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Partial differential equations / *Équations aux dérivées partielles*

Characterization of balls via solutions of the modified Helmholtz equation

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Abstract. A theorem characterizing analytically balls in the Euclidean space \mathbb{R}^m is proved. For this purpose positive solutions of the modified Helmholtz equation are used instead of harmonic functions applied in previous results. The obtained Kuran type theorem is based on the volume mean value property of solutions to this equation.

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1. Introduction and main result

In 1972 Kuran [5] proved the following inverse of the volume mean value theorem for harmonic functions:

Let D be a domain (= connected open set) of finite (Lebesgue) measure in the Euclidean space \mathbf{R}^m where $m \geq 2$. Suppose that there exists a point P_0 in D such that, for every function h harmonic in D and integrable over D , the volume mean of h over D equals $h(P_0)$. Then D is an open ball (disk when $m = 2$) centred at P_0 .

The result was originally obtained by Epstein [2] for a simply connected two-dimensional D . Armitage and Goldstein [1] proved this result assuming that the mean value equality holds only for positive harmonic functions which are L^p -integrable, $p \in (0, n/(n-2))$. Hansen and Netuka [3] considered some particular class of potentials as the set of test harmonic functions in Kuran's theorem. A slight modification of his considerations shows that Kuran's theorem is valid even if D is disconnected; see [8, p. 377].

In the survey article [8], one finds also a discussion of applications of Kuran's theorem and a possibility of similar results involving some kinds of average over ∂D , where D is a bounded domain. One of them (due to Kosmodem'yanskii [4]) is based on the relation similar to that between the mean values over balls and spheres and reads as follows:

Let $D \subset \mathbb{R}^2$ be a bounded, convex C^2 -domain. If the equality

$$\frac{1}{|D|} \int_D u(x) \, dx = \frac{1}{|\partial D|} \int_{\partial D} u(x) \, dS_x$$

holds for every function $u \in C^2(D) \cap C^1(\overline{D})$ which is harmonic in D , then D is an open disc.

Here and below $|D|$ is the domain's area (volume if $D \subset \mathbb{R}^m$, $m \geq 3$), whereas $|\partial D|$ is the boundary's length (area if $D \subset \mathbb{R}^m$, $m \geq 3$), and $|B_r| = \omega_m r^m$ is the volume of a ball B_r of radius r ; here $\omega_m = 2\pi^{m/2} / [m\Gamma(m/2)]$ is the volume of unit ball, whereas Γ denotes the Gamma function.

In this note, we prove new analytic characterization of balls. Like Kuran's theorem, it is based, on the m -dimensional volume mean value equality, but instead of harmonic functions uses solutions of the modified Helmholtz equation:

$$\nabla^2 u - \lambda^2 u = 0, \quad \lambda \in \mathbb{R} \setminus \{0\}; \tag{1}$$

$\nabla = (\partial_1, \dots, \partial_m)$ is the gradient operator and $\partial_i = \partial/\partial x_i$. Solutions are assumed to be real; indeed, the obtained results can be extended to complex-valued functions by considering the real and imaginary part separately.

Before giving the precise formulation of the main result, let us introduce some notation. By $B_r(x) = \{y : |y - x| < r\}$ we denote the open ball of radius r centred at $x \in \mathbb{R}^m$; if $D \subset \mathbb{R}^m$ is a domain of finite Lebesgue measure and a function f is integrable over D , then

$$M(f, D) = \frac{1}{|D|} \int_D f(x) \, dx$$

is its volume mean value over D . Also, we need the following function

$$a(t) = \Gamma\left(\frac{m}{2} + 1\right) \frac{I_{m/2}(t)}{(t/2)^{m/2}}, \tag{2}$$

where I_ν stands for the modified Bessel function of order ν . The relation

$$[z^{-\nu} I_\nu(z)]' = z^{-\nu} I_{\nu+1}(z) \quad (\text{see [12, p. 79]}), \tag{3}$$

where the right-hand side is positive for $z > 0$ and vanishes at $z = 0$, implies that the function a increases monotonically on $[0, \infty)$ from $a(0) = 1$ to infinity; the latter is a consequence of the asymptotic formula valid as $|z| \rightarrow \infty$:

$$I_\nu(z) = \frac{e^z}{\sqrt{2\pi z}} [1 + O(|z|^{-1})], \quad |\arg z| < \pi/2 \quad (\text{see [12, p. 80]}).$$

The function a arises in the m -dimensional mean value formula for balls

$$a(\lambda r) u(x) = M(u, B_r(x)), \quad x \in D, \tag{4}$$

which holds, for example, if $u \in C^0(\overline{D})$ is a solution of (1) in D and $B_r(x) \subset D$. This equality was obtained by the author recently; see [7, p. 95]. Before that only the three-dimensional mean value formula for spheres had been derived by C. Neumann (see his book [9, Chapter 9, Section 3], published in 1896), whereas the m -dimensional formula for spheres was given without proof in [11]; its derivation see in the author's note [6].

Now, we are in a position to formulate the main result.

Theorem 1. *Let for a bounded domain $D \subset \mathbb{R}^m$, $m \geq 2$, a positive r be such that $|B_r| = |D|$, and let $x_0 \in D$. If the equality $u(x_0) a(\lambda r) = M(u, D)$ holds for all solutions of equation (1) in an open domain containing D , then $D = B_r(x_0)$.*

2. Proof of Theorem 1 and discussion

Prior to proving Theorem 1, we introduce the following function

$$U(x) = \Gamma\left(\frac{m}{2}\right) \frac{I_{(m-2)/2}(\lambda|x|)}{(\lambda|x|/2)^{(m-2)/2}}, \quad x \in \mathbb{R}^m, \tag{5}$$

where the coefficient is chosen so that $U(0) = 1$. Let us consider some of its properties. According to (3), this spherically symmetric function monotonically increases as $|x|$ goes from zero to infinity. Also, it solves equation (1) in \mathbb{R}^m ; indeed, the representation

$$U(x) = \frac{2\Gamma(m/2)}{\sqrt{\pi}\Gamma((m-1)/2)} \int_0^1 (1-s^2)^{(m-3)/2} \cosh(\lambda|x|s) \, ds, \tag{6}$$

is easy to differentiate, thus verifying (1). This formula for U is a consequence of Poisson's integral (see [10], p. 223):

$$I_\nu(z) = \frac{(z/2)^\nu}{\sqrt{\pi}\Gamma(\nu+1/2)} \int_{-1}^1 (1-s^2)^{\nu-1/2} \cosh zs \, ds.$$

Moreover, (6) takes particularly simple form for $m = 3$, namely, $U(x) = (\lambda|x|)^{-1} \sinh \lambda|x|$.

Proof of Theorem 1. Without loss of generality, we suppose that the domain D is located so that x_0 coincides with the origin. If we assume that $D \neq B_r(0)$, then $G_i = D \setminus \overline{B_r(0)}$ and $G_e = B_r(0) \setminus \overline{D}$ are bounded open sets such that $|G_e| = |G_i| \neq 0$ by the definition of r .

To obtain a contradiction from this assumption we write the volume mean equality for U over D as follows:

$$|D| a(\lambda r) = \int_D U(y) \, dy; \tag{7}$$

here the condition $U(0) = 1$ is taken into account. Since property (4) holds for U over $B_r(0)$, we write it in the same way:

$$|B_r(0)| a(\lambda r) = \int_{B_r(0)} U(y) \, dy. \tag{8}$$

Subtracting (8) from (7) and using the definition of r , we obtain

$$0 = \int_{G_i} U(y) \, dy - \int_{G_e} U(y) \, dy > 0.$$

Here the difference is positive because $U(y)$ (positive and monotonically increasing with $|y|$) is greater (less) than $[U(y)]_{|y|=r}$ in G_i (G_e , respectively), whereas $|G_i| = |G_e|$. This contradiction proves the theorem. \square

In the limit $\lambda \rightarrow 0$, equation (1) turns into Laplace's, whose solutions are harmonic functions; moreover, the assumption about r becomes superfluous in this case. Thus, letting $\lambda \rightarrow 0$ in Theorem 1 leads to an improved formulation of Kuran's theorem because only positive harmonic functions are involved; see also [1].

In the case of sufficiently smooth ∂D , the integral $\int_D u(y) \, dy$ can be replaced by the flux integral $\int_{\partial D} \partial u / \partial n_y \, dS_y$ in the formulation of Theorem 1; here n is the exterior unit normal. Indeed, we have

$$\int_D u(y) \, dy = \lambda^{-2} \int_D \nabla^2 u(y) \, dy = \lambda^{-2} \int_{\partial D} \partial u / \partial n_y \, dS_y.$$

This suggests that the following mean flux equality

$$\frac{\lambda^2 r \Gamma(m/2)}{2\Gamma(\frac{m}{2} + 1)} a(\lambda r) u(x_0) = \frac{1}{|\partial D|} \int_{\partial D} \frac{\partial v}{\partial n_y} \, dS_y$$

(cf. [7, formula (31)]) may also characterize balls provided there exists $x_0 \in D$ such that this equality is valid for all sufficiently smooth solutions of equation (1) in D with a smooth boundary, whereas $\lambda, r > 0$ satisfy assumptions similar to those in Theorem 1.

In conclusion we notice that the equality (see [7, Theorem 8])

$$mI_{m/2}(\lambda r) \int_{\partial B_r(x)} u(y) \, dS_y = \lambda r I_{(m-2)/2}(\lambda r) \int_{B_r(x)} u(y) \, dy$$

holds for every point x belonging to a domain $D \subset \mathbb{R}^m$ and all r such that $\overline{B_r(x)} \subset D$ if and only if u is a solution of equation (1) in D . This is analogous to the equality of the mean values over spheres and balls for harmonic functions. In view of Kosmodem'yanskii's theorem, one might expect that this equality with $B_r(x)$ changed to D characterizes balls in \mathbb{R}^m provided it is valid for all solutions of equation (1) in D .

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