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Abstract. For any complete C-algebraic variety Y and its underlying compact C-analytic space Y, it follows from the well known GAGA principle that the algebraic Picard group \( \text{Pic}(Y) \) and the analytic Picard group \( \text{Pic}^a(Y) \) are isomorphic. Our main purpose here is to provide a simple proof of an analogous situation for non complete C-algebraic varieties, namely C-algebraic affine hypersurfaces with at most isolated singularities.

1. Introduction

Unless the contrary is explicitly stated, all C-analytic spaces \( \mathcal{X} \) are assumed to be equipped with an analytic structural sheaf \( \mathcal{O}_{\mathcal{X}} \). For any C-algebraic variety \( X \), let us denote by \( \text{Pic}(X) \) (resp. \( \text{Pic}^a(X) = H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}^*) \)), the algebraic (resp. analytic) Picard group of \( X \) (resp. of \( \mathcal{X} \)), where \( \mathcal{X} \) is the C-analytic space associated to \( X \). Any 1-dimensional C-analytic spaces will be referred to as curves. Assume that a given compact C-analytic space \( \mathcal{Y} \) is biholomorphic to an underlying topological space of some complete C-algebraic variety \( Y \); since there is a 1-1 correspondence between linear equivalent classes of Cartier divisors and locally free sheaves of rank 1, it follows from Serre GAGA principle, (see e.g. [7, Chapitre XII, Théorème 4.4] that the analytic Picard group \( \text{Pic}^a(Y) \) and the algebraic Picard group \( \text{Pic}(Y) \) are isomorphic.

On the other hand, let \( \mathcal{X} \) be a C-analytic space which is an underlying topological space of some affine algebraic variety \( X \) defined over C. Then it is well known that

1. \( \mathcal{X} \) is Stein.
2. we have the following exact sequence
   \[ 0 \to \mathcal{O}_X^* \to \mathcal{O}_{\mathcal{X}}^* \to 0 \]
3. \( \text{Pic}(X) = H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}^*) \simeq H^2(X, \mathbb{Z}) \).

In that direction, we have the following well known result:
Theorem 1. Let $\mathbb{A}^n$ (resp. $\mathbb{C}^n$) be the affine $n$-space (resp. the complex $n$-space). Then

\[ \text{Pic}(\mathbb{A}^n) \quad \text{and} \quad \text{Pic}(\mathbb{C}^n) \] are trivial.

As far as reduced non-singular affine algebraic curves are concerned, a glimpse of GAGA principle does enter into this picture, namely

Proposition 2 ([16, Corollary 2.2]). Biholomorphically equivalent non-singular affine algebraic curves are algebraically isomorphic.

Unfortunately, all the similarities cease from there; indeed, one has

Example 3. Let $C$ be a fixed non-singular projective curve of genus $g \geq 0$ together with a finite set of points $p_j \in C$ and let $A := C \setminus \cup_{j \geq 1} p_j$ be the affine curve. By abuse of notations, let us denote also by $A$ its associated (non compact) Riemann surface.

Therefore, since $H^2(A, \mathbb{Z}) = 0$, we infer from (3) that
\[ \text{Pic}(A) = 0 \]

On the other hand, one has

Proposition 4 ([8, Corollary 1.3]). $\text{Pic}(A) = 0$ iff $g = 0$.

Example 5. Let $X := \mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^1 \setminus \{0\}$ and $\mathcal{X} = \mathbb{C} \setminus \{0\} \times \mathbb{C} \setminus \{0\}$. Then it is easy to see that
\[ \text{Pic}(X) = 0 \quad \text{and} \quad \text{Pic}(\mathcal{X}) \cong \mathbb{Z} \]

Example 6. Let $B$ be a fixed non-singular affine curve of genus $g > 0$. For $i = 1, 2$, let $L_i \in \text{Pic}(B)$ be 2 non-equivalent algebraic line bundles. Let $X_i$ be the total space of $L_i$ and let $\mathcal{X}_i$ be its associated Stein surfaces. Then, from [16], one obtains the following biholomorphisms
\[ \mathcal{X}_1 \cong \mathbb{C} \times B \cong \mathcal{X}_2 \] (1)

Therefore, from (1) one has
\[ \text{Pic}(\mathcal{X}_1) = \text{Pic}(\mathcal{X}_2) \]

However, in contrast with Proposition 2, it is known that $X_i$ are not algebraically isomorphic [16, Proposition 3.1]. In spite of this fact and against all expectations, we have the following interesting result which was communicated to us by the referee which we gratefully acknowledge.

Theorem 7. $\text{Pic}(X_1) \simeq \text{Pic}(B) \simeq \text{Pic}(X_2)$

Proof. Let $V$ be an algebraic variety over an algebraically closed field $k$. Let $k_V^*$ be the constant sheaf on $V$ associated to $k^*$, let $G_{m, V}$ be the units sheaf on $V$, and let $U_{k, V} :=$ the presheaf cokernel of $(k_V^* \to G_{m, V})$. Then it is known [14, Lemma 2] that

(1) $U_{k, V}$ is a sheaf on $V$,

(2) $\text{Pic}(V) = H^1(V, U_{k, V}) = H^1(V, G_{m, V})$, and

(3) for a smooth curve $B$ and a Zariski fibration [14, Definition 3] $f : E \to B$ with fibre $F$, one has [14, Theorem 5] the following exact sequence

\[ 0 \to U_{k, B}(B) \to U_{k, F}(E) \to U_{k, F}(F) \to \text{Pic}(B) \to \text{Pic}(E) \to \text{Pic}(F) \] (2)

provided, for all sufficiently small open sets $W$ of $B$ the natural map
\[ \text{Pic}(F) \times \text{Pic}(W) \to \text{Pic}(F \times W) \quad \text{is an isomorphism.} \] (3)

Now, from a more general result in [4, Corollary 6, p. 11] we have, for any smooth algebraic variety $V$,
\[ \text{Pic}(\mathbb{A}^n \times V) \cong \text{Pic}(V) \] (4)
From (4) it follows that, the assumption (3) is fulfilled for any line bundle $E \rightarrow B$; in particular for $X_i$ with $i = 1$ or 2. Therefore the exact sequence (2) can be applied. Furthermore in our case $F = \mathbb{A}^1$, the groups $U_{k,F}(F)$ and $	ext{Pic}(F)$ are trivial. Hence one obtains $\text{Pic}(X_1) = \text{Pic}(B) = \text{Pic}(X_2)$.

Remark 8. Confronted with this state of affairs, we are looking at a class of affine algebraic hypersurfaces $X$ with $\dim X \geq 3$.

2. The non-singular hypersurfaces

Despite such an adverse situation, one has the following important result:

Theorem 9 ([11, Corollary 2.3]). Let $Y \subset \mathbb{P}_{n+1}$ with $n \geq 3$, be a non-singular hypersurface, let $\Gamma \subset Y$ be a transverse hyperplane section and let $X := Y \setminus \Gamma$. Then one has

$$H^i(X, \mathbb{Z}) = 0 \quad \text{for} \quad i \neq 0, n.$$ 

Since the underlying $\mathbb{C}$-analytic variety of $X =: \mathcal{X}$ is a Stein manifold, we infer from Theorem 9, the following:

Corollary 10. Let $X$ be as in Theorem 9. Then

$$\text{Pic}(\mathcal{X}) := H^1(\mathcal{X}, \mathcal{O}_X^*) \simeq H^2(X, \mathbb{Z})$$

is trivial.

Dimensionwise, this result is optimal. In fact, let us look at the following

Example 11. Let $Y_0 \subset \mathbb{P}_3$ be a non-singular hypersurface and let $\Gamma \subset Y_0$ be a transverse hyperplane section. Then it is known [2, Lemma 1.2] that $X_0 := Y_0 \setminus \Gamma$ is homeomorphic to the Milnor fiber of the singularity $(\mathcal{C}, 0)$ where $\mathcal{C}$ is the affine cone over $\Gamma$ with 0 as its vertex. Consequently

$$\text{Pic}(X_0) = H^2(X_0, \mathbb{Z}) = \mathbb{Z}^\mu$$

where $\mu$ is the Milnor number of $(\mathcal{C}, 0)$.

As far as an algebraic analogue of Corollary 10 is concerned, we have the following result:

Proposition 12. Let $Y \subset \mathbb{P}_{n+1}$ be a non-singular hypersurface, let $\Gamma \subset Y$ be a transverse hyperplane section and let $X := Y \setminus \Gamma$. Then

$$\text{Pic}(X) = 0$$

provided $n = \dim Y \geq 3$.

Proof. By [9, Chapter IV, Corollary 3.2] one has $\text{Pic}(Y) = \mathbb{Z} \Gamma$. Then from the following exact sequence [10, Chapter II, Proposition 6.5 (c)]

$$\mathbb{Z} \rightarrow \text{Pic}(Y) \rightarrow \mathbb{Z} \frac{\delta}{\delta} \text{Pic}(X) \rightarrow 0$$

$$1 \rightarrow 1, \Gamma$$

and the surjectivity of $\delta$, we infer the exact sequence

$$0 \rightarrow \text{Pic}(X) \rightarrow 0$$

Hence our desired conclusion will follow.

Remark 13. The proof of Theorem 9 relies heavily on Poincare duality for $\Gamma$ and Alexander duality for the pair $(Y, Y \setminus \Gamma)$ [5] which depend entirely on the fact that both $\Gamma$ and $Y$ are non-singular and the transversal intersection of $\Gamma$. In this situation, it is natural to wonder how such results could be generalized to the context of an ambient space $Y \subset \mathbb{P}_{n+1}$ with only mild singularities; that is the purpose of the next section.
3. Hypersurfaces with isolated singularities

With those examples as guidelines, various endeavors were devoted to generalize Theorem 9 within the framework of hypersurfaces $Y \subset \mathbb{P}_N$ with only isolated singularities and with $N \geq 4$. In the same spirit as Theorem 9, we are now in a position to provide an elementary and complete proof of the following result:

Theorem 14. Let $Y \subset \mathbb{P}_{n+1}$ be an irreducible hypersurface, with only isolated singularities, say $(p_j)_{1 \leq j \leq k}$ and with $n \geq 3$. Let $\Gamma \subset Y$ be a transverse hyperplane section, in particular $p_j \notin \Gamma$ for $1 \leq j \leq k$ and let $X := Y \setminus \Gamma$. Then one has

1. $H_i(X, \mathbb{Z}) = 0$ for $1 \leq i \leq n - 2$.
2. 

$$H_{n-1}(X, \mathbb{Z}) = \begin{cases} 
0 & \text{if } n \text{ is odd.} \\
1 & \text{or finite cyclic if } n \text{ is even.}
\end{cases}$$

Proof.

(Step 1) For simplicity, let us assume that $Y$ has only 1 isolated singularity, say $(p)$. Now let $h : \mathbb{C}^{n+2} \rightarrow \mathbb{C}$ be a homogeneous polynomial of degree $d$, defining the $\mathbb{C}$-projective hypersurface

$$Y = \{x \in \mathbb{P}_{n+1} | h(x) = 0\}$$

In view of the Sard theorem, there exist $\epsilon > 0$ and a general homogeneous polynomial of degree $d$, say $h_d$, so that for any $s \in \Delta := \{|s| \in \mathbb{C} | 0 \leq |s| < \epsilon\}$, the total space of the pencil

$$\mathcal{M} = \{(x, t) \in \mathbb{P}_{n+1} \times \Delta | h + s h_d = 0\}$$

is a one-parameter smoothing of degree $d$, for $Y$.

From the second projection

$$\pi : \mathbb{P}_{n+1} \times \Delta \rightarrow \Delta$$

let $\pi := \pi_2 \mid \mathcal{M} : \mathcal{M} \rightarrow \Delta$ be its restriction. Then one can check that

(a) $\pi^{-1}(0) = Y$ and

(b) $Y_s := \pi^{-1}(s)$ is a smooth $\mathbb{C}$-projective hypersurface of degree $d$, for any $s \neq 0$.

Now by identifying the unique singular point $(p)$ with the origin $0 \in \mathbb{C}^{n+1}$, the singularity $(Y, 0)$ can be defined by $\{f = 0\}$ where $f : (\mathbb{B}_r \subset \mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ is an analytic function germ with an isolated critical point at $0 \in \mathbb{C}^{n+1}$ and $\mathbb{B}_r$ is a ball centered at $0$ with sufficiently small radius $r$. Then [3, § 3].

(c) For $r \neq 0$, $\Xi := f^{-1}(t) \cap \mathbb{B}_r$ is the Milnor fibre of the isolated singularity germ $(Y, 0)$.

(d) $\Xi$ has the homotopy type of a bouquet of $n$-spheres $\mathbb{S}_n$ say, $\bigvee ^n \mathbb{S}_n$, where

$$\dim \mathbb{C} \frac{\mathcal{O}_0(\mathbb{C}^{n+1})}{J_f},$$

$\mathcal{O}_0(\mathbb{C}^{n+1})$, is the local ring of holomorphic functions at $0 \in \mathbb{C}^{n+1}$ and $J_f := (\partial f / \partial x_0, \ldots, \partial f / \partial x_n)$ is the Jacobian ideal of the singularity of $f$.

(e) The ball $\mathbb{B}_r$ has $Y \cap \mathbb{B}_r$ as a deformation retract.

(Step 2) Now let $r$ be such a retraction and let $i$ be the inclusion [15, § I.7]

$$f^{-1}(t) \cap \mathbb{B}_r \hookrightarrow \mathbb{B}_r$$

Then the composite $r \circ i$ gives a map

$$f^{-1}(t) \cap \mathbb{B}_r \rightarrow Y \cap \mathbb{B}_r$$
which contracts $\Xi$ to $\{p\}$.

Now let $\mathcal{B} \subset \mathcal{M}$ be a sufficiently small ball centered at $\{p\}$. Then $\pi^{-1}(s) \cap \mathcal{B}$ can be identified with the Milnor fibre of the isolated hypersurface singularity germ $(Y,0)$, for $s \neq 0$.

Notice that such a contraction can be extended [18, Chapter V, § 14, Exercices(3) p. 332] to a continuous map

$$\Phi : Y_s \rightarrow Y$$

(Step 3) Now let $\phi := \Phi|_{X_s}$, $X_s := Y_s \setminus \phi^{-1}(\Gamma)$ and let us consider the following commutative diagram of integral homology groups with exact rows

\[
\begin{array}{ccccccccc}
H_k(\{p\}) & \longrightarrow & H_k(X) & \longrightarrow & H_k(X, \{p\}) & \longrightarrow & H_{k-1}(\{p\}) & \longrightarrow \\
\uparrow & & \uparrow \phi_* & & \uparrow \delta_* & & \uparrow & \\
H_k(\Xi) & \longrightarrow & H_k(X_s) & \longrightarrow & H_k(X_s, \Xi) & \longrightarrow & H_{k-1}(\Xi) & \\
\end{array}
\]

Since, for any $k > 1$, $\delta_*$ is an isomorphism, we deduce from the above commutative diagram, the following exact sequence

$$H_k(\Xi) \longrightarrow H_k(X_s) \xrightarrow{\phi_*} H_k(X) \longrightarrow H_{k-1}(\Xi)$$

(5)

Now it follows from (2)(b), that

$$H_j(\Xi) = 0 \quad \text{for} \quad 1 \leq j \leq n - 1.$$

Therefore we infer from (5) that $\phi_*$ is an isomorphism, provided $2 \leq k \leq n - 1$ and our conclusion follows from [11, Theorem 9].

(Step 4) Since $H_0(X) \simeq \mathbb{Z} \simeq H_0(X_s)$, we infer from the above commutative diagram, the following exact sequence

$$H_1(\Xi) \longrightarrow H_1(X_s) \xrightarrow{\alpha} H_1(X) \longrightarrow H_0(\Xi) \xrightarrow{\gamma} H_0(\{p\}) \longrightarrow 0$$

(6)

Notice that

(a) In view of (2)(b), $\alpha$ is injective.

(b) Since $H_0(\Xi) \simeq \mathbb{Z} \simeq H_0(\{p\})$, $\gamma$ is bijective; consequently $\alpha$ is also surjective.

Therefore we infer from (6) that

$$H_1(X) = H_1(X_s) = 0.$$

(Step 5) Now let $Y$ with arbitrary isolated singularities $\{p_j\}$. Then exactly as in (Step 1), one can exhibit [1, § 3] a family

$$\pi : \mathcal{M} \rightarrow \Delta$$

such that

(a) $\pi^{-1}(0) = Y$

(b) $Y_s := \pi^{-1}(s)$ for any $s \neq 0$, is a smooth $\mathbb{C}$-projective hypersurface of degree $d$ which is a smooth deformation of $Y$.

(Step 6) Then a construction of the specialization map

$$\Phi : Y_s \rightarrow Y$$

which contracts each Milnor fibre $\Xi_j$ to $p_j$, will be proceeded exactly as carried out in detail in [1, § 3]. Finally the same arguments as in (Step 3) and (Step 4) above will complete our proof. $\square$
By using the Universal coefficient Theorem, we infer from Theorem 14 the following result

**Corollary 15.** Let \( X \) be as in Theorem 14 and let \( \mathcal{X} \) be its associated \( \mathbb{C} \)-analytic space. Then one has

1. \( H^i(X, \mathbb{Z}) = 0 \) if \( 1 \leq i \leq n - 1 \).
2. \( \text{Pic}(\mathcal{X}) \) is trivial.

**Remark 16.** Notice that, the transversal hypothesis of \( \Gamma \) in Theorem 14 is crucial here, as shown by the following

**Example 17** ([11, § 4 p. 213]). Let \( Y_2 := \{x^2 + y^2 + z^2 + w^2 = 0\} \subset \mathbb{P}_4 \{x : y : z : w : t\} \) be a quadric hypersurface with a single (isolated) singular point \( q := (0 : 0 : 0 : 0 : 1) \) and let \( A_2 := Y_2 \cap \{x \neq 0\} \). Then it is clear that \( A_2 \cong \mathbb{P}(\mathbb{C}^2, \mathbb{Z}) \) is a non-singular affine algebraic variety, where \( \zeta := \frac{y}{x}, \xi := \frac{z}{x}, \nu := \frac{w}{x}, \) and \( \tau := \frac{t}{x} \). Certainly \( A_2 \) is homotopically equivalent to \( A_2 \cap \{\tau = 0\} \) which has the same homotopy type as the 2-sphere \( S^2 \); consequently, one has

\[
\text{Pic}(A_2) \cong H^2(S^2, \mathbb{Z}) \cong \mathbb{Z}
\]

where \( A_2 \) is the Stein 3-fold associated to \( A_2 \).

### 3.1. Question

Let \( X \) be as in Theorem 14. Does one also have

\[
\text{Pic}(X) = 0?
\]

In this direction, we would like to provide a positive answer to this question. i.e. an algebraic analogue to our Corollary 15.

**Theorem 18** ([17]). Let \( X \) and \( Y \) be as in Theorem 14. Then one has

\[
\text{Pic}(X) \text{ is trivial.}
\]

Consequently, by using Corollary 15, we obtain the following

**Corollary 19.** Let \( Y \subset \mathbb{P}_{n+1} \) with \( n \geq 3 \), be a non-singular hypersurface, let \( \Gamma \subset Y \) be a transverse hyperplane section and let \( X := Y \setminus \Gamma \). Then one has

\[
\text{Pic}(X) \text{ and } \text{Pic}(\mathcal{X}) \text{ are trivial.}
\]

### 4. Proper hyperplane sections

Motivated by Example 17, throughout this section, let us consider the following:

**Definition 20.** Let \( Y \subset \mathbb{P}_{n+1} \) be an irreducible hypersurface and let \( \mathcal{H} \subset \mathbb{P}_{n+1} \) be a non-singular hyperplane. Then

\[
\mathcal{H} := Y \cap \mathcal{H}
\]

will be referred to as a proper hyperplane section, if \( \dim_x \mathcal{H} = n - 1 \), for any \( x \in \mathcal{H} \).

**Example 21.** Let \( Y_2 \subset \mathbb{P}_4(z_0 : z_1 : z_2 : z_3 : z_4) \) be a singular quadric hypersurface defined by \( \sum_{i=0}^{3} z_i^2 = 0 \) with a single isolated singular point \( p := (0 : 0 : 0 : 0 : 1) \). Let \( Cl(Y_2) := \) the Divisor class group of \( Y_2 \). It is known [10, Example 6.5, p. 147] that

\[
Cl(Y_2) \cong H_4(Y_2, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}
\]

(7)

On the other hand

\[
\text{Pic}(Y_2) \cong H^2(Y_2, \mathbb{Z}) \cong \mathbb{Z}
\]

(8)

From (7) and (8), we have a well known fact that the local ring \( \mathcal{O}_{Y_2, p} \) is not \( \mathbb{Q} \)-factorial.
Remark 22. In sharp contrast with Example 21, one has the following important result

**Theorem 23 ([6, Chapter XI, p. 314]).** Let $Y \subset \mathbb{P}_{n+1}$ be a hypersurface with only isolated singularities $\{p_j\}$. Assume that $\dim Y \geq 4$. Then the local rings $\mathcal{O}_{Y, p_j}$ are factorial for any $j$ (i.e. any Weil divisor on $Y$ is also Cartier).

Now we infer from this result, the Universal Coefficient Theorem and the proof of Theorem 18 the following

**Theorem 24 ([17]).** Let $Y \subset \mathbb{P}_{n+1}$ be an irreducible hypersurface with only isolated singularities and let $\mathcal{H} \subset Y$ be a proper hyperplane section. Let $X := Y \setminus \mathcal{H}$ and let $\mathcal{X}$ be its associated analytic space. Then

$$\text{Pic}(X) \text{ and } \mathbb{Pic}(\mathcal{X}) \text{ are trivial}$$

provided $n \geq 4$.

**Remark 25.** Example 17 shows that the bound given in this Theorem is quite sharp.

5. The transverse hypersurface sections

Throughout this section, let us consider exclusively a non-singular hypersurface $Y \subset \mathbb{P}_{n+1}$ and its transverse hypersurface section $\mathcal{H} \subset Y$ i.e. $\mathcal{H} := Y \cap H_v$, for some non-singular hypersurface $H_v \subset \mathbb{P}_{n+1}$, of degree $v \geq 1$. Then, from seminal works by Kato in [12] and [13], one derives from his far reaching result [13, Theorem 6.3], the following

**Theorem 26.** Let $Y \subset \mathbb{P}_{n+1}$ be a non-singular hypersurface with $n \geq 3$, let $\mathcal{H} \subset Y$ be a transverse hypersurface section and let $X := Y \setminus \mathcal{H}$. Then one has

$$H_i(X, \mathbb{Z}) = \begin{cases} \mathbb{Z}_v & \text{if } i \text{ is odd and } 1 \leq i \leq n - 1, \\ 0 & \text{if } i \text{ is even and } 2 \leq i \leq n - 1. \end{cases}$$

We are now in a position to provide the following result which generalizes Corollary 19.

**Theorem 27 ([17]).** Let $Y \subset \mathbb{P}_{n+1}$ be a non-singular hypersurface with $n \geq 3$ and let $\mathcal{H} \subset Y$ be a transverse hypersurface section. Let $X := Y \setminus \mathcal{H}$ and let $\mathcal{X}$ be its associated Stein manifold. Then one has

$$\text{Pic}(X) \cong \mathbb{Pic}(\mathcal{X}) \cong \mathbb{Z}_v$$

provided $n \geq 3$.

**Remark 28.** Notice that, dimensionwise, Theorem 27 is optimal; indeed besides Example 3 and Proposition 4, let us consider the following:

**Example 29.** Let $\mathcal{C} \subset \mathbb{P}_2$ be a non-singular cubic plane curve (i.e. $g(\mathcal{C}) = 1$) and let $X = \mathbb{P}_2 \setminus \mathcal{C}$. Then [10, Chapter II, Example 6.5.1] one has

$$\text{Pic}(X) \cong \mathbb{Z}_3$$

On the other hand, from the following exact sequence of integral cohomology groups of the pair $(\mathbb{P}_2, X)$

$$H^2(X) \xrightarrow{\lambda} H^3(\mathbb{P}_2, X) \longrightarrow H^3(\mathbb{P}_2) \cong 0$$

since $H^3(\mathbb{P}_2, X) \cong H_1(\mathcal{C})$ and $\text{Rank} \, H_1(\mathcal{C}) = 2g(\mathcal{C}) = 2$, we infer from the surjectivity of $\lambda$, that

$$\text{Pic}(X) \cong H^2(X, \mathbb{Z}) \neq \mathbb{Z}_3 \cong \text{Pic}(X)$$
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