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## *Mathématique*


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Volume 359, issue 8 (2021), p. 959-968

<<https://doi.org/10.5802/crmath.253>>

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Partial differential equations / *Équations aux dérivées partielles*

# Anisotropic Choquard problems with Stein–Weiss potential: nonlinear patterns and stationary waves

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**Abstract.** Weighted inequality theory for fractional integrals is a relatively less known branch of calculus that offers remarkable opportunities to simulate interdisciplinary processes. Basic weighted inequalities are often associated to Hardy, Littlewood and Sobolev [6, 11], Caffarelli, Kohn and Nirenberg [4], respectively to Stein and Weiss [12]. A key attempt in the present paper is to prove a Stein–Weiss inequality with lack of symmetry and variable exponents. We quantify the defect of symmetry of the potential by considering the gap between the minimum and the maximum of the variable exponent. We conclude our work with a section dealing with the existence of stationary waves for a class of nonlocal problems with Choquard nonlinearity and anisotropic Stein–Weiss potential.

**2020 Mathematics Subject Classification.** 35A23, 47J20, 58E05, 58E35.

**Funding.** The research of Youpei Zhang and Vicențiu D. Rădulescu was supported by a grant of the Ministry of Research, Innovation and Digitization, CNCS/CCCDI–UEFISCDI, project number PCE 137/2021, within PNCDI III. This research is partially supported by the National Natural Science Foundation of China (No. 11971485) and the Fundamental Research Funds for the Central Universities of Central South University (Nos. 2019zzts211, 2021zzts0041). This paper has been completed while Youpei Zhang was visiting University of Craiova (Romania) with the financial support of China Scholarship Council (No. 201906370079).

*Manuscript received 29th June 2021, accepted 3rd August 2021.*

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### 1. A weighted anisotropic Stein–Weiss inequality

This paper is concerned with the extension and generalization of a classic inequality first considered by Stein and Weiss [12], which is a two-weight counterpart of the Hardy–Littlewood–Sobolev inequality (see [6, 11]). In the present work, this new weighted inequality is established in the non-symmetric and anisotropic setting described by potentials with variable exponent. We offer a new look to anisotropic differential inequalities by controlling the gap between the minimum and the maximum of the variable exponent. This enables us to quantify the defect of symmetry of the potential. Finally, we apply the new anisotropic Stein–Weiss inequality to the study of a nonlocal Choquard problem with variable growth and lack of compactness.

The features of this paper are the following:

- (i) we establish a *non-symmetric* Stein–Weiss inequality with *variable potential*;
- (ii) in order to quantify the *defect of symmetry* of the potential, we prove more general estimates by considering the gap between the minimum and the maximum of the variable exponent;
- (iii) the analysis developed in this paper is concerned with the *combined effects* of a nonhomogeneous differential operator with unbalanced growth and a Choquard nonlinearity with variable exponent;
- (iv) our analysis combines the *nonlocal* nature of the Choquard nonlinearity with the *local* perturbation in the absorption term.

Let us first recall the following classical Stein–Weiss inequality [12].

**Theorem 1.** *Let  $1 < p, q < +\infty$ ,  $0 < \lambda < N$ ,  $\alpha + \beta \geq 0$  and  $\alpha + \beta + \lambda \leq N$ . Then the following properties hold.*

- (i) *If  $1/p + 1/q + (\alpha + \beta + \lambda)/N = 2$  and  $1 - 1/p - \lambda/N < \alpha/N < 1 - 1/p$ , then there exists a constant  $C_0 = C_0(p, q, \alpha, \beta, \lambda, N) < \infty$  such that*

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x|^\alpha |x - y|^\lambda |y|^\beta} dx dy \right| \leq C_0 \|f\|_{L^p(\mathbb{R}^N)} \|g\|_{L^q(\mathbb{R}^N)}, \tag{1}$$

*for all  $f \in L^p(\mathbb{R}^N)$ ,  $g \in L^q(\mathbb{R}^N)$ , where  $C_0$  is independent of  $f, g$ .*

- (ii) *For all  $f \in L^p(\mathbb{R}^N)$  there exists a constant  $C_1 = C_1(p, q, \alpha, \beta, \lambda, N) < \infty$  independent of  $f$ , such that*

$$\left\| \int_{\mathbb{R}^N} \frac{f(y)}{|x|^\alpha |x - y|^\lambda |y|^\beta} dy \right\|_{L^q(\mathbb{R}^N)} \leq C_1 \|f\|_{L^p(\mathbb{R}^N)}, \tag{2}$$

*where  $1 + 1/q = 1/p + (\alpha + \beta + \lambda)/N$  and  $\alpha/N < 1/q < (\alpha + \lambda)/N$ .*

All hypotheses in the previous theorem are sharp. In fact, these conditions are necessary either to ensure integrability or they follow from the scaling of the inequality, which is a special feature of the power-weights case. In the case of radially symmetric functions, the condition  $\alpha + \beta \geq 0$  can be relaxed and  $\alpha + \beta$  is allowed to assume negative values, for instance  $\alpha + \beta \geq -(N - 1)|p^{-1} - q^{-1}|$ ; see Rubin [10].

In what follows, we set  $C^+(\mathbb{R}^N) := \{r \in C(\mathbb{R}^N) : 1 < r^- := \inf_{\mathbb{R}^N} r \leq r^+ := \sup_{\mathbb{R}^N} r < +\infty\}$ .

The main result in this section establishes the following Stein–Weiss inequality with variable exponents.

**Theorem 2.** *Let  $p, q \in C^+(\mathbb{R}^N)$ ,  $f \in L^{p^+}(\mathbb{R}^N) \cap L^{p^-}(\mathbb{R}^N)$ ,  $g \in L^{q^+}(\mathbb{R}^N) \cap L^{q^-}(\mathbb{R}^N)$ ,  $\alpha + \beta \geq 0$  and  $\lambda : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$  be a continuous function such that*

$$0 < \lambda^- := \inf_{\mathbb{R}^N \times \mathbb{R}^N} \lambda \leq \lambda^+ := \sup_{\mathbb{R}^N \times \mathbb{R}^N} \lambda < N$$

*and  $0 < \alpha + \beta + \lambda^- \leq \alpha + \beta + \lambda^+ \leq N$ . Then the following properties hold.*

(i) *There exists a sharp constant  $C_3 = C_3(p^\pm, q^\pm, \alpha, \beta, \lambda^\pm, N) < \infty$ , independent of  $f, g$ , such that*

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x|^\alpha |x-y|^{\lambda(x,y)} |y|^\beta} dx dy \right| \leq C_3 \|f\|_{L^{p^+}(\mathbb{R}^N)} \|g\|_{L^{q^+}(\mathbb{R}^N)} + C_3 \|f\|_{L^{p^-}(\mathbb{R}^N)} \|g\|_{L^{q^-}(\mathbb{R}^N)}, \tag{3}$$

where

$$\max \left\{ 1 - \frac{1}{p^+} - \frac{\lambda^+}{N}, 1 - \frac{1}{p^-} - \frac{\lambda^-}{N} \right\} < \frac{\alpha}{N} < 1 - \frac{1}{p^-}$$

and

$$\frac{1}{p(x)} + \frac{1}{q(y)} + \frac{\alpha + \beta + \lambda(x,y)}{N} = 2, \quad \forall x, y \in \mathbb{R}^N.$$

(ii) *Moreover, there exist constants*

$$C_4 = C_4(p^+, q^+, \alpha, \beta, \lambda^+, N) < \infty \quad \text{and} \quad C_5 = C_5(p^-, q^-, \alpha, \beta, \lambda^-, N) < \infty,$$

*independent of  $f$ , such that*

$$\left\| \int_{\mathbb{R}^N} \frac{f(y)}{|x|^\alpha |x-y|^{\lambda^+} |y|^\beta} dy \right\|_{L^{q^+}(\mathbb{R}^N)} \leq C_4 \|f\|_{L^{p^+}(\mathbb{R}^N)} \tag{4}$$

and

$$\left\| \int_{\mathbb{R}^N} \frac{f(y)}{|x|^\alpha |x-y|^{\lambda^-} |y|^\beta} dy \right\|_{L^{q^-}(\mathbb{R}^N)} \leq C_5 \|f\|_{L^{p^-}(\mathbb{R}^N)}, \tag{5}$$

where

$$1 + \frac{1}{q^+} = \frac{1}{p^+} + \frac{\alpha + \beta + \lambda^+}{N}, \quad 1 + \frac{1}{q^-} = \frac{1}{p^-} + \frac{\alpha + \beta + \lambda^-}{N}$$

and

$$\frac{\alpha}{N} < \frac{1}{q^+} \leq \frac{1}{q^-} < \frac{\alpha + \lambda^-}{N}.$$

**Proof.** (i). We first observe that

$$\begin{aligned} \alpha + \beta + \lambda(x,y) &\leq N \left( 2 - \frac{1}{p^+} - \frac{1}{q^+} \right), \quad \forall x, y \in \mathbb{R}^N, \\ \implies \alpha + \beta + \lambda^+ &\leq N \left( 2 - \frac{1}{p^+} - \frac{1}{q^+} \right). \end{aligned}$$

On the other hand, we have

$$p^+ = \sup_{\mathbb{R}^N} p \quad \text{and} \quad q^+ = \sup_{\mathbb{R}^N} q,$$

$$\implies \text{there exist } \{x_n\}_{n \in \mathbb{N}}, \{y_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}^N \text{ such that } p(x_n) \rightarrow p^+, q(y_n) \rightarrow q^+ \text{ as } n \rightarrow \infty.$$

It follows that

$$\alpha + \beta + \lambda(x_n, y_n) \rightarrow N \left( 2 - \frac{1}{p^+} - \frac{1}{q^+} \right) \text{ as } n \rightarrow \infty.$$

We conclude that

$$\frac{1}{p^+} + \frac{1}{q^+} + \frac{\alpha + \beta + \lambda^+}{N} = 2. \tag{6}$$

In a similar way, we obtain

$$\frac{1}{p^-} + \frac{1}{q^-} + \frac{\alpha + \beta + \lambda^-}{N} = 2. \tag{7}$$

Taking into account the elementary inequality

$$\frac{1}{|x|^\alpha |x-y|^{\lambda(x,y)} |y|^\beta} \leq \frac{1}{|x|^\alpha |x-y|^{\lambda^+} |y|^\beta} + \frac{1}{|x|^\alpha |x-y|^{\lambda^-} |y|^\beta}, \quad \forall x, y \in \mathbb{R}^N,$$

we deduce that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(x)||g(y)|}{|x|^\alpha |x-y|^{\lambda(x,y)} |y|^\beta} dx dy \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(x)||g(y)|}{|x|^\alpha |x-y|^{\lambda^+} |y|^\beta} dx dy + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(x)||g(y)|}{|x|^\alpha |x-y|^{\lambda^-} |y|^\beta} dx dy. \tag{8}$$

Next, combining the inequality

$$\max \left\{ 1 - \frac{1}{p^+} - \frac{\lambda^+}{N}, 1 - \frac{1}{p^-} - \frac{\lambda^-}{N} \right\} < \frac{\alpha}{N} < 1 - \frac{1}{p^-},$$

with relations (6)–(8) and Theorem 1, we infer that

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x|^\alpha |x-y|^{\lambda(x,y)} |y|^\beta} dx dy \right| \leq C_3 \|f\|_{L^{p^+}(\mathbb{R}^N)} \|g\|_{L^{q^+}(\mathbb{R}^N)} + C_3 \|f\|_{L^{p^-}(\mathbb{R}^N)} \|g\|_{L^{q^-}(\mathbb{R}^N)}$$

for some constant  $C_3 = C_3(p^\pm, q^\pm, \alpha, \beta, \lambda^\pm, N) < \infty$ , which is independent of  $f, g$ .

(ii). It follows from Theorem 1. □

For  $p \in C^+(\mathbb{R}^N)$ , we consider the following Lebesgue space with variable exponent

$$L^{p(x)}(\mathbb{R}^N) := \left\{ u : \mathbb{R}^N \rightarrow \mathbb{R} \text{ is a measurable function; } \int_{\mathbb{R}^N} |u(x)|^{p(x)} dx < \infty \right\},$$

equipped with the Luxemburg norm

$$\|u\|_{L^{p(x)}(\mathbb{R}^N)} = \inf \left\{ \eta > 0 : \int_{\mathbb{R}^N} \left| \frac{u(x)}{\eta} \right|^{p(x)} dx \leq 1 \right\}.$$

The variable exponent Sobolev space  $W^{1,p(x)}(\mathbb{R}^N)$  is the subspace of functions  $u \in L^{p(x)}(\mathbb{R}^N)$  whose distributional gradient exists almost everywhere and satisfies  $|\nabla u| \in L^{p(x)}(\mathbb{R}^N)$ . More precisely, we have

$$W^{1,p(x)}(\mathbb{R}^N) := \{u \in L^{p(x)}(\mathbb{R}^N) : |\nabla u| \in L^{p(x)}(\mathbb{R}^N)\},$$

which is a Banach space under the norm

$$\|u\|_{W^{1,p(x)}(\mathbb{R}^N)} = \|u\|_{L^{p(x)}(\mathbb{R}^N)} + \|\nabla u\|_{L^{p(x)}(\mathbb{R}^N)}.$$

In the sequel, we set  $h \ll s$  if and only if  $h, s \in C(\mathbb{R}^N, \mathbb{R})$  and  $\inf\{s(x) - h(x) : x \in \mathbb{R}^N\} > 0$ .

**Theorem 3.** Assume that  $p : \mathbb{R}^N \rightarrow \mathbb{R}$  is a Lipschitz function with  $1 < p^- \leq p^+ < N$  and  $t \in C^+(\mathbb{R}^N)$ . Then  $W^{1,p(x)}(\mathbb{R}^N)$  is continuously embedded into  $L^{t(x)}(\mathbb{R}^N)$  for any  $p \leq t \leq p^*$  and  $W^{1,p(x)}(\mathbb{R}^N)$  is compactly embedded into  $L^{t(x)}_{loc}(\mathbb{R}^N)$  for any  $p \leq t \ll p^*$ , where  $p^*(x) := Np(x)/(N - p(x))$  for all  $x \in \mathbb{R}^N$ .

We refer to [5] for a proof of this result and more details.

In what follows, we assume that  $p : \mathbb{R}^N \rightarrow \mathbb{R}$  is a Lipschitz continuous function with  $1 < p^- \leq p^+ < N$ .

**Corollary 4.** Let  $q \in C^+(\mathbb{R}^N)$ ,  $\alpha \geq 0$ , and assume that  $\lambda : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$  is a continuous function such that  $0 < \lambda^- := \inf_{\mathbb{R}^N \times \mathbb{R}^N} \lambda \leq \lambda^+ := \sup_{\mathbb{R}^N \times \mathbb{R}^N} \lambda < N$ ,  $0 < 2\alpha + \lambda^- \leq 2\alpha + \lambda^+ \leq N$ ,

$$\max \left\{ 1 - \frac{1}{q^+} - \frac{\lambda^+}{N}, 1 - \frac{1}{q^-} - \frac{\lambda^-}{N} \right\} < \frac{\alpha}{N} < 1 - \frac{1}{q^-}$$

and

$$\frac{1}{q(x)} + \frac{1}{q(y)} + \frac{2\alpha + \lambda(x,y)}{N} = 2, \quad \forall x, y \in \mathbb{R}^N.$$

Let  $u \in W^{1,p(x)}(\mathbb{R}^N)$  and  $r \in \mathcal{G}$ , where

$$\mathcal{G} := \{r \in C^+(\mathbb{R}^N) : p(x) \leq r(x)q^- \leq r(x)q^+ \leq p^*(x), \quad \forall x \in \mathbb{R}^N\}. \tag{9}$$

Then  $|u(\cdot)|^{r(\cdot)} \in L^{q^+}(\mathbb{R}^N) \cap L^{q^-}(\mathbb{R}^N)$ .

Moreover, we have

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{r(x)}|u(y)|^{r(y)}}{|x|^\alpha|x-y|^{\lambda(x,y)}|y|^\alpha} dx dy \right| \leq C_6 \left( \| |u(\cdot)|^{r(\cdot)} \|_{L^{q^+}(\mathbb{R}^N)}^2 + \| |u(\cdot)|^{r(\cdot)} \|_{L^{q^-}(\mathbb{R}^N)}^2 \right) \tag{10}$$

for some  $C_6 = C_6(q^\pm, \alpha, \lambda^\pm, N) < \infty$ , where  $C_6$  does not depend on  $u \in W^{1,p(x)}(\mathbb{R}^N)$ .

**Proof.** Taking into account that  $r \in \mathcal{G}$  and using Theorem 3, we obtain that  $|u(\cdot)|^{r(\cdot)} \in L^{q^+}(\mathbb{R}^N) \cap L^{q^-}(\mathbb{R}^N)$ . Then we can use Theorem 2 to obtain the desired result.  $\square$

## 2. Choquard problems with weighted anisotropic Stein–Weiss potential

In this section, we focus on the existence of solutions for the following nonlocal problem with anisotropic Stein–Weiss convolution term:

$$\left\{ \begin{array}{l} -\Delta_{p(x)}u + V(x)|u|^{p(x)-2}u = \frac{1}{|x|^\alpha} \left( \int_{\mathbb{R}^N} \frac{F(y, u(y))}{|x-y|^{\lambda(x,y)}|y|^\alpha} dy \right) f(x, u(x)) \text{ in } \mathbb{R}^N, \\ u \in W^{1,p(x)}(\mathbb{R}^N), \end{array} \right\} \tag{P_\lambda}$$

where  $\alpha \geq 0$ ,  $0 < \lambda^- \leq \lambda^+ < N$ ,  $0 < 2\alpha + \lambda^- \leq 2\alpha + \lambda^+ \leq N$ ,  $V \in C(\mathbb{R}^N, \mathbb{R})$ ,  $\lambda \in C(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R})$ ,  $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ ,  $F(x, t) := \int_0^t f(x, \tau) d\tau$ , and  $\Delta_{p(x)}u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ .

We assume that the positive potential  $V$  is bounded from below and is coercive, that is,

$$V \in C(\mathbb{R}^N, \mathbb{R}), \quad \inf_{\mathbb{R}^N} V := V_0 > 0 \quad \text{and} \quad V(x) \rightarrow +\infty \text{ as } |x| \rightarrow +\infty. \tag{V_0}$$

Next, we introduce the weighted Sobolev space  $W_V^{1,p(x)}(\mathbb{R}^N)$  defined by

$$W_V^{1,p(x)}(\mathbb{R}^N) := \overline{C_0^\infty(\mathbb{R}^N)}^{\|\cdot\|},$$

where

$$\|u\| = \|\nabla u\|_{L^{p(x)}(\mathbb{R}^N)} + \|V(\cdot)^{1/p(\cdot)}u\|_{L^{p(x)}(\mathbb{R}^N)} := \|\nabla u\|_{L^{p(x)}(\mathbb{R}^N)} + \|u\|_{L^{p(x)}(\mathbb{R}^N), V}.$$

From Lemma 4.2 of [1], we know that  $W_V^{1,p(x)}$  is compactly embedded into  $L^{t(x)}(\mathbb{R}^N)$  for all  $t \in C^+(\mathbb{R}^N)$  and  $p \ll t \ll p^*$ .

In addition, the following assumptions are required on the reaction  $f : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ :

- (f<sub>1</sub>)  $|f(x, t)| \leq C_7 (|t|^{r(x)-1} + |t|^{s(x)-1})$  for all  $(x, t) \in \mathbb{R}^N \times \mathbb{R}$ , where  $C_7 > 0$  and  $r, s \in \mathcal{G}$  ( $\mathcal{G}$  is defined by (9)) with  $p \ll rq^- \leq rq^+ \ll q^*$ ,  $p \ll sq^- \leq sq^+ \ll q^*$  and  $r^-, s^- > p^+/2$ ;
- (f<sub>2</sub>) there exists  $\vartheta > p^+$  such that  $0 < \vartheta F(x, t) \leq 2f(x, t)t$  for all  $t > 0$ ;
- (f<sub>3</sub>) there exist constants  $\ell > 0$  and  $C_\ell > 0$  such that  $F(x, \ell) \geq C_\ell$  for all  $x \in \mathbb{R}^N$ .

The main result of this section establishes the following existence property for problem (P<sub>λ</sub>).

**Theorem 5.** Assume that (V<sub>0</sub>) and (f<sub>1</sub>)–(f<sub>3</sub>) are fulfilled, and that  $p : \mathbb{R}^N \rightarrow \mathbb{R}$  is a Lipschitz function with  $1 < p^- \leq p^+ < N$ . Then problem (P<sub>λ</sub>) has at least one nontrivial solution.

The proof of Theorem 5 relies on a variational method. For this purpose, we introduce the energy (Euler) functional  $\mathcal{E} : W_V^{1,p(x)}(\mathbb{R}^N) \rightarrow \mathbb{R}$  defined by

$$\mathcal{E}(u) := \int_{\mathbb{R}^N} \frac{1}{p(x)} (|\nabla u|^{p(x)} + V(x)|u|^{p(x)}) dx - \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(x, u(x))F(y, u(y))}{|x|^\alpha|x-y|^{\lambda(x,y)}|y|^\alpha} dx dy$$

for all  $u \in W_V^{1,p(x)}(\mathbb{R}^N)$ .

Arguing as in the proof of Lemma 3.2 in [2], Theorem 2 combined with hypotheses  $(V_0)$ ,  $(f_1)$  and the definition of  $F$  implies that  $\mathcal{E}$  is well-defined and  $\mathcal{E} \in C^1(W_V^{1,p(x)}(\mathbb{R}^N), \mathbb{R})$ , with

$$\begin{aligned} \langle \mathcal{E}'(u), \varphi \rangle &= \int_{\mathbb{R}^N} (|\nabla u|^{p(x)-2} \nabla u \nabla \varphi + V(x)|u|^{p(x)-2} u \varphi) \, dx \\ &\quad - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(y, u(y))f(x, u(x))\varphi(x)}{|x|^\alpha |x-y|^{\lambda(x,y)} |y|^\alpha} \, dx dy \text{ for all } \varphi \in W_V^{1,p(x)}(\mathbb{R}^N). \end{aligned}$$

As in many situations, the energy  $\mathcal{E}$  is unbounded from above and below, so that it has no maximum or minimum. This forces us to look for saddle points, which are obtained by minimax arguments that go back to the mountain pass theorem of Ambrosetti and Rabinowitz [3] (see also Pucci and Rădulescu [9] for a survey). The original proof of this classical result relies on some deep deformation techniques developed by Palais and Smale [8], who put the main ideas of the Morse theory into the framework of differential topology on infinite dimensional manifolds.

We first show that the energy (Euler) functional  $\mathcal{E}$  has a mountain pass geometry. The next property establishes the existence of a “mountain” near the origin, while the second property implies the existence of a “valley” on the other side of this mountain.

**Lemma 6.** *The following properties are fulfilled for the functional  $\mathcal{E}$  :*

- (a) *there exist  $\delta, \rho > 0$  such that  $\mathcal{E}(u) \geq \delta$  for  $u \in W_V^{1,p(x)}(\mathbb{R}^N)$  with  $\|u\| = \rho$ ;*
- (b) *there exists  $e \in W_V^{1,p(x)}(\mathbb{R}^N)$  such that  $\|e\| > \rho$  and  $\mathcal{E}(e) < 0$ .*

**Proof.** (a). We first observe that, using condition  $(f_1)$  and the definition of  $F$ , we have, for all  $u \in W_V^{1,p(x)}(\mathbb{R}^N)$ :

$$\begin{aligned} \|F(\cdot, u)\|_{L^{q^+}(\mathbb{R}^N)}^2 &\leq C_8 \left( \int_{\mathbb{R}^N} (|u|^{q^+ r(x)} + |u|^{q^+ s(x)}) \, dx \right)^{2/q^+} \\ &\leq C_9 \left( \int_{\mathbb{R}^N} |u|^{q^+ r(x)} \, dx \right)^{2/q^+} + C_9 \left( \int_{\mathbb{R}^N} |u|^{q^+ s(x)} \, dx \right)^{2/q^+}, \end{aligned}$$

for some constants  $C_8, C_9 > 0$ .

It follows that

$$\begin{aligned} \|F(\cdot, u)\|_{L^{q^+}(\mathbb{R}^N)}^2 &\leq C_9 \max \left\{ \|u\|_{L^{q^+ r(x)}(\mathbb{R}^N)}^{2r^+}, \|u\|_{L^{q^+ r(x)}(\mathbb{R}^N)}^{2r^-} \right\} \\ &\quad + C_9 \max \left\{ \|u\|_{L^{q^+ s(x)}(\mathbb{R}^N)}^{2s^+}, \|u\|_{L^{q^+ s(x)}(\mathbb{R}^N)}^{2s^-} \right\} \\ &\leq C_{10} \max \left\{ \|u\|_{W^{1,p(x)}(\mathbb{R}^N)}^{2r^+}, \|u\|_{W^{1,p(x)}(\mathbb{R}^N)}^{2r^-} \right\} \\ &\quad + C_{10} \max \left\{ \|u\|_{W^{1,p(x)}(\mathbb{R}^N)}^{2s^+}, \|u\|_{W^{1,p(x)}(\mathbb{R}^N)}^{2s^-} \right\} \quad (\text{by Theorem 3}), \end{aligned} \tag{11}$$

where  $C_{10}$  is a finite constant.

In a similar way, we have

$$\begin{aligned} \|F(\cdot, u)\|_{L^{q^-}(\mathbb{R}^N)}^2 &\leq C_{11} \max \left\{ \|u\|_{W^{1,p(x)}(\mathbb{R}^N)}^{2r^+}, \|u\|_{W^{1,p(x)}(\mathbb{R}^N)}^{2r^-} \right\} \\ &\quad + C_{11} \max \left\{ \|u\|_{W^{1,p(x)}(\mathbb{R}^N)}^{2s^+}, \|u\|_{W^{1,p(x)}(\mathbb{R}^N)}^{2s^-} \right\}, \end{aligned} \tag{12}$$

for some finite constant  $C_{11}$ .

Using Theorem 2 and relations (9), (11) and (12), we deduce that, for all  $u \in W_V^{1,p(x)}(\mathbb{R}^N)$  with  $\|u\|_{W^{1,p(x)}(\mathbb{R}^N)} \leq 1$ , we have

$$\begin{aligned} \mathcal{E}(u) &\geq C_{12} \left( \|\nabla u\|_{L^{p(x)}(\mathbb{R}^N)}^{p^+} + \|u\|_{L^{p(x)}(\mathbb{R}^N)}^{p^+} \right) - C_{13} \left( \|\nabla u\|_{L^{p(x)}(\mathbb{R}^N)}^{2r^+} + \|u\|_{L^{p(x)}(\mathbb{R}^N)}^{2r^+} \right) \\ &\quad - C_{13} \left( \|\nabla u\|_{L^{p(x)}(\mathbb{R}^N)}^{2r^-} + \|u\|_{L^{p(x)}(\mathbb{R}^N)}^{2r^-} \right) - C_{13} \left( \|\nabla u\|_{L^{p(x)}(\mathbb{R}^N)}^{2s^+} + \|u\|_{L^{p(x)}(\mathbb{R}^N)}^{2s^+} \right) \\ &\quad - C_{13} \left( \|\nabla u\|_{L^{p(x)}(\mathbb{R}^N)}^{2s^-} + \|u\|_{L^{p(x)}(\mathbb{R}^N)}^{2s^-} \right), \end{aligned}$$

for some finite positive constants  $C_{12}, C_{13}$  that are independent of  $u$ .

Using  $2r^- , 2s^- > p^+$  and taking  $\|u\|_{W^{1,p(x)}(\mathbb{R}^N)} \leq C_{14}\|u\| := C_{14}\varrho$  (where  $C_{14} > 0$  is independent of  $u$ ) with  $\varrho \leq 1/C_{14}$  sufficiently small, we see that (a) holds true.

(b). Using hypotheses  $(f_2)$ – $(f_3)$ , there exists a constant  $C_{15} = C_{15}(\ell, \vartheta) > 0$  such that  $F(x, t) \geq C_{15}t^{\frac{\vartheta}{2}}$  for all  $(x, t) \in \mathbb{R}^N \times [\ell, +\infty)$ . Then we can choose a function  $v \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}$  and use the last inequality to infer that  $\mathcal{E}(tv) < 0$  for  $t > 0$  sufficiently large. The proof of the lemma is complete.  $\square$

Using Lemma 6, we can define the following minimax level:

$$0 < c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \mathcal{E}(\gamma(t)), \tag{13}$$

where

$$\Gamma := \left\{ \gamma \in C([0, 1], W_V^{1,p(x)}(\mathbb{R}^N)) : \gamma(0) = 0, \mathcal{E}(\gamma(1)) < 0 \right\}.$$

**Lemma 7.** *The Palais–Smale sequence  $\{u_n\}_{n \in \mathbb{N}} \subset W_V^{1,p}(\mathbb{R}^N)$  of the functional  $\mathcal{E}$  at the level  $c$  is bounded.*

**Proof.** Since  $\{u_n\}_{n \in \mathbb{N}} \subset W_V^{1,p}(\mathbb{R}^N)$  is a Palais–Smale sequence at the level  $c$ , we have

$$\mathcal{E}(u_n) = c + o_n(1) \text{ and } \mathcal{E}'(u_n) = o_n(1) \text{ as } n \rightarrow \infty.$$

Using  $(f_2)$ , for large enough  $n \in \mathbb{N}$  we see that

$$\begin{aligned} c(1 + \|u_n\|) &\geq \mathcal{E}(u_n) - \frac{1}{\vartheta} \langle \mathcal{E}'(u_n), u_n \rangle \\ &\geq \int_{\mathbb{R}^N} \left( \frac{1}{p^+} - \frac{1}{\vartheta} \right) (|\nabla u_n|^{p(x)} + V(x)|u_n|^{p(x)}) \, dx \\ &\quad + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(y, u_n(y))(2f(x, u_n(x))u_n(x) - \vartheta F(x, u_n(x)))}{2\vartheta|x|^\alpha|x-y|^{\lambda(x,y)}|y|^\alpha} \, dx dy \\ &\geq \left( \frac{1}{p^+} - \frac{1}{\vartheta} \right) \int_{\mathbb{R}^N} (|\nabla u_n|^{p(x)} + V(x)|u_n|^{p(x)}) \, dx, \end{aligned}$$

which implies that the sequence  $\{u_n\}_{n \in \mathbb{N}} \subset W_V^{1,p}(\mathbb{R}^N)$  is bounded.  $\square$

### 2.1. Proof of Theorem 5 completed

Let  $\{u_n\}_{n \in \mathbb{N}} \subset W_V^{1,p}(\mathbb{R}^N)$  be a Palais–Smale sequence at the level  $c$ , that is,  $\mathcal{E}(u_n) = c + o_n(1)$  and  $\mathcal{E}'(u_n) = o_n(1)$  as  $n \rightarrow \infty$ . From Lemma 7 it follows that  $\{u_n\}_{n \in \mathbb{N}}$  is bounded in  $W_V^{1,p(x)}(\mathbb{R}^N)$ . So, we may suppose that  $u_n \rightharpoonup u$  in  $W_V^{1,p}(\mathbb{R}^N)$ ,  $u_n \rightarrow u$  in  $L^{t(x)}(\mathbb{R}^N)$  for all  $t \in C^+(\mathbb{R}^N)$  and  $p \ll t \ll p^*$  and  $u_n(x) \rightarrow u(x)$  a.e. in  $\mathbb{R}^N$  as  $n \rightarrow \infty$ . Since  $\mathcal{E}(u_n) \rightarrow c > 0$  and  $\mathcal{E}'(u_n) \rightarrow 0$  as  $n \rightarrow \infty$ , then we can deduce that  $u \neq 0$ . Assume by contradiction that  $u = 0$ . Using the fact that  $u_n \rightarrow 0$  in  $L^{t(x)}(\mathbb{R}^N)$  for all  $t \in C^+(\mathbb{R}^N)$  and  $p \ll t \ll p^*$ , together with  $(f_1)$  and the definition of  $F$ , we deduce that

$$\begin{aligned} F(\cdot, u_n) &\rightarrow 0 \text{ in } L^{q^+}(\mathbb{R}^N) \text{ as } n \rightarrow \infty, \\ F(\cdot, u_n) &\rightarrow 0 \text{ in } L^{q^-}(\mathbb{R}^N) \text{ as } n \rightarrow \infty, \\ f(\cdot, u_n)u_n &\rightarrow 0 \text{ in } L^{q^+}(\mathbb{R}^N) \text{ as } n \rightarrow \infty, \\ f(\cdot, u_n)u_n &\rightarrow 0 \text{ in } L^{q^-}(\mathbb{R}^N) \text{ as } n \rightarrow \infty. \end{aligned}$$

The above limits and Theorem 2 yield

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(y, u_n(y))f(x, u_n(x))u_n(x)}{|x|^\alpha|x-y|^{\lambda(x,y)}|y|^\alpha} \, dx dy \rightarrow 0 \text{ as } n \rightarrow \infty.$$



Now, we can use the above limit and the fact that  $\langle \mathcal{E}'(u_n), u_n \rangle = o_n(1)$  as  $n \rightarrow \infty$  to conclude that

$$\int_{\mathbb{R}^N} (|\nabla u_n|^{p(x)} + V(x)|u_n|^{p(x)}) dx \rightarrow 0 \text{ as } n \rightarrow \infty.$$

So, it follows that  $u_n \rightarrow 0$  in  $W_V^{1,p}(\mathbb{R}^N)$  as  $n \rightarrow \infty$ . Therefore,  $\mathcal{E}(u_n) \rightarrow 0$  as  $n \rightarrow \infty$ . Since  $\mathcal{E}(u_n) \rightarrow c > 0$  as  $n \rightarrow \infty$ , we reach a contradiction.

Next, we prove that the weak limit  $u$  is a critical point of the functional  $\mathcal{E}$ .

We first show that the following limit holds true up to a subsequence:

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(F(y, u_n(y)) - F(y, u(y))) f(x, u(x)) \varphi(x)}{|x|^\alpha |x - y|^{\lambda(x,y)} |y|^\alpha} dx dy \rightarrow 0 \tag{14}$$

as  $n \rightarrow \infty$ , for all  $\varphi \in C_0^\infty(\mathbb{R}^N)$ . In fact, since  $L^{q^+}(\mathbb{R}^N)$  and  $L^{q^-}(\mathbb{R}^N)$  are uniformly convex, then the Banach space  $(L^{q^+}(\mathbb{R}^N) \cap L^{q^-}(\mathbb{R}^N), \max\{\|\cdot\|_{L^{q^+}(\mathbb{R}^N)}, \|\cdot\|_{L^{q^-}(\mathbb{R}^N)}\})$  is also uniformly convex (and therefore, reflexive). In addition, from the definition of  $F$  and the boundedness of  $\{u_n\}_{n \in \mathbb{N}} \subset W_V^{1,p(x)}(\mathbb{R}^N)$  it follows that the sequence  $\{F(\cdot, u_n)\}_{n \in \mathbb{N}} \subset L^{q^+}(\mathbb{R}^N) \cap L^{q^-}(\mathbb{R}^N)$  is bounded. Then we may assume that there exists a function  $w \in L^{q^+}(\mathbb{R}^N) \cap L^{q^-}(\mathbb{R}^N)$  such that  $F(\cdot, u_n) \xrightarrow{w} w$  (up to a subsequence) in  $L^{q^+}(\mathbb{R}^N) \cap L^{q^-}(\mathbb{R}^N)$  as  $n \rightarrow \infty$ . Using Proposition 5.4.7 of Willem [13], we see that  $F(\cdot, u_n) \xrightarrow{w} F(\cdot, u)$  in  $L^{q^+}(\mathbb{R}^N)$  as  $n \rightarrow \infty$ . Then we can conclude that  $w(\cdot) = F(\cdot, u(\cdot))$  a.e. in  $\mathbb{R}^N$ .

For some fixed  $\varphi \in C_0^\infty(\mathbb{R}^N)$ , we define the following continuous linear functional

$$I_\varphi(v) := \int_{\mathbb{R}^N} v \varphi dx \text{ for all } v \in L^{q^+}(\mathbb{R}^N) \cap L^{q^-}(\mathbb{R}^N).$$

Consequently, we have

$$I_\varphi(F(\cdot, u_n)) \rightarrow \int_{\mathbb{R}^N} w \varphi dx \text{ as } n \rightarrow \infty.$$

By Theorem 2, we can define the following continuous linear functional:

$$\mathcal{L}(v) := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{v(y) f(x, u(x)) \varphi(x)}{|x|^\alpha |x - y|^{\lambda(x,y)} |y|^\alpha} dx dy \text{ for all } v \in L^{q^+}(\mathbb{R}^N) \cap L^{q^-}(\mathbb{R}^N).$$

Finally, since  $\mathcal{L}$  is linear, continuous and  $F(\cdot, u_n) \xrightarrow{w} F(\cdot, u)$  in  $L^{q^+}(\mathbb{R}^N) \cap L^{q^-}(\mathbb{R}^N)$  as  $n \rightarrow \infty$ , we obtain (14).

Combining the fact that  $f$  has a subcritical growth with Theorem 2, the boundedness of  $\{u_n\}_{n \in \mathbb{N}} \subset W_V^{1,p(x)}(\mathbb{R}^N)$ , and the Lebesgue Dominated Convergence Theorem, we have

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(y, u_n(y))(f(x, u_n(x)) - f(x, u(x))) \varphi(x)}{|x|^\alpha |x - y|^{\lambda(x,y)} |y|^\alpha} dx dy \rightarrow 0 \tag{15}$$

as  $n \rightarrow \infty$ , for all  $\varphi \in C_0^\infty(\mathbb{R}^N)$ .

Let  $R > 0$  and  $\psi \in C_0^\infty(\mathbb{R}^N)$  such that  $\psi \in [0, 1]$  and  $\psi(x) = 1$  for  $x \in B_R(0)$ . Using  $\langle \mathcal{E}'(u_n), u_n \psi \rangle \rightarrow 0$  and  $\langle \mathcal{E}'(u_n), u \psi \rangle \rightarrow 0$  as  $n \rightarrow \infty$ , we have

$$\begin{aligned} & \int_{\mathbb{R}^N} (|\nabla u_n|^{p(x)-2} \nabla u_n - |\nabla u|^{p(x)-2} \nabla u) (\nabla u_n - \nabla u) \psi dx + o_n(1) \\ &= \langle \mathcal{E}'(u_n), u_n \psi \rangle - \langle \mathcal{E}'(u_n), u \psi \rangle - \int_{\mathbb{R}^N} (u_n - u) |\nabla u_n|^{p(x)-2} \nabla u_n \nabla \psi dx \\ & \quad - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(y, u_n(y)) f(x, u_n(x)) (u_n(x) - u(x)) \psi(x)}{|x|^\alpha |x - y|^{\lambda(x,y)} |y|^\alpha} dx dy \\ & \quad - \int_{\mathbb{R}^N} V(x) |u_n|^{p(x)-2} u_n (u_n - u) \psi dx - \int_{\mathbb{R}^N} |\nabla u|^{p(x)-2} \nabla u (\nabla u_n - \nabla u) \psi dx. \end{aligned}$$

It is easy to check that

$$\int_{\mathbb{R}^N} (|\nabla u_n|^{p(x)-2} \nabla u_n - |\nabla u|^{p(x)-2} \nabla u) (\nabla u_n - \nabla u) \psi dx \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{16}$$

From Proposition 3.3 of [7], we know that the following estimate

$$(|\xi|^{p(x)-2}\xi - |\zeta|^{p(x)-2}\zeta, \xi - \zeta)_{\mathbb{R}^N} \geq \begin{cases} (|\xi| + |\zeta|)^{p(x)-2}|\xi - \zeta|^2 & \text{if } 1 < p(x) < 2, \\ 4^{1-p^+}|\xi - \zeta|^{p(x)} & \text{if } p(x) \geq 2 \end{cases} \tag{17}$$

holds true for all  $\xi, \zeta \in \mathbb{R}^N$ .

Combining relations (16) and (17), we deduce that for all  $R > 0$  we have  $\nabla u_n \rightarrow \nabla u$  in  $L^{p(x)}(B_R(0))$  as  $n \rightarrow \infty$ . Since  $R > 0$  is arbitrary, passing to a subsequence we can infer that  $\nabla u_n(x) \rightarrow \nabla u(x)$  a.e. in  $\mathbb{R}^N$  as  $n \rightarrow \infty$ .

In addition, since  $\{|\nabla u_n|^{p(x)-2}\nabla u_n\}_{n \in \mathbb{N}} \subset L^{p(x)/(p(x)-1)}(\mathbb{R}^N)$  is bounded, arguing as the proof of Proposition 5.4.7 of Willem [13], we have

$$|\nabla u_n|^{p(x)-2}\nabla u_n \xrightarrow{w} |\nabla u|^{p(x)-2}\nabla u \text{ in } (L^{p(x)/(p(x)-1)}(\mathbb{R}^N))_{\mathbb{R}^N} \text{ as } n \rightarrow \infty. \tag{18}$$

Additionally, applying the Lebesgue Dominated Convergence Theorem, we also get

$$\int_{\mathbb{R}^N} V(x)|u_n|^{p(x)-2}u_n\varphi \, dx \rightarrow \int_{\mathbb{R}^N} V(x)|u|^{p(x)-2}u\varphi \, dx \text{ as } n \rightarrow \infty. \tag{19}$$

Combining relations (14), (15), (18), (19) and the fact that

$$\langle \mathcal{E}'(u_n), \varphi \rangle \rightarrow 0 \text{ for all } \varphi \in C_0^\infty(\mathbb{R}^N) \text{ as } n \rightarrow \infty,$$

we deduce that

$$\langle \mathcal{E}'(u), \varphi \rangle = 0 \text{ for all } \varphi \in C_0^\infty(\mathbb{R}^N).$$

Since  $\overline{C_0^\infty(\mathbb{R}^N)}^{\|\cdot\|} = W_V^{1,p(x)}(\mathbb{R}^N)$ , we conclude that  $u$  is a critical point of  $\mathcal{E}$ , hence a nontrivial solution of problem  $(P_\lambda)$ . □

### Acknowledgments

The authors would like to thank Professor Petru Mironescu for the careful reading of this paper and for his useful remarks on the initial version of the present work. Youpei Zhang would like to thank the China Scholarship Council and the Embassy of the People's Republic of China in Romania.

### References

- [1] C. O. Alves, "Existence of radial solutions for a class of  $p(x)$ -Laplacian equations with critical growth", *Differ. Integral Equ.* **23** (2010), no. 1-2, p. 113-123.
- [2] C. O. Alves, L. S. Tavares, "A Hardy–Littlewood–Sobolev-type inequality for variable exponents and applications to quasilinear Choquard equations involving variable exponent", *Mediterr. J. Math.* **16** (2019), no. 2, article no. 55 (27 pages).
- [3] A. Ambrosetti, P. H. Rabinowitz, "Dual variational methods in critical point theory and applications", *J. Funct. Anal.* **14** (1973), p. 349-381.
- [4] L. Caffarelli, R. Kohn, L. Nirenberg, "First order interpolation inequalities with weights", *Compos. Math.* **53** (1984), no. 3, p. 259-275.
- [5] X. Fan, J. Shen, D. Zhao, "Sobolev embedding theorems for spaces  $W^{k,p(x)}(\Omega)$ ", *J. Math. Anal. Appl.* **262** (2001), no. 2, p. 749-760.
- [6] G. H. Hardy, J. E. Littlewood, "Some properties of fractional integrals. I", *Math. Z.* **27** (1928), no. 1, p. 565-606.
- [7] I. H. Kim, Y.-H. Kim, "Mountain pass type solutions and positivity of the infimum eigenvalue for quasilinear elliptic equations with variable exponents", *Manuscr. Math.* **147** (2015), no. 1-2, p. 169-191.
- [8] R. S. Palais, S. Smale, "A generalized Morse theory", *Bull. Am. Math. Soc.* **70** (1964), p. 165-171.
- [9] P. Pucci, V. D. Rădulescu, "The impact of the mountain pass theory in nonlinear analysis: a mathematical survey", *Boll. Unione Mat. Ital.* (9) **3** (2010), no. 3, p. 543-584.
- [10] B. Rubin, "One-dimensional representation, inversion and certain properties of Riesz potentials of radial functions", *Mat. Zametki* **34** (1983), p. 521-533, translation in *Math. Notes* **34** (1983), p. 751-757.

- [11] S. Sobolev, "On a theorem of functional analysis", *Mat. Sb., N. Ser.* **4** (1938), p. 471-479, translation in *Amer. Math. Soc. Transl. Ser.* **2** **34** (1963), p. 39-68.
- [12] E. M. Stein, G. Weiss, "Fractional integrals on  $n$ -dimensional Euclidean space", *J. Math. Mech.* **7** (1958), p. 503-514.
- [13] M. Willem, *Functional Analysis. Fundamentals and Applications*, Cornerstones, Birkhäuser, 2013.