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Projective bundles and blowing ups

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\textbf{Abstract.} We study the blowing up $\tilde{X}$ of a smooth projective variety $X$ along a smooth center $B$ that is equipped with a projective bundle structure over a variety $Z$. If $B$ is a point, then $X$ is a projective space. If the Picard number $\rho(X)$ is 1, then $\dim Z = \dim X - \dim B - 1$. Moreover, when $\dim Z = \dim X - \dim B - 1$, $X$ is a projective space and $B$ is a linear subspace in $X$. If $X$ is a projective space $\mathbb{P}_n$ and $B$ is a curve, then either $n = 3$ and $B$ is a twisted cubic curve or $n$ is an arbitrary integer and $B$ is a line in $\mathbb{P}_n$. If $X$ is a quadric $Q_n$ and $B$ is a curve, then $n$ is 3 and $B$ is a line in $Q_3$.

\textbf{Résumé.} Nous étudions l’éclatement $\tilde{X}$ d’une variété projective lisse $X$ le long d’un centre lisse $B$, munie d’une structure de fibré projectif. Si $B$ est un point, $X$ est un espace projectif. Si le nombre de Picard $\rho(X)$ est 1, alors $\dim Z = \dim X - \dim B - 1$. De plus, lorsque $\dim Z = \dim X - \dim B - 1$, $X$ est un espace projectif et $B$ est un sous-espace linéaire dans $X$. Si $X$ est l’espace projectif $\mathbb{P}_n$ et $B$ est une courbe, ou $n$ est égale à 3 et $B$ est une courbe cubique tordue, ou $n$ est un entier arbitraire et $B$ est une ligne droite dans $\mathbb{P}_n$. Si $X$ est une quadrique et $B$ est une courbe, alors $n$ est égale à 3 et $B$ est une ligne droite dans $Q_3$.

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1. Introduction

Let $L$ be a linear subspace of dimension $l$ in a projective space $\mathbb{P}_n$. We consider the rational map $\pi_L : \mathbb{P}_n \dashrightarrow \Gamma$ given by the linear projection from $L$ to $\Gamma$, where $\Gamma$ is a linear subspace of dimension $n - l - 1$ disjoint from $L$. Then the blowing up $\text{Bl}_L(\mathbb{P}_n)$ of $\mathbb{P}_n$ along $L$ is the graph of $\pi_L$. More precisely, $\text{Bl}_L(\mathbb{P}_n) \subseteq \mathbb{P}_n \times \Gamma$ is the closed subvariety of $\mathbb{P}_n \times \Gamma$ defined as $\text{Bl}_L(\mathbb{P}_n) = \{(p, q) \in \mathbb{P}_n \times \Gamma \mid p \in \langle L, q \rangle\}$, where $\langle L, q \rangle$ is the linear subspace generated by $L$ and $q$. The projection $p_2 : \text{Bl}_L(\mathbb{P}_n) \rightarrow \Gamma$ to the second factor is a projective bundle. Actually, for any point $q$ in $\Gamma$, the fiber $p_2^{-1}(q)$ is the linear space $\langle L, q \rangle$. So any blowing up of a projective space along a linear subspace is equipped with a projective bundle structure. It is interesting to find more examples of blowing ups that are equipped with projective bundle structures. In [1, Section 4] and [2], there are examples of blowing ups of projective spaces along non-linear subvarieties that are equipped with projective bundle structures.

In this article, we aim to prove some classification results about $(X, B)$ (where $X$ and $B$ are projective smooth varieties and $B$ is a subvariety of $X$) such that the blowing up $\text{Bl}_B(X)$ is a...
projective bundle over some variety $Z$. In Proposition 1, we show that if $B$ is a point, then $X$ is a projective space. In Theorem 4, we show that if the Picard number $\rho(X)$ is 1 and $\dim Z = \dim X - \dim B - 1$, then $X$ is a projective space and $B$ is a linear subspace in $X$. Finally, in Theorem 6, we show that if $X$ is $\mathbb{P}_n$ and $B$ is a curve, then either $n$ is 3 and $B$ is a twisted cubic curve or $n$ is an arbitrary positive integer and $B$ is a line in $\mathbb{P}_n$; if $X$ is a quadric $Q_n (n \geq 3)$ and $B$ is a curve, then $n$ is 3 and $B$ is a line in $Q_3$.

**Convention.** A complex variety is an irreducible integral scheme of finite type defined over $\mathbb{C}$. In this article, all varieties considered are complex projective varieties.

### 2. Main results

**Setup.** We assume that $X$ is a smooth projective variety of dimension $n$ and we denote the blowing up $\widetilde{X}$ of $X$ along a closed smooth subvariety $B$ by $\varphi : \widetilde{X} \to X$. Assume that $\pi : \widetilde{X} \to Z$ is a projective bundle over a variety $Z$. We summarize all the morphisms in a diagram as follows:

\[
\begin{array}{c}
\widetilde{X} \\
\downarrow \varphi \\
X
\end{array}
\quad \rightarrow 
\begin{array}{c}
Z \\
\downarrow \pi \\
\end{array}
\]

We introduce some notations here: we let $E$ be the exceptional divisor of $\varphi$, let $F_1$ be a line in a fiber $\pi^{-1}(Z)$ (we have $\pi^{-1}(Z) = \mathbb{P}_{n-\dim Z}$) and let $F_2$ be a line in a fiber $\varphi|_E^{-1}(b)$ (we have $\varphi|_E^{-1}(b) = \mathbb{P}_{n-\dim B-1}$).

In the remaining part of this section, we keep the assumptions in the Setup and we keep using the notations there.

We firstly consider the blowing up at a point that is equipped with a projective bundle structure.

**Proposition 1.** If $B$ is a point, then $X$ is a projective space $\mathbb{P}_n$.

**Proof.** We consider the morphism $\pi|_E : E \to Z$. Since $E$ is a projective space $\mathbb{P}_{n-1}$, the morphism $\pi|_E : E \to Z$ is quasi-finite. Otherwise, $\pi|_E$ will contract $E$ to a single point. So $\dim Z = n-1$ and hence $\pi|_E$ is surjective. Note that $Z$ is smooth and projective, by the Lazarsfeld’s theorem in [5], $Z$ is a projective space $\mathbb{P}_{n-1}$. By [4, Lemma V.3.7.8], $X$ is a projective space. \(\square\)

In the remaining part of this section, we keep an additional assumption that the Picard number $\rho(X)$ of $X$ is 1.

Now let us prove a lemma about the lower bound of $\dim Z$.

**Lemma 2.** The morphism $\pi|_E : E \to Z$ is surjective and the varieties $Z$, $\widetilde{X}$, $X$ are Fano varieties. Moreover, $\dim Z$ is at least $n - \dim B - 1$. If $\dim Z = n - \dim B - 1$, then $Z$ is a projective space.

**Proof.** Note that $E$ is a divisor in $\widetilde{X}$. The codimension $\text{codim}(\pi(E), Z)$ is at most 1. If $\text{codim}(\pi(E), Z)$ is 1, the divisor $E$ is the pull-back of $\pi(E)$. Since the Picard number of $Z$ is 1, $E$ is a nef divisor, which contradicts to the fact that $E$ is covered by negative curves. Then $\pi|_E$ is surjective. Now $Z$ is a uniruled variety whose Picard number is 1, so $Z$ is Fano.

To prove $\widetilde{X}$ is Fano, we have the canonical bundle formulas:

\[-K_{\widetilde{X}} = \varphi^*(-K_X) - (n - \dim B - 1)E = \pi^*(-K_Z - \text{det} \mathcal{E}) + \mathcal{O}_\pi(n - \dim Z + 1)\]

where $\mathcal{O}_\pi(1)$ is the tautological line bundle of $\pi$. Since the Picard number of $\widetilde{X}$ is 2, the cone $NE(\widetilde{X})$ is generated by two extremal rays as $\mathbb{R}_{\geq 0}[F_1] + \mathbb{R}_{\geq 0}[F_2]$, where $[F_i](i = 1, 2)$ are the numerically...
equivalent classes of $F_l$. By calculating the intersection numbers of $-K_X$ with $F_1$ and $F_2$, we deduce that $\bar{X}$ is Fano. Since $X$ is a uniruled variety whose Picard number is 1, $X$ is Fano.

Let $Y \cong \mathbb{P}_{n-dim B - 1}$ be a fiber of $\varphi|_E : E \to B$. The morphism $\pi|_Y : Y \to Z$ is quasi-finite onto its image. Otherwise, there would be a curve $C_Y$ contracted by $\pi$. But $C_Y$ is numerically equivalent to a positive multiple of a line in $Y$, then the extremal ray of $NE(\bar{X})$ spanned by $|F_2|$ is contracted by $\pi$, which is impossible. So dim $Z$ is at least $n - dim B - 1$. Moreover, if dim $Z$ is $n - dim B - 1$, $\pi|_Y$ is surjective. By the Lazarsfeld's theorem in [5], $Z$ is a projective space. \hfill \Box

Now we prove a simple lemma about the intersection numbers, which is very useful in the proof of our main results.

**Lemma 3.** Let $H_1$ be the pull-back of the ample generator $H_Z$ of Pic($Z$) and let $H_2$ be the pull-back of the ample generator $H_X$ of Pic($X$). We denote by $a, b, c, d$ the intersection numbers $H_1 \cdot F_2$, $\varnothing(1) \cdot F_2$, $H_2 \cdot F_1$, $E \cdot F_1$ in the following diagram of intersection numbers:

<table>
<thead>
<tr>
<th></th>
<th>$H_1$</th>
<th>$\varnothing(1)$</th>
<th>$H_2$</th>
<th>$E$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_1$</td>
<td>0</td>
<td>1</td>
<td>c</td>
<td>d</td>
</tr>
<tr>
<td>$F_2$</td>
<td>a</td>
<td>b</td>
<td>0</td>
<td>-1</td>
</tr>
</tbody>
</table>

Then $a$ equals to $c$ and $a$ divides $1 + bd$.

**Proof.** The torsion free abelian group Pic($\bar{X}$) has two bases $(H_1, \varnothing(1))$ and $(H_2, E)$ which satisfy the relation $\begin{pmatrix} H_1 \\ \varnothing(1) \end{pmatrix} = \begin{pmatrix} da/c \\ 1+bd/c \end{pmatrix} \begin{pmatrix} H_2 \\ E \end{pmatrix}$.

The matrix $A = \begin{pmatrix} da/c \\ 1+bd/c \end{pmatrix}$ is an element of $SL_2(\mathbb{Z})$, so $c$ divides $1 + bd$ and the determinant $\det(A) = \frac{c}{d}$ is 1. \hfill \Box

In the remaining part of this article, we keep using the notations in Lemma 3.

If dim $Z$ is $n - dim B - 1$, we have the following classification.

**Theorem 4.** Let dim $B$ be $m$. If dim $Z$ is $n - m - 1$, then $X$ is a projective space $\mathbb{P}_n$, $B \cong \mathbb{P}_m (\subseteq \mathbb{P}_n)$ is a linear subspace in $\mathbb{P}_n$ and $\pi \circ \varphi^{-1} : \mathbb{P}_n \to \mathbb{P}_n$ is the linear projection from $B$.

**Proof.** Let $R_z$ be the fiber $\pi^{-1}(z) (\cong \mathbb{P}_{m+1})$ of some general point $z \in Z$. Then the intersection $Y_z = E \cap R_z$ is a hypersurface of degree $d$ in $R_z$. We claim that $d$ must equal 1 (the proof of this claim mainly follows from [7, Lemma 2.1]). Actually, suppose that $d$ is at least 2, then for a general line $l$ in $R_z$, the intersection $l \cap Y_z$ consists of at least two distinct points $y_1$ and $y_2$. Since $\varphi|_{R_z}$ is quasi-finite onto its image, $\varphi(Y_z)$ is $B$. So we can assume that $b_i = \varphi(y_i) (i = 1, 2)$ are distinct points in $B$. By varying $R_z$ and $Y_z$, we can construct a one-dimensional family of lines $\{l_t\}_{t \in C}$ in $\bar{X}$ such that the intersection of every $l_t$ with each $\varphi^{-1}(b_i)$ is not empty. Thus the surface $S = \cup_{t \in C} l_t$ is a ruled surface. Let $\varphi^{-1}(b_i) \cap S be C_i$. The curves $C_i (i = 1, 2)$ satisfy $C_1 \cap C_2 = \emptyset$ and $\varphi(C_i) = b_i$. By the construction of $S$, $S$ is not contained in $E$. So $\varphi|_S$ is a birational morphism. Hence $C_i (i = 1, 2)$ are exceptional curves, which is impossible.

Now $d$ is 1. Note that $E$ doesn’t contain any fiber of $\pi$. Otherwise, there will be a morphism $\varphi|_{R_z} : R_z \to B$ where $R_z \cong \mathbb{P}_{m+1}$ is some fiber of $\pi$. Then $\varphi$ contracts $R_z$ to a point, which is impossible. Hence for any $z \in Z$, the intersection $E \cap R_z = Y_z \cong \mathbb{P}_m$ is a linear subspace in $R_z$. Since $\varphi(Y_z)$ is $B$, the variety $B$ is a projective space. Then $E$ has two projective bundle structures over projective spaces. So by [6, Theorem A], the morphism $(\varphi|_E, \pi|_E) : E \to B \times Z$ is an isomorphism.

Hence $\varphi|_Y : Y \to B$ is an isomorphism. So the intersection number $H_2 \cdot F_1 (= H_1 \cdot F_2)$ equals to 1. Suppose that $\pi : \bar{X} \to Z$ is given by $[a H_2 - \beta E]$. We have equalities $H_1 \cdot F_2 = (a H_2 - \beta E) \cdot F_2 = 1$ and $H_1 \cdot F_1 = (a H_2 - \beta E) \cdot F_1 = 0$. So both $a$ and $\beta$ equal to 1. Note the identities $m + 2 = -K_X \cdot F_2 = i_X - (n - m - 1)$ (where $i_X$ is the index of $X$). We deduce that $i_X$ is $n + 1$. By [3, Corollary of...
Theorem 1.1], $X$ is a projective space. Since $\dim H^0(\tilde{X}, \mathcal{O}_X(H_2 - E)) = \dim H^0(X, I_B(1))$ is at least $n - m$, the subvariety $B$ is a linear subspace in $X \cong \mathbb{P}_n$. \hfill $\square$

When $B$ is a smooth curve, we have the following criterion.

**Proposition 5.** If $B$ is a smooth curve and $n$ is at least 3, the following conditions are equivalent:

1. $\pi$ maps $F_2$ birationally to a line in $Z$,
2. $\varphi$ maps $F_1$ birationally to a line in $X$,
3. $(X, B)$ is one of the following cases:
   a. $(\mathbb{P}_n, \text{line}) (\pi \circ \varphi^{-1} : \mathbb{P}_n \dashrightarrow \mathbb{P}_{n-2} \text{ is the linear projection from } B).
   b. $(\mathbb{P}_3, \text{twisted cubic curve}) (\pi \circ \varphi^{-1} : \mathbb{P}_3 \dashrightarrow \mathbb{P}_2 \text{ is given by sections of } |\mathcal{O}_{\mathbb{P}_3}(2)| \text{ vanishing along } B).
   c. $(Q_3, \text{line}) (\pi \circ \varphi^{-1} : Q_3 \dashrightarrow \mathbb{P}_2 \text{ is the linear projection from } B).

**Proof.** By Lemma 3, conditions (1) and (2) are equivalent. It is obvious that condition (3) implies conditions (1) and (2).

Now suppose condition (1) or (2) holds. By Lemma 2, dim $Z$ is at least $n - 2$. If dim $Z$ is $n - 2$, then by Theorem 4, $(X, B)$ is case (a).

If dim $Z$ is $n - 1$, then $\pi$ is a $\mathbb{P}_1$-bundle. So we have identities $-K_{\tilde{X}} \cdot F_1 = i_X - d(n - 2) = 2$. Then $i_X$ is $2 + d(n - 2)$, which is at most $n + 1$. Hence there are only two possibilities: when $d$ is 2, $n$ is 3 and $i_X$ is 4; when $d$ is 1, $i_X$ is $n$.

If $d$ is 2, then it is easy to see that $X$ is $\mathbb{P}_3$ and $Z$ is $\mathbb{P}_2$. Note that $\pi$ is given by the linear system $|2H_2 - E|$. Then dim $H^0(\mathbb{P}_3, I_B(2))$ is at least 3. We claim that $B$ is not a plane curve. Otherwise, there exists a plane $L$ in $\mathbb{P}_3$ containing $B$ and the strict transform $\tilde{L}$ of $L$ is numerically equivalent to $H_2 - kE$ for some positive integer $k$. So we have equalities $1 = H_2 \cdot F_1 = \tilde{L} \cdot F_1 + kE \cdot F_1 = \tilde{L} \cdot F_1 + 2k$, which is impossible. If deg $B$ is 4, then $B$ is a complete intersection of two quadrics and dim $H^0(\mathbb{P}_3, I_B(2))$ is 2. So deg $B$ is 3.

If $d$ is 1, then $i_X$ is $n$, hence by [3, Corollary of Theorem 2.1], $X$ is a quadric $Q_n$. We consider the morphism $\pi|_E : E \to Z$. Since $d$ is 1, $\pi|_E$ is a birational morphism. Note that $\pi$ is given by the linear system $|2H_2 - E|$. So dim $H^0(Q_3, I_B(1))$ is at least $n$, which implies that $B$ is a line. The exceptional divisor $E$ of $\varphi$ is isomorphic to $\mathbb{P}_1(\mathcal{O} \oplus \mathcal{O}(1)^{\oplus(n-2)})$. So the birational morphism $\pi|_E$ contracts the minimal section of $\varphi|_E$ corresponding to the surjection $\mathcal{O} \oplus \mathcal{O}(1)^{\oplus(n-2)} \to \mathcal{O} \to 0$. Since $Z$ is smooth, the exceptional locus of $\pi|_E$ should be a divisor, which implies the equality $n - 2 = 1$. So $X$ is $Q_3$ and $B$ is a line in $Q_3$. \hfill $\square$

Now let us prove another main result of this section.

**Theorem 6.** Assume that $X$ is $\mathbb{P}_n$ and $B$ is a curve, then either $n$ is 3 and $B$ is a twisted cubic curve or $n$ is an arbitrary integer and $B$ is a line in $\mathbb{P}_n$. Assume that $X$ is $Q_n$ and $B$ is a curve, then $n$ is 3 and $B$ is a line in $Q_3$.

**Proof.** If $X$ is $\mathbb{P}_n$, there are equalities: $1 = H_2^n = \varphi^*(H_X^n) = (-bH_1 + a\mathcal{O}_X(1))^n = (-b)^n H_1^n + a(\sum_{k=0}^{n-1} \mathcal{O}_X^n H_1^{n-k} C_n^k \cdot \mathcal{O}_X(n-k))$. Since $H_1^n$ vanishes, the integer $a$ divides 1, hence $a$ is 1. Then by Proposition 5, either $n$ is 3 and $B$ is a twisted cubic curve or $n$ is an arbitrary integer and $B$ is a line in $\mathbb{P}_n$.

If $X$ is $Q_3$, then there are equalities: $2 = H_2^3 = \varphi^*(H_X^3) = (-bH_1 + a\mathcal{O}_X(1))^3 = a(\sum_{k=0}^{n-1} \mathcal{O}_X^3 H_1^{3-k} C_3^k \cdot \mathcal{O}_X(3-k))$. Suppose that $a$ is 2, then $\sum_{k=0}^{n-1} \mathcal{O}_X^3 H_1^{3-k} C_3^k \cdot \mathcal{O}_X(3-k)$ is 1. Assume that $\mathcal{E}$ is a vector bundle such that $\pi : \tilde{X} \to Z$ is the projection of $\mathcal{E}$. If $\text{rk} \mathcal{E}$ is 2, then by the canonical bundle formulas:

$$-K_{\tilde{X}} = \varphi^*(-K_X) - (n - 2)E = \pi^*(-K_Z - \text{det} \mathcal{E}) + \text{rk} \mathcal{E} \cdot \mathcal{O}_Z(1),$$

we have $(i_X H_2 - (n - 2)E) \cdot F_2 = (i_Z - \text{deg} \mathcal{E}) H_2 + 2\mathcal{O}_Z(1) \cdot F_2$. By Lemma 3, $n$ equals to $2(i_Z - \text{deg} \mathcal{E} + b + 1)$, hence $n$ is an even number. Then $\sum_{k=0}^{n-1} \mathcal{O}_X^3 H_1^{3-k} C_3^k \cdot \mathcal{O}_X(3-k)$ is $(n-bH_1)^{n-k}$.
\[ \mathcal{O}_X(1) + a \sum_{k=0}^{n-2} a^{n-k-2} C_k^n (-bH_1)^k \cdot \mathcal{O}_X(1)^{n-k}, \] which is an even number. So \( \text{rk}\mathcal{E} \) is 3. By Theorem 4, \( X \) should be a projective space, which is impossible. So \( a = 1 \). By Proposition 5, \( X \) is \( Q_3 \) and \( B \) is a line in \( Q_3 \).

\[ \square \]

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