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Comptes Rendus

Mathématique


Óscar Domínguez and Mario Milman

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Volume 359, issue 8 (2021), p. 1059-1069

<<https://doi.org/10.5802/crmath.252>>

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Sparse Brudnyi and John–Nirenberg Spaces

Óscar Domínguez^a and Mario Milman^b

^a O. Domínguez, Departamento de Análisis Matemático y Matemática Aplicada,
Facultad de Matemáticas, Universidad Complutense de Madrid, Plaza de Ciencias 3,
28040 Madrid, Spain.

^b M. Milman, Instituto Argentino de Matematica, Buenos Aires, Argentina

URL: <https://sites.google.com/site/mariomilman/>

E-mails: oscar.dominguez@ucm.es, mario.milman@icloud.com

Abstract. A generalization of the theory of Y. Brudnyi [7], and A. and Y. Brudnyi [5, 6], is presented. Our construction connects Brudnyi's theory, which relies on local polynomial approximation, with new results on sparse domination. In particular, we find an analogue of the maximal theorem for the fractional maximal function, solving a problem proposed by Kruglyak–Kuznetsov. Our spaces shed light on the structure of the John–Nirenberg spaces. We show that SJN_p (sparse John–Nirenberg space) coincides with L^p , $1 < p < \infty$. This characterization yields the John–Nirenberg inequality by extrapolation and is useful in the theory of commutators.

2020 Mathematics Subject Classification. 42B35, 42B25, 46E30, 46E35.

Funding. The first named author is supported in part by MTM2017-84058-P (AEI/FEDER, UE).

Manuscript received 11th July 2021, accepted 1st August 2021.

1. Preamble

The function spaces we use in Analysis can be described and characterized in different qualitative and quantitative ways (e.g. by duality, as being part of an interpolation scale, through the boundedness of suitable functionals, by the rate of approximation of their elements with respect to a fixed class of approximants, etc.). To have a complete catalog of different characterizations at hand is of fundamental importance to understand the structure of the spaces and facilitates their use in applications (cf. [28, 32]).

Yuri and Alexander Brudnyi (cf. [5–7]) have proposed the concept of best local polynomial approximation as a unifying characteristic to understand the structure of classical function spaces as diverse as, BMO, John–Nirenberg spaces JN_p , Sobolev spaces, Besov spaces, Morrey spaces, Jordan–Wiener spaces, etc. Their massive theory can be seen as a complement of the theories of function spaces that have evolved through the work of many authors, including names such as Coifman–Meyer, Frazier–Jawerth, Peetre, Triebel (cf. [15, 28, 32] and the references therein), where the underlying unifying themes and tools are wavelet approximations, representation theorems, maximal inequalities, interpolation, etc. A distinguished feature of Brudnyi's constructions is the

fact that instead of explicitly defining oscillations, these appear in Brudnyi's theory as the solution of variational problems. For example, best local approximation by constants in L^1 , amounts to replace $\int_Q |f(x) - f_Q| dx$ by $\inf_c \int_Q |f(x) - c| dx$, for each cube Q . Higher order oscillations can be handled replacing constants by polynomials of a given order and one can go on to accommodate different geometries, different approximants, etc.

In our work on fractional maximal operators and commutators we found it very useful to combine the constructions of Brudnyi's theory of function spaces with new developments in Harmonic Analysis connected with Covering Lemmas, more specifically "sparseness" (cf. Lerner [23]). Our analysis led us to modify the original constructions and formally introduce a new class of spaces that we call *Sparse Brudnyi spaces* (SB spaces). As we shall see, SB spaces can be used to provide new characterizations of classical function spaces as well as clarify, simplify and solve some open questions. In a different direction, our spaces build a new bridge that should also benefit local polynomial approximation theory.

The problem of understanding the structure of the John–Nirenberg JN_p spaces continues to attract attention to this day (cf. [11] for a recent account). It was therefore surprising for us to discover that the SJN_p spaces (Sparse John–Nirenberg spaces) admit a simple characterization (cf. Theorem 1 below). Let Q_0 be a cube in \mathbb{R}^n , then

$$SJN_p(Q_0) = \begin{cases} L(\log L)(Q_0), & p = 1, \\ L^p(Q_0), & 1 < p < \infty, \\ BMO(Q_0), & p = \infty. \end{cases}$$

Moreover, this characterization is useful in applications to a number of problems in Analysis, where JN_p type conditions appear naturally but, for which, the usual embedding into weak- L^p , i.e.,

$$JN_p(Q_0) \subset L(p, \infty)(Q_0), \quad 1 < p < \infty,$$

only leads to weaker inequalities (pun intended.)

As an example, we answer a question proposed by Kruglyak–Kuznetsov [22] concerning the fractional maximal operator. Let M be the classical maximal operator, a version of the Hardy–Littlewood theorem can be formulated as²

$$\|Mf\|_{L^p(Q_0)} \approx \|f\|_{L^p(Q_0)}, \quad 1 < p \leq \infty. \quad (1)$$

Kruglyak–Kuznetsov [22], ask for an analogue for maximal fractional operators (cf. (16) below). Using the parameters of Sobolev's inequality, namely, $\lambda \in (0, n)$, $p \in (1, \frac{n}{\lambda})$ and $\frac{1}{q} = \frac{1}{p} - \frac{\lambda}{n}$, we only have the one direction inequality

$$\|M_\lambda f\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}. \quad (2)$$

The problem thus is to find spaces that could turn (2) into an equivalence, and Kruglyak–Kuznetsov proposed certain spaces defined in terms of capacities. Using SB spaces instead, we can now provide a complete answer (cf. Theorem 5 below)

$$\|f\|_{SV_{p,q}^{k,\lambda}(Q_0)} \approx \|M_{q,\lambda,Q_0}(f - P_{Q_0}^k f)\|_{L^p(Q_0)}$$

provided that $k \in \mathbb{N}$, $\lambda \in [0, n)$ and $p, q \in [1, \infty)$.

The proofs of the results announced in this note are given in [12].

¹ Where as usual, f_Q denotes the integral average of f over Q , i.e., $f_Q = \frac{1}{|Q|} \int_Q f$.

² Here the symbol $f \approx g$ indicates the existence of a universal constant $c > 0$ (independent of all parameters involved) such that $(1/c)f \leq g \leq cf$. Likewise the symbol $f \lesssim g$ will mean that there exists a universal constant $c > 0$ (independent of all parameters involved) such that $f \leq cg$.

2. John–Nirenberg Spaces: A primer

Initially introduced by John–Nirenberg [20] in 1961 to study some problems in elasticity, BMO (cf. (5)) has been often at the center of important developments in Analysis ever since.

It was quickly realized, probably starting with Moser [26], that BMO could be used to substitute L^∞ , as an endpoint space for the scale of L^p spaces. Indeed, many important inequalities in PDEs and Harmonic Analysis that fail for the smaller space L^∞ are valid for BMO. Interpolation theorems by Stampachia [30, 31], Campanato [9] and others, soon confirmed the rôle of BMO as a natural endpoint space for the L^p scale. A major inflection point establishing the importance of BMO in Analysis, was Fefferman’s celebrated duality theorem [13] and the real variable approach to the theory of H^p spaces by Fefferman and Stein [14]. These developments were naturally connected with probabilistic interpretations of BMO using martingale theory (cf. [16]). In the last 50 years or so the applications have multiplied in many directions including the study of PDEs, Singular Integrals (T1 theorem, Commutators), Sobolev spaces, Noncommutative Analysis and Operator Theory. BMO conditions have been studied in many unexpected settings, for example, in the Theory of Semigroups (probably starting in [33]) just to name a few of these developments.

An early fundamental result of BMO theory is the John–Nirenberg inequality [20] that gives the exponential decay of the distribution function of elements of BMO, and implies the embedding

$$\text{BMO} \subset e^L. \tag{3}$$

Moreover, in the same paper, John–Nirenberg introduced the JN_p spaces, whose definition we now recall. Let Q_0 be a fixed cube in \mathbb{R}^n with sides parallel to the coordinate axes, the $JN_p(Q_0)$ spaces can be defined using packings³ $\Pi(Q_0)$, as follows. For each $\pi = \{Q_i\}_{i \in I} \in \Pi(Q_0)$, let $f_\pi(x) = \sum_{i \in I} (\frac{1}{|Q_i|} \int_{Q_i} |f - f_{Q_i}|) \mathbb{1}_{Q_i}(x)$, then $JN_p(Q_0)$, $1 \leq p < \infty$, is defined requiring the following functionals to be finite,

$$\begin{aligned} \|f\|_{JN_p(Q_0)} &= \sup_{\pi \in \Pi(Q_0)} \|f_\pi\|_{L^p(Q_0)} \\ &= \sup_{\pi \in \Pi(Q_0)} \left(\sum_{i \in I} \left(\frac{1}{|Q_i|} \int_{Q_i} |f(x) - f_{Q_i}| \, dx \right)^p |Q_i| \right)^{1/p}. \end{aligned} \tag{4}$$

For $p = \infty$ we simply have,

$$\|f\|_{JN_\infty(Q_0)} = \sup_{Q \subset Q_0} \frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx = \|f\|_{\text{BMO}(Q_0)}. \tag{5}$$

The decay of the distribution functions of elements of $JN_p(Q_0)$ can be obtained from the embeddings

$$L_p(Q_0) \subset JN_p(Q_0) \subset L(p, \infty)(Q_0), \quad 1 < p < \infty. \tag{6}$$

Bennett–DeVore–Sharpley [4], showed that the sharp limiting result obtains replacing e^L by $L(\infty, \infty) =$ the rearrangement invariant hull of BMO^4 in (3).

The initial motivation for our investigation was the method to prove (6) given in [17, 25], and the theory of Garsia–Rodemich spaces⁵ that has evolved since (cf. [2] and the references therein) suggesting that perhaps there was a simpler structure behind the JN_p spaces.

³ Packings $\Pi(Q_0)$ are simply countable collections of subcubes of Q_0 with pairwise disjoint interiors.

⁴ The set $L(\infty, \infty)$ can be described by the finiteness of the nonlinear functional

$$\|f\|_{L(\infty, \infty)} = \sup_{t > 0} \{f^{**}(t) - f^*(t)\}.$$

⁵ Initially (cf. [25]) these spaces were denoted by GaRo_p , but after the definition was extended to r.i. spaces the notation GaRo_X was adopted (cf. [2]). In particular, in this notation, $\text{GaRo}_{L(p, \infty)} := \text{GaRo}_p$.

Recall that the space $\text{GaRo}_{L(p,\infty)}(Q_0)$, $1 < p \leq \infty$, is defined in terms of the functional

$$\|f\|_{\text{GaRo}_{L(p,\infty)}(Q_0)} = \sup_{(Q_i)_{i \in I} \in \Pi(Q_0)} \frac{\sum_{i \in I} \int_{Q_i} |f(x) - f_{Q_i}| \, dx}{(\sum_{i \in I} |Q_i|)^{1/p'}}.$$

As usual, p' denotes the dual exponent of p given by $\frac{1}{p} + \frac{1}{p'} = 1$. Therefore, since for any $(Q_i)_{i \in I} \in \Pi(Q_0)$,

$$\begin{aligned} \frac{\sum_{i \in I} \int_{Q_i} |f(x) - f_{Q_i}| \, dx}{(\sum_{i \in I} |Q_i|)^{1/p'}} &= \frac{\sum_{i \in I} \left(\frac{1}{|Q_i|} \int_{Q_i} |f(x) - f_{Q_i}| \, dx \right) |Q_i|^{1/p} |Q_i|^{1/p'}}{(\sum_{i \in I} |Q_i|)^{1/p'}} \\ &\leq \left(\sum_{i \in I} \left(\frac{1}{|Q_i|} \int_{Q_i} |f(x) - f_{Q_i}| \, dx \right)^p |Q_i| \right)^{1/p}, \end{aligned}$$

it follows that

$$\|f\|_{\text{GaRo}_{L(p,\infty)}(Q_0)} \leq \|f\|_{JN_p(Q_0)}. \tag{7}$$

The remarkable result here (cf. [17, 25]) is that

$$\text{GaRo}_{L(p,\infty)}(Q_0) = L(p, \infty)(Q_0), \quad 1 < p < \infty, \tag{8}$$

and with proper definitions (cf. [2])

$$\text{GaRo}_{L^p}(Q_0) = \begin{cases} L^p(Q_0), & 1 < p < \infty \\ \text{BMO}(Q_0), & p = \infty. \end{cases} \tag{9}$$

Therefore the second embedding in (6) can be alternatively achieved as a combination of (7) and (8).

Another relevant result for us, is the classical theorem of Riesz [29] asserting⁶ that

$$\|f\|_{L^p(Q_0)} \approx \sup_{(Q_i)_{i \in I} \in \Pi(Q_0)} \left(\sum_{i \in I} \left(\frac{1}{|Q_i|} \int_{Q_i} |f(x)| \, dx \right)^p |Q_i| \right)^{1/p}.$$

The previous discussion shows that this equivalence fails when dealing with oscillations (cf. (4))

$$\|f - f_{Q_0}\|_{L^p(Q_0)} \not\approx \|f\|_{JN_p(Q_0)} = \sup_{(Q_i)_{i \in I} \in \Pi(Q_0)} \left(\sum_{i \in I} \left(\frac{1}{|Q_i|} \int_{Q_i} |f(x) - f_{Q_i}| \, dx \right)^p |Q_i| \right)^{1/p}.$$

This sets up the stage for the new ingredient in our construction.

3. Sparse John–Nirenberg spaces

Let $\mathcal{D}(Q_0)$ be the collection of all dyadic subcubes in Q_0 . We say that $\mathcal{S}(Q_0) \subset \mathcal{D}(Q_0)$ is *sparse* if for every $Q \in \mathcal{S}(Q_0)$,

$$\sum_{Q' \in \mathbf{Ch}_{\mathcal{S}(Q_0)}(Q)} |Q'| \leq \frac{1}{2} |Q|, \tag{10}$$

where $\mathbf{Ch}_{\mathcal{S}(Q_0)}(Q)$ denotes the set of maximal (with respect to inclusion) cubes in $\mathcal{S}(Q_0)$, which are strictly contained in Q .

The concept of sparse family has its roots in the classical Calderón–Zygmund decomposition lemma [8, Lemma 1]. The related sparse domination principle, which essentially establishes pointwise bounds of general Calderón–Zygmund operators by a supremum of a special collection of dyadic and positive operators (the so-called *sparse operators*), has been recently developed into a powerful tool by Lerner [23]. Over the last few years, sparse domination has been further

⁶ One estimate follows by Jensen's inequality and the other using dyadic partitions and Lebesgue differentiation theorem.

extended and refined to deal with many classical operators in Analysis. In this regard, we only mention [24] and the references within.

Let $p \in [1, \infty]$, the *sparse John–Nirenberg space* $SJN_p(Q_0)$ is defined as the set of all $f \in L^1(Q_0)$ such that

$$\|f\|_{SJN_p(Q_0)} = \sup_{(Q_i)_{i \in I} \in \mathcal{S}(Q_0)} \left\| \sum_{i \in I} \left(\frac{1}{|Q_i|} \int_{Q_i} |f(x) - f_{Q_i}| dx \right) \mathbb{1}_{Q_i \setminus \cup_{Q' \in \text{Ch}_{\mathcal{S}(Q_0)}(Q_i)} Q'} \right\|_{L^p(Q_0)} < \infty. \tag{11}$$

Clearly, if $p = \infty$ then $SJN_\infty(Q_0) = \text{BMO}(Q_0)$. Since every sparse family $(Q_i)_{i \in I} = \mathcal{S}(Q_0)$ is, in particular, *weakly sparse*, i.e., for every Q_i there exists Borel sets $E_{Q_i} \subset Q_i$ with the properties that $|E_{Q_i}| \geq \frac{1}{2}|Q_i|$ and such that the collection $\{E_{Q_i} : Q_i \in \mathcal{S}(Q_0)\}$ is pairwise disjoint. Indeed, take $E_{Q_i} = Q_i \setminus \cup_{Q' \in \text{Ch}_{\mathcal{S}(Q_0)}(Q_i)} Q'$ (cf. (10)). Thus we readily see that if $p \in [1, \infty)$, the expression for $\|f\|_{SJN_p(Q_0)}$, simplifies to

$$\|f\|_{SJN_p(Q_0)} \approx \sup_{(Q_i)_{i \in I} \in \mathcal{S}(Q_0)} \left\{ \sum_{i \in I} \left(\frac{1}{|Q_i|} \int_{Q_i} |f(x) - f_{Q_i}| dx \right)^p |Q_i| \right\}^{1/p}. \tag{12}$$

Comparing (12) with (4) we see that the only difference is that the class of admissible families of cubes is more restrictive for the former, and therefore,

$$\|f\|_{JN_p(Q_0)} \leq \|f\|_{SJN_p(Q_0)}.$$

Furthermore, $JN_p(Q_0)$ is not rearrangement invariant (cf. [11]). However, the situation is dramatically different for the $SJN_p(Q_0)$ spaces. Indeed, we have

Theorem 1. *Let $p \in [1, \infty]$. Then*

$$SJN_p(Q_0) = \begin{cases} L^p(Q_0), & 1 < p < \infty, \\ \text{BMO}(Q_0), & p = \infty, \\ L(\log L)(Q_0), & p = 1. \end{cases}$$

Remark 2. When we finished the first version of this paper we came across the interesting preprint [1] by Airta–Hytönen–Li–Martikainen–Oikari concerning mapping properties of bi-commutators on mixed norm spaces. Sparseness is also used in their method to control expressions involving oscillations. More precisely, Proposition 3.2 in [1] is very close in spirit to Theorem 1 with $p \in (1, \infty)$, in as much as the use of weak sparse families to control JN_p conditions. But the authors use a linearization argument combined with duality, that moreover involves the Hardy–Littlewood maximal theorem. In particular, these issues preclude them from obtaining endpoint results. At any rate they do not define SJN_p or, more generally, sparse variants of the Brudnyi’s constructions as will appear in Section 4 below. In this regard, it is important to point out that the concept of weak sparseness is not well adapted, but one needs the stronger version of sparseness provided in (10)⁷.

Remark 3. The definition of the SJN_p provided by (11) can be extended by means of replacing L^p with a general Banach function space (e.g., Lorentz space, Orlicz space, weighted Lebesgue spaces, etc.). This is not clear at all if we use instead the right-hand side of (12) to define SJN_p .

As a by-product of the previous theorem and (8), (9), we can write (6) as follows

$$\text{GaRo}_{L^p}(Q_0) = SJN_p(Q_0) \subset JN_p(Q_0) \subset \text{GaRo}_{L(p, \infty)}(Q_0), \quad p \in (1, \infty).$$

Theorem 1 is actually a special case of a more general result involving SB spaces, which we now introduce.

⁷ One only has that every weakly sparse collection decomposes into a disjoint union of finitely many sparse subcollections; cf. [24, Lemma 6.6].

4. Sparse Brudnyi Spaces

First we review briefly the Brudnyi spaces treated in detail in [5–7], and for this we need to recall the concept of *best local polynomial approximation*.

For $k \in \mathbb{N}$ and $f \in L^q(Q_0)$, $1 \leq q \leq \infty$, we consider the set function

$$E_k(f; Q_0)_q = \inf_{m \in \mathcal{P}_{k-1}^n} \|f - m\|_{L^q(Q_0)},$$

where \mathcal{P}_{k-1}^n is the set of all polynomials in \mathbb{R}^n of degree at most $k - 1$. Let $\lambda \in \mathbb{R}$, and $1 \leq p \leq \infty$. We let $V_{p,q}^{k,\lambda}(Q_0)$ denote⁸ the set of all functions $f \in L^q(Q_0)$ such that

$$\|f\|_{V_{p,q}^{k,\lambda}(Q_0)} = \sup_{(Q_i)_{i \in I} \in \Pi(Q_0)} \left(\sum_{i \in I} \left(|Q_i|^{\frac{\lambda}{n} - \frac{1}{q}} E_k(f; Q_i)_q \right)^p |Q_i| \right)^{1/p} < \infty \tag{13}$$

(with the usual modification if $p = \infty$). One of the main features of the $V_{p,q}^{k,\lambda}(Q_0)$ scale is that it can be used to provide a unified treatment (e.g., duality assertions and structural properties) of many classical spaces in Analysis. Since $E_1(f; Q)_q \approx (\int_Q |f - f_Q|^q)^{1/q}$, we see that the spaces $JN_p(Q_0)$, $p \in (1, \infty)$, are distinguished elements of this scale, namely,

$$JN_p(Q_0) = V_{p,q}^{1,0}(Q_0), \quad q \in [1, p).$$

The list of examples of Brudnyi spaces $V_{p,q}^{k,\lambda}(Q_0)$ also includes, for suitable choices of the parameters, $BMO(Q_0)$, $BV(Q_0)$, $\dot{W}^{k,p}(Q_0)$ and $M_q^\lambda(Q_0)$ (Morrey spaces), among others.

Imitating the construction of the $SJN_p(Q_0)$ spaces given above (cf. (11)), we can introduce general SB spaces as follows. Let $k \in \mathbb{N}$, $\lambda \in \mathbb{R}$ and $p, q \in [1, \infty]$. The space $SV_{p,q}^{k,\lambda}(Q_0)$ is the set of all functions $f \in L^q(Q_0)$ such that

$$\|f\|_{SV_{p,q}^{k,\lambda}(Q_0)} = \sup_{(Q_i)_{i \in I} \in \mathcal{S}(Q_0)} \left\| \sum_{i \in I} |Q_i|^{\frac{\lambda}{nq} - \frac{1}{q}} E_k(f; Q_i)_q \mathbb{1}_{Q_i \setminus \cup_{Q' \in \text{Ch}_{\mathcal{S}(Q_0)}(Q_i)} Q'} \right\|_{L^p(Q_0)} < \infty. \tag{14}$$

Representative examples are given

$$SV_{p,1}^{1,0}(Q_0) = SJN_p(Q_0), \quad p \in [1, \infty) \quad (\text{cf. (11)}), \tag{15}$$

$$SV_{\infty,1}^{1,\lambda}(Q_0) = \begin{cases} BMO(Q_0), & \text{if } \lambda = 0, \\ \mathcal{C}^{-\lambda}(Q_0), & \text{if } \lambda \in (-1, 0), \\ M_1^\lambda(Q_0), & \text{if } \lambda \in (0, n). \end{cases}$$

⁹ Moreover, we obviously have

$$SV_{p,q}^{k,\lambda}(Q_0) \subset V_{p,q}^{k,\lambda}(Q_0) \quad \text{and} \quad \|f\|_{V_{p,q}^{k,\lambda}(Q_0)} \leq \|f\|_{SV_{p,q}^{k,\lambda}(Q_0)}.$$

As a first application of the SB spaces we indicate a solution to a problem concerning the fractional maximal operator.

For $\lambda \in [0, n)$ and $q \in [1, \infty)$, the (*dyadic*) *local fractional maximal operator* M_{q,λ,Q_0} is defined for $f \in L^q(Q_0)$, by

$$M_{q,\lambda,Q_0} f(x) = \sup_{\substack{Q \ni x \\ Q \in \mathcal{D}(Q_0)}} \left(|Q|^{\frac{\lambda}{n} - 1} \int_Q |f(y)|^q dy \right)^{\frac{1}{q}}, \quad x \in Q_0. \tag{16}$$

⁸ We warn the reader that we have slightly changed the notation used in [5, 6], more precisely, the space $V_{p,q}^{k,\lambda}(Q_0)$ defined here corresponds with $V_{p,q}^{k, -\frac{1}{p} - \frac{\lambda}{nq} + \frac{1}{q}}(Q_0)$ in those papers. We feel this parametrization helps in providing a more clear formulation of our results.

⁹ $\mathcal{C}^{-\lambda}(Q_0)$ denotes classical Hölder–Zygmund spaces

In particular, if $q = 1$ we simply write M_{λ, Q_0} . If in addition $\lambda = 0$, then the classical maximal function M_{Q_0} is obtained, i.e.,

$$M_{Q_0} f(x) = \sup_{\substack{Q \ni x \\ Q \in \mathcal{D}(Q_0)}} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

Kruglyak and Kuznetsov [22, p. 310] posed the following

Question 4. *What is a good analogue of the Hardy–Littlewood maximal theorem (1) for the fractional maximal function? That is, can one construct function spaces X and Y such that $L^p \subset X, Y \subset L^q$ and $\|M_{\lambda, Q_0} f\|_Y \approx \|f\|_X$?*

A first attempt to this question is given in Theorem 3 of [22] where the authors proposed an interpolation based approach to deal with the case $Y = L^q(\mathbb{R}^n, C)$ where C is a certain fractional capacity.

It turns out that the extension of Theorem 1 to function spaces with smoothness can be used to give an answer to Question 4.

Theorem 5. *Let $k \in \mathbb{N}, \lambda \in [0, n), p \in [1, \infty)$ and $q \in [1, \infty)$. Then*

$$SV_{p,q}^{k,\lambda}(Q_0) = M_{q,\lambda,Q_0} L^p(Q_0).$$

More precisely, if $P_{Q_0}^k f \in \mathcal{P}_{k-1}^n$ denotes a nearly best polynomial approximation of f in $L^q(Q_0)$ (i.e., $E_k(f; Q_0)_q \approx (\int_{Q_0} |f - P_{Q_0}^k f|^q)^{1/q}$) then

$$\|M_{q,\lambda,Q_0}(f - P_{Q_0}^k f)\|_{L^p(Q_0)} \leq c_n \|M_{Q_0}\|_{L^{p'}(Q_0) \rightarrow L^{p'}(Q_0)} \|f\|_{SV_{p,q}^{k,\lambda}(Q_0)} \tag{17}$$

and

$$\|f\|_{SV_{p,q}^{k,\lambda}(Q_0)} \leq 2 \|M_{q,\lambda,Q_0}(f - P_{Q_0}^k f)\|_{L^p(Q_0)}.$$

Here, c_n denotes a purely dimensional constant.

Theorem 1 is an immediate consequence of the previous result with $\lambda = 0, k = 1$ and $q = 1$ and the Hardy–Littlewood maximal theorem (cf. (1)).

Remark 6. A similar comment as in Remark 3 also applies to the previous theorem.

In fact, the estimate (17) can be sharpened if we replace the functional $\|f\|_{SV_{p,q}^{k,\lambda}(Q_0)}$ appearing in the right-hand side by a weaker functional involving a fractional variant of the sparse condition (10). This will be explained in more detail in the next section.

5. Fractional Capacities and Sparseness

The following definition is motivated by fractional capacities of sets of cubes (cf. [21]). Let $\lambda \in (0, 1]$. We say that $\mathcal{S}^\lambda(Q_0) \subset \mathcal{D}(Q_0)$ is *sparse of order λ* if for every $Q \in \mathcal{S}^\lambda(Q_0)$,

$$\sum_{Q' \in \text{Ch}_{\mathcal{S}^\lambda(Q_0)}(Q)} |Q'|^\lambda \leq \frac{1}{2} |Q|^\lambda. \tag{18}$$

Clearly, $\mathcal{S}^1(Q_0) = \mathcal{S}(Q_0)$ and

$$\mathcal{S}^{\lambda_0}(Q_0) \subset \mathcal{S}^{\lambda_1}(Q_0), \quad \lambda_0 < \lambda_1 \tag{19}$$

where this inclusion must be appropriately interpreted (i.e., if a given family of cubes satisfies the condition (18) with λ_0 then the corresponding condition with λ_1 also holds). Accordingly,

we can now introduce the class of function spaces $\widetilde{SV}_{p,q}^{k,\lambda}(Q_0)$, $\lambda \in [0, n)$, formed by all functions $f \in L^q(Q_0)$ such that

$$\|f\|_{\widetilde{SV}_{p,q}^{k,\lambda}(Q_0)} = \sup_{(Q_i)_{i \in I} \in \mathcal{S}^{1-\frac{\lambda}{n}}(Q_0)} \left\| \sum_{i \in I} |Q_i|^{\frac{\lambda}{nq} - \frac{1}{q}} E_k(f; Q_i)_q \mathbb{1}_{Q_i \setminus \cup_{Q' \in \text{Ch}_{\mathcal{S}'}(Q_0)}(Q_i)} Q' \right\|_{L^p(Q_0)} < \infty.$$

Here a new phenomenon appears, namely, the supremum runs over families of sparse cubes depending on the smoothness parameter λ . In particular, this will enable us to capture better the smoothness properties of the space and is in sharp contrast with the classical constructions (13) where the supremum is taken with respect to all possible families of cubes. Clearly

$$\widetilde{SV}_{p,q}^{k,0}(Q_0) = SV_{p,q}^{k,0}(Q_0),$$

and for general $\lambda \in [0, n)$, by (19),

$$SV_{p,q}^{k,\lambda}(Q_0) \subset \widetilde{SV}_{p,q}^{k,\lambda}(Q_0) \quad \text{and} \quad \|f\|_{\widetilde{SV}_{p,q}^{k,\lambda}(Q_0)} \leq \|f\|_{SV_{p,q}^{k,\lambda}(Q_0)}.$$

An improvement of (17) (cf. (20) below) is contained in the following

Theorem 7. *Let $k \in \mathbb{N}$, $\lambda \in [0, n)$, $p \in [1, \infty)$ and $q \in [1, \infty)$. Then*

$$\widetilde{SV}_{p,q}^{k,\lambda}(Q_0) = M_{q,\lambda,Q_0} L^p(Q_0).$$

More precisely, if $P_{Q_0}^k f \in \mathcal{P}_{k-1}^n$ denotes a nearly best polynomial approximation of f in $L^q(Q_0)$ then

$$\|M_{q,\lambda,Q_0}(f - P_{Q_0}^k f)\|_{L^p(Q_0)} \leq c_n \|M_{Q_0}\|_{L^{p'}(Q_0) \rightarrow L^{p'}(Q_0)} \|f\|_{\widetilde{SV}_{p,q}^{k,\lambda}(Q_0)} \tag{20}$$

and

$$\|f\|_{\widetilde{SV}_{p,q}^{k,\lambda}(Q_0)} \leq 2 \|M_{q,\lambda,Q_0}(f - P_{Q_0}^k f)\|_{L^p(Q_0)}.$$

6. Some applications

In this section we present further selected applications of the spaces $SV_{p,q}^{k,\lambda}(Q_0)$.

6.1. A unified theory of commutators

To simplify our presentation, the results given in this section are only stated for the Hilbert transform H on \mathbb{R} , but corresponding results for smooth Calderón–Zygmund operators on \mathbb{R}^n also hold. An important family of commutators in Complex Analysis, Nonlinear PDE’s, Operator Theory and Interpolation Theory is given by

$$[H, b]f = H(bf) - bH(f), \quad b \in L^1_{\text{loc}}(\mathbb{R}).$$

The mapping properties of this operator between Lebesgue spaces are collected in the following

Theorem 8. *Let $1 < p, q < \infty$. Then*

$$[H, b] : L^p(\mathbb{R}) \rightarrow L^q(\mathbb{R})$$

if and only if

- (i) $p = q$ and $b \in \text{BMO}(\mathbb{R})$,
- (ii) $p < q$ and $b \in \mathcal{C}^\alpha(\mathbb{R})$ with $\alpha = \frac{1}{p} - \frac{1}{q}$,
- (iii) $p > q$ and $b \in L^r(\mathbb{R})$ with $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$.

The regime $p \leq q$ in the previous theorem is classical. Specifically, the diagonal case $p = q$ corresponds to the celebrated Coifman–Rochberg–Weiss theorem [10] (with [27] as a forerunner), while the case $p < q$ is due to Janson [19]. On the other hand, the case $q < p$ was only achieved recently by Hytönen [18] and it is intimately connected with the Jacobian equation. Note that the non-trivial assertion in this case concerns the necessity of $b \in L^r(\mathbb{R})$ and, in particular, it shows that the cancellation inherited to the commutator does not play a role. The results of our paper can now be applied to give a unified treatment of these three cases and, in particular, they show that $SV_{p,q}^{k,\lambda}(\mathbb{R}^n)$ spaces (naturally defined using (14)) seem to be appropriate spaces to state the commutator theorems. For instance, Theorem 8 can now be rewritten as follows.

Theorem 9. *Assume $1 < p, q < \infty$. Let $\lambda = -(\frac{1}{p} - \frac{1}{q})_+$ and $\frac{1}{r} = (\frac{1}{q} - \frac{1}{p})_+$. Then*

$$[H, b] : SV_{p,1}^{1,0}(\mathbb{R}) \rightarrow SV_{q,1}^{1,0}(\mathbb{R}) \iff b \in SV_{r,1}^{1,\lambda}(\mathbb{R}).$$

The previous statement should be adequately interpreted since $SV_{p,1}^{1,0}(\mathbb{R})$ coincides with $L^p(\mathbb{R})$ modulo constants (cf. Theorem 1 and (15)).

The reformulation of the commutator theorem given in Theorem 9 paves the way to further lines of research (cf. [12]). For instance, it is natural to investigate what is the role played in the commutator theorem by the parameters k, λ, p and q that appear in $SV_{p,q}^{k,\lambda}(\mathbb{R}^n)$; the pair (L^p, L^q) can be replaced by more general pairs of Banach function spaces; the Hilbert transform can be replaced by another classical operators in Analysis such as maximal functions (cf. [3].)

6.2. John–Nirenberg inequalities

We can give an elementary proof of the John–Nirenberg embedding (3) via the spaces $SJN_p(Q_0)$. Indeed, given $p > 1$, it follows from Theorem 1 (cf. also Theorem 5 with $k = 1, \lambda = 0$ and $q = 1$) that

$$\|f - f_{Q_0}\|_{L^p(Q_0)} \leq c_n \|M_{Q_0}\|_{L^{p'}(Q_0) \rightarrow L^{p'}(Q_0)} \|f\|_{SJN_p(Q_0)}.$$

Furthermore, a well-known interpolation argument yields that

$$\|M_{Q_0}\|_{L^{p'}(Q_0) \rightarrow L^{p'}(Q_0)} \leq \frac{p'}{p' - 1}.$$

Note that $\frac{p'}{p' - 1} = p$. Therefore

$$\|f - f_{Q_0}\|_{L^p(Q_0)} \lesssim p \|f\|_{SJN_p(Q_0)} \leq p |Q_0|^{1/p} \|f\|_{BMO(Q_0)}$$

and by classical extrapolation we arrive at (3).

The above argument can be applied *mutatis mutandis* to derive John–Nirenberg-type inequalities for Morrey spaces. Further details are left to the reader.

6.3. Sobolev inequalities

Theorem 5 can be applied to give an elementary proof of the local counterpart of the Sobolev inequality (2). Indeed, let $\lambda \in (0, n)$, $p \in (1, \frac{n}{\lambda})$ and $\frac{1}{q} = \frac{1}{p} - \frac{\lambda}{n}$, we have

$$\begin{aligned} \|M_{\lambda, Q_0}(f - f_{Q_0})\|_{L^q(Q_0)} &\approx \|f\|_{SV_{q,1}^{\lambda}(Q_0)} \\ &= \sup_{(Q_i)_{i \in I} \in \mathcal{S}(Q_0)} \left\| \sum_{i \in I} |Q_i|^{\frac{\lambda}{n}-1} E_1(f; Q_i)_1 \mathbb{1}_{Q_i \setminus \cup_{Q' \in \text{Ch}_{\mathcal{S}(Q_0)}(Q_i)} Q'} \right\|_{L^q(Q_0)} \\ &\leq \sup_{(Q_i)_{i \in I} \in \mathcal{S}(Q_0)} \left(\sum_{i \in I} \left(|Q_i|^{\frac{\lambda}{n}-1} E_1(f; Q_i)_1 \right)^q |Q_i| \right)^{1/q} \\ &\leq \sup_{(Q_i)_{i \in I} \in \mathcal{S}(Q_0)} \left(\sum_{i \in I} \left(|Q_i|^{-1} E_1(f; Q_i)_1 \right)^p |Q_i| \right)^{1/p} \\ &= \|f\|_{SJN_p(Q_0)} \approx \|f - f_{Q_0}\|_{L_p(Q_0)}. \end{aligned}$$

Acknowledgments

We thank Emiel Lorist for useful discussions concerning the concept of sparse family of cubes.

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