \cdot [D] \geq \sum_i \left( \mu_i^{(2)} - \mu_i^{(d)} \right) [\mathcal{L}_0^{(d)}] \cdot [D].
\]

This proves that \( \kappa_2 \) is nef. That \( \kappa_1 \) is nef is proved in Proposition 29. This completes the proof of the theorem. \( \square \)

Corollary 42. Let \( n \geq d \). Then the class \( \mathcal{O}_\mathcal{D}(1) + \mu_0^{(2)} \mathcal{L}_0^{(d)} \in \mathcal{N}^{1}(\mathcal{D}) \) is nef but not ample.

Proof. By Proposition 41 we have that \( \mathcal{O}_\mathcal{D}(1) + \mu_0^{(2)} \mathcal{L}_0^{(d)} \) is nef. Now recall that when \( n \geq d \) we have the curve \( \delta \rightarrow \mathcal{D} \) (8). From the definition of \( \delta \) and Lemma 16 we have \( \mathcal{O}_\mathcal{D}(1) \cdot [\delta] = 0 \). Also \( \Phi_\delta \delta = \delta \). Hence \( \mathcal{L}_0^{(d)} \cdot [\delta] = \mathcal{L}_0^{(d)} \cdot [\delta] = 0 \). From this we get \( \mathcal{O}_\mathcal{D}(1) + \mu_0^{(2)} \mathcal{L}_0^{(d)} \cdot [\delta] = 0 \) and hence \( \mathcal{O}_\mathcal{D}(1) + \mu_0^{(2)} \mathcal{L}_0^{(d)} \) is not ample. \( \square \)

As a corollary we get the following result. When \( g = 1 \) note that \( \mu_0^{(2)} = 1 \).

Theorem 43. Let \( g = 1 \), \( n \geq 1 \) and \( \mathcal{D} = \mathcal{O}(n,d) \). Then the class \( \mathcal{O}_\mathcal{D}(1) + [\Delta_d/2] \in \mathcal{N}^{1}(\mathcal{D}) \) is nef. Moreover,

\[
\text{Nef}(\mathcal{D}) = \mathbb{R}_{\geq 0} \left( \mathcal{O}_\mathcal{D}(1) + [\Delta_d/2] \right) + \mathbb{R}_{\geq 0} [\theta_d] + \mathbb{R}_{\geq 0} [\Delta_d/2].
\]
8. Curves over the small diagonal

Throughout this section the genus of the curve $C$ will be $g(C) \geq 2$ and $C$ is a very general curve. Recall that $\Phi : \mathcal{O} \to C^{(d)}$ is the Hilbert–Chow map.

**Proposition 44.** Let $f : D \to \mathcal{O}(n, d)$ be such that $\Phi \circ f$ factors through the small diagonal. Then $|\mathcal{O}_{\mathcal{O}}(1)| \cdot |D| \geq 0$.

**Proof.** Since $\Phi \circ f$ factors through the small diagonal, there is a map $g : D \to C$ such that if $\Gamma := \Gamma_g$ denotes the graph of $g$ in $C \times D$, and $\mathcal{O}_{C \times D} ^n \to \mathcal{B}_D$ is the quotient on $C \times D$, then $\mathcal{B}_D$ is supported on $\mathcal{O}_{C \times D} / \mathcal{I}(\Gamma)^n$. Denote $\mathcal{I} := \mathcal{I}(\Gamma)$. Then $\mathcal{B}_D / \mathcal{I} \mathcal{B}_D$ is a globally generated sheaf on $D$ and so its determinant has degree $\geq 0$. Now consider the sheaf

$$\mathcal{I}^i \mathcal{B}_D / \mathcal{I}^{i+1} \mathcal{B}_D \cong (\mathcal{I} / \mathcal{I}^2)^{\otimes i} \otimes \mathcal{B}_D / \mathcal{I} \mathcal{B}_D.$$  

Using adjunction it is easily seen that $\mathcal{I} / \mathcal{I}^2 \cong g^* \omega_C$. Since $\det(\mathcal{B}_D / \mathcal{I} \mathcal{B}_D)$ has degree $\geq 0$, it follows that $\det(\mathcal{I}^i \mathcal{B}_D / \mathcal{I}^{i+1} \mathcal{B}_D)$ has degree $\geq 0$. From the filtration

$$\mathcal{B}_D \supset \mathcal{I} \mathcal{B}_D \supset \mathcal{I}^2 \mathcal{B}_D \supset \ldots \supset \mathcal{I}^d \mathcal{B}_D = 0$$

we easily conclude that $|\mathcal{O}_{\mathcal{O}}(1)| \cdot |D| \geq 0$.

**Lemma 45.** Let $D \to C^{(d)}$ be a morphism. Then we can find a cover $\tilde{D} \to D$ such that the composite $\tilde{D} \to D \to C^{(d)}$ factors through $C^d$.

**Proof.** Let $D_1$ be a component of $D \times C^{(d)}$ which dominates $D$. Take $\tilde{D}$ to be a resolution of $D_1$.

**Corollary 46.** Let $D \to \mathcal{O}$ be a morphism. Replacing $D$ by a cover $\tilde{D}$ we may assume that the map $\tilde{D} \to D \to \mathcal{O} \to C^{(d)}$ factors through $C^d$.

In view of the above, given a map $D \to Q$ we may assume that the composite $D \to \mathcal{O} \to C^{(d)}$ factors through $C^d$. Let each component be given by a map $f_i : D \to C$. Denote by $i_D : \Gamma \hookrightarrow C \times D$ the pullback of the universal subscheme $\Sigma \hookrightarrow C \times C^{(d)}$ to $C \times D$. The ideal sheaf of $\Gamma$ is the product $\mathcal{I}(\Gamma_f)$, the ideal sheaves of the graphs $\Gamma_f \subset C \times D$. Moreover, $\mathcal{B}_D$ is supported on $\Gamma$. Let $g_1, g_2, \ldots, g_r$ be the distinct maps in the set $\{f_1, f_2, \ldots, f_d\}$ and assume that $g_i$ occurs $d_i$ many times. Then we have $\mathcal{I}(\Gamma) = \prod_{i=1}^r \mathcal{I}(\Gamma_{g_i})^{d_i}$. There is a natural map

$$\psi : \mathcal{B}_D \to \bigoplus_{i} \mathcal{B}_D / \mathcal{I}(\Gamma_{g_i})^{d_i} \mathcal{B}_D.$$ 

**Lemma 47.** Let $f : D \to \mathcal{O}$ be such that $\Phi \circ f$ factors through $C^d \to C^d$. If $\psi$ is an isomorphism then $|\mathcal{O}_{\mathcal{O}}(1)| \cdot |D| \geq 0$.

**Proof.** Since $\mathcal{B}_D$ is a quotient of $\mathcal{O}_{C \times D} ^n$ it follows that each $\mathcal{B}_D / \mathcal{I}(\Gamma_{g_i})^{d_i} \mathcal{B}_D$ is a quotient of $\mathcal{O}_{C \times D} ^n$. Thus, each $\mathcal{B}_D / \mathcal{I}(\Gamma_{g_i})^{d_i} \mathcal{B}_D$ defines a map $D \to \mathcal{O}(n, d_i)$ such that the image under the map $\Phi : \mathcal{O}(n, d_i) \to C^{(d_i)}$ is the small diagonal. By Proposition 44 it follows that degree of $\det(p_{D^*}(\mathcal{B}_D / \mathcal{I}(\Gamma_{g_i})^{d_i} \mathcal{B}_D))$ is $\geq 0$. Since $\psi$ is an isomorphism it follows that degree of $\det(p_{D^*}(\mathcal{B}_D))$ is $\geq 0$.

We can use the above method to prove a result similar to Theorem 37 when $d = 3$.

**Corollary 48.** Let $d = 3$. Consider the Quot scheme $\mathcal{Q} = \mathcal{Q}(n, 3)$. Let $\mu_0^{(3)} = \frac{g+2}{3g}$. Then $|\mathcal{O}_{\mathcal{O}}(1)| + \mu_0^{(3)} |\mathcal{Q}^{(3)}|$ is nef.

**Proof.** If $d = 3$ there are only these types of quotients:

1. $\mathcal{O}_{C} / \mathcal{O}_{C} / \mathcal{M}_{c_1} \mathcal{M}_{c_2} \mathcal{M}_{c_3}$,
(2) $\mathcal{O}_C^n \rightarrow \mathcal{O}_{C, c_1} / \mathfrak{m}_{C, c_1} \oplus \mathcal{O}_C / \mathfrak{m}_{C, c_1} \mathfrak{m}_{C, c_2}$,

(3) $\mathcal{O}_C^n \rightarrow \frac{\mathcal{O}_{C, c}}{\mathfrak{m}_{C, c}} \oplus \frac{\mathcal{O}_{C, c}}{\mathfrak{m}_{C, c}} \oplus \frac{\mathcal{O}_{C, c}}{\mathfrak{m}_{C, c}}$.

Let $f : D \rightarrow \mathcal{D}$ be a map. If $D$ contains a quotient of type (1) or (3) then $D$ meets $U$ or $U'$ (see Definition 24 and Definition 35). Thus, in these cases $(|\mathcal{O}_{\mathcal{D}}(1)| + \mu_0^{(3)} [L_0^{(3)}]) \cdot [D] \geq 0$ by Corollary 27 and Lemma 36.

Now consider the case when all points in the image of $D$ are of type (2). After replacing $D$ by a cover, using Corollary 46, we may assume that the map $D \rightarrow \mathcal{D}$ factors through $C^3$. Since the images of points of $D$ represent quotients of type (2), we may assume that the map from $D \rightarrow C^3$ looks like $d \rightarrow (g_1(d), g_1(d), g_2(d))$. Now consider a general section $\mathcal{O}_{C \times D} \rightarrow \mathcal{B}_D$. Arguing as in the proof of Lemma 26 we get a diagram as in equation (12), such that $\mathcal{O}_{\mathcal{D}}$ defines a map $D \rightarrow C^{(2)}$ and $\mathcal{F}' = \mathcal{F} / T_0(\mathcal{F})$ is a line bundle on $D$ which is globally generated. Hence

$$[\mathcal{O}_{\mathcal{D}}(1) \cdot [D] \geq \mathcal{O}(-\Delta_2/2) \cdot [D] + \left[ c_1 \left( p_{D*}(\mathcal{F}) \right) \right] \cdot [D]$$

$$\geq -\mu_0^{(2)} \left[ L_0^{(2)} \right] \cdot [D].$$

One easily checks using the definition of $L_0$ that in this case $L_0^{(3)} \cdot [D] = 2[L_0^{(2)}] \cdot [D]$. Thus,

$$\left( |\mathcal{O}_{\mathcal{D}}(1)| + \mu_0^{(2)} \left[ L_0^{(3)} \right] \right) \cdot [D] \geq \left[ 2\mu_0^{(3)} - \mu_0^{(2)} \right] \left[ L_0^{(2)} \right] \cdot [D] \geq 0.$$

This completes the proof of the Corollary 48. □

Combining this with Proposition 20 we get the following result.

**Theorem 49.** Let $C$ be a very general curve of genus $2 \leq g(C) \leq 4$. Let $n \geq 3$ and let $\mathcal{D} = \mathcal{D}(n, 3)$. Let $\mu_0 = \frac{g + 2}{3g}$ Then

$$\text{Nef}(\mathcal{D}) = \mathbb{R}_{>0} \left( |\mathcal{O}_{\mathcal{D}}(1)| + \mu_0 \left[ L_0^{(3)} \right] \right) + \mathbb{R}_{\geq 0} [\theta_d] + \mathbb{R}_{\geq 0} \left[ L_0^{(3)} \right].$$

References


