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Operator theory / *Théorie des opérateurs*

# Fredholm conditions for operators invariant with respect to compact Lie group actions

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**Abstract.** Let $G$ be a compact Lie group acting smoothly on a smooth, compact manifold $M$, let $P \in \psi^m(M;E_0,E_1)$ be a $G$–invariant, classical pseudodifferential operator acting between sections of two $G$-equivariant vector bundles $E_i \to M$, $i = 0,1$, and let $\alpha$ be an irreducible representation of the group $G$. Then $P$ induces a map $\pi_\alpha(P) : H^s(M;E_0)_\alpha \to H^{s-m}(M;E_1)_\alpha$ between the $\alpha$-isotypical components. We prove that the map $\pi_\alpha(P)$ is Fredholm if, and only if, $P$ is *transversally $\alpha$-elliptic*, a condition defined in terms of the principal symbol of $P$ and the action of $G$ on the vector bundles $E_i$.

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## 1. Introduction

We let $G$ be a *compact Lie group* acting smoothly on a smooth, *compact* riemannian manifold $M$. There is no loss of generality to assume that $M$ is endowed with a $G$-invariant metric. Let $\psi^m(M;E_0,E_1)$ be the space of order $m$, classical pseudodifferential operators acting between sections of two $G$-equivariant vector bundles $E_i \to M$, $i = 0,1$, and let $\alpha$ be an irreducible representation of the group $G$. Then $P \in \psi^m(M;E_0,E_1)^G$ induces, by $G$-invariance, a map

$$\pi_\alpha(P) : H^s(M;E_0)_\alpha \to H^{s-m}(M;E_1)_\alpha \tag{1}$$

between the $\alpha$-isotypical components of the corresponding Sobolev spaces of sections. In this short note, we obtain necessary and sufficient conditions for $\pi_\alpha(P)$ to be Fredholm. Fredholm operators are important in many applications to PDEs and geometry. Our result generalizes to the case of compact Lie groups the results of [3,4], which dealt with finite groups. We assume the reader is familiar with [3] and here we just explain the main differences from the case of finite groups.

## 1.1. *Notation.*

In order to state our main result, we need to set up some notation, some of which is classical and some of which was already introduced in [3, 4]. If $G$ acts on $X$, we let $X^G$ be the set of fixed points of $G$, we let $G_x$ be the stabilizer of a point $x \in X$ (in $G$), and, finally, we let $X_{(H)}$ be the set of points whose stabilizer is conjugated to $H$. Also, as usual, $\widehat{G}$ denotes the set of equivalence classes of irreducible $G$-modules (or representations, agreeing that all our representations are strongly continuous). In general, if $T : V_0 \to V_1$ is a $G$-equivariant linear map of $G$-modules and $\alpha \in \widehat{G}$, we let

$$\pi_\alpha(T) : V_{0\alpha} \to V_{1\alpha} \tag{2}$$

denote the induced $G$-linear map between the $\alpha$-isotypical components of the $G$-modules $V_i$, $i = 0,1$. Since $P$ is $G$-invariant, its principal symbol $\sigma_m(P)$ belongs to $\mathscr{C}^\infty(T^*M \smallsetminus \{0\}; \mathrm{Hom}(E_0,E_1))^G$, where $\pi : T^*M \to M$ is the cotangent bundle projection. Let $G_\xi$ and $G_x$ denote the isotropy subgroups of $\xi \in T^*_x M$ and $x \in M$, as usual. Then $G_\xi \subset G_x$ acts linearly on the fibers $E_{0x}$ and $E_{1x}$. Let $\mathfrak{g}$ denote the Lie algebra of $G$. Then any $Y \in \mathfrak{g}$ defines a canonical vector field $Y_M$ on $M$. Let the *$G$-transverse cotangent space of $M$* be given by

$$T^*_G M := \left\{\xi \in T^*M \,\middle|\, \forall\, Y \in \mathfrak{g},\ \langle \xi, Y_M(\pi(\xi))\rangle = 0\right\}, \tag{3}$$

as in [2]. As usual, $S^*M$ denotes the unit cosphere bundle of $M$. Let $S^*_G M := S^*M \cap T^*_G M$ denote the set of unit covectors in the $G$-transverse cotangent space $T^*_G M$. The group $G$ acts on $\{G_\xi \mid \xi \in T^*M\}$ by $g \cdot G_\xi := G_{g\xi} = gG_\xi g^{-1}$. For $\rho \in \widehat{G}_\xi$ define $g \cdot \rho \in \widehat{G}_{g\xi}$ by $(g \cdot \rho)(h) = \rho(g^{-1}hg)$, for all $h \in G_{g\xi}$. The characterization of Fredholm operators can be reduced to each component of the orbit space $M/G$, and therefore we can and will assume $M/G$ to be *connected*. Under this hypothesis there exists a minimal isotropy subgroup $K$ such that any isotropy subgroup of $G$ acting on $M$ contains a subgroup conjugated to $K$ and the set of points $M_{(K)}$ with stabilizer conjugated to $K$ is an open dense submanifold $M_{(K)}$ of $M$ called the *principal orbit bundle of $M$*, see [9].

## 1.2. *The $\alpha$-principal symbol and $\alpha$-ellipticity*

Let $E = E_0 \oplus E_1$ and $\Omega_M(E) := \left\{(\xi,\rho) \in S^*_G M \times \widehat{G}_\xi \,\middle|\, E_{\xi\rho} \neq 0\right\}$. Then

$$\begin{aligned} \sigma^G_m(P)\ :\ \Omega_M(E) &\to \bigcup_{(\xi,\rho)\in\Omega_M(E)} \mathrm{Hom}(E_{0\xi\rho}, E_{1\xi\rho})^{G_\xi}, \\ \sigma^G_m(P)(\xi,\rho) &:= \pi_\rho\big(\sigma_m(P)(\xi)\big) \in \mathrm{Hom}(E_{0\xi\rho}, E_{1\xi\rho})^{G_\xi},\ \ \xi \in (S^*_G M)_x, \end{aligned} \tag{4}$$

is the *$G$-principal symbol $\sigma^G_m(P)$* of $P$. (Here we have used that $E_x = E_\xi$ for any $\xi \in T^*_x M$, by the definition of the pullback $\pi^*(E)$ of $E \to M$ to $T^*M$.) If $A$ and $B$ are compact groups, if $H$ is a subgroup of both $A$ and $B$, and if $\mathrm{Hom}_H(\alpha,\beta) \neq 0$, then $\alpha \in \widehat{A}$ and $\beta \in \widehat{B}$ are said *$H$-associated* [4]. Let us fix a minimal isotropy group $K \subset G$ for $M$ and let

$$\Omega^\alpha_M(E) := \left\{(\xi,\rho) \in \Omega_M(E) \,\middle|\, \exists\, g \in G,\ g\cdot\rho \text{ and } \alpha \text{ are } K\text{-associated}\right\}. \tag{5}$$

In the definition of the space $\Omega^\alpha_M(E)$ above, it is implicit that the group element $g \in G$ is such that $K \subset g \cdot G_\xi$.

**Definition 1.** *The* $\alpha$-principal symbol $\sigma^\alpha_m(P)$ *of $P$ is $\sigma^\alpha_m(P) := \sigma^G_m(P)|_{\Omega^\alpha_M(E)}$. We shall say that $P \in \psi^m(M;E_0,E_1)^G$ is* transversally $\alpha$-elliptic *if its $\alpha$-principal symbol $\sigma^\alpha_m(P)$ is invertible everywhere on its domain of definition.*

The transversal $\mathbb{1}$-ellipticity is related with transversal ellipticity on (singular) foliations [1,7,8]. We can now formulate our main result.

**Theorem 2.** *Let $m \in \mathbb{R}$, $P \in \psi^m(M; E_0, E_1)^G$ and $\alpha \in \widehat{G}$. Then*

$$\pi_\alpha(P) : H^s(M; E_0)_\alpha \to H^{s-m}(M; E_1)_\alpha$$

*is Fredholm if, and only if $P$ is transversally $\alpha$-elliptic.*

For $G$ finite, our main result was proved before [3, 4]. This is the first paper that deals with the non-discrete case. As far as the statement of the result goes, the case non-discrete is different from the discrete case in that $S^*_G M \neq S^* M$, in general. The proof in the non-discrete case, is, however, significantly different from the one in the discrete case. Extending our results to the case of compact Lie groups is motivated, in part, by questions in Index Theory and also by the recent preprint [6] and the reference therein. The techniques used in this paper to obtain Fredholm conditions are related also to the ones in [16], used for $G$-opeators, and the ones in [5], used for complexes of operators. See also [8, 12, 14, 15].

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## 2. Background material

This section is devoted to background material and results. The reader can find more details in [3, 4]. We concentrate only on the material that is significantly different from the discrete case, for which we refer to [3]. There is no loss of generality to assume that $M/G$ is connected (recall that $G$ is a compact Lie group acting by isometries on the compact Riemannian manifold $M$).

One of the main differences in the non-discrete case is that we need two versions of induction. Let $H \subset G$ be a closed subgroup and $V$ be an $H$-module, we define, as usual, the *continuous induced representation* by

$$c_0\text{-}\mathrm{Ind}_H^G(V) := \mathscr{C}(G, V)^H = \left\{ f \in \mathscr{C}(G, V) \,\middle|\, f(gh^{-1}) = hf(g) \right\}. \tag{6}$$

Assume that $V$ is a Hilbert space and that the representation is unitary. Then we let the *hilbertian induced representation* $L^2\text{-}\mathrm{Ind}_H^G(V)$ be the completion of $c_0\text{-}\mathrm{Ind}_H^G(V)$ with respect to the induced Hilbert space norm. Then $G$ acts by left translation on $c_0\text{-}\mathrm{Ind}_H^G(V)$ and $L^2\text{-}\mathrm{Ind}_H^G(V)$. Frobenius reciprocity holds for both types of induction and that is used in the proof.

Let us summarize two important properties of the $G$-transversal spaces $T^*_G M$ and $S^*_G M$.

**Lemma 3.** *Let $H$ be a closed subgroup of $G$ and $S$ be a $H$-manifold. Then $T^*_G(G \times_H S) \cong G \times_H T^*_H S$. If $M/G$ is connected with minimal isotropy group $K$, then $S^*_G M_{(K)}$ is dense in $S^*_G M$ and $T^*_G M$ also has $K$ as minimal isotropy group.*

**Proof.** Let us show that $K$ is a minimal isotropy group for $T^*_G M$. We only need to show that there is $\xi_0 \in T^*_G M$ such that $G_{\xi_0} = K$ and that, for all $\xi \in T^*_G M$, a conjugate of $K$ will be contained in $G_\xi$. Since $(x, 0) \in T^*_G M$ it follows that $G_{(x,0)} = K$ for all $x \in M_{(K)}$ with $G_x = K$. Now let $(x, \xi) \in T^*_G M$ and identify $T^* M$ and $TM$ via the Riemannian metric. We have $G_{(x,\xi)} = G_{(x,\lambda\xi)}$, for any $\lambda \in \mathbb{R}^*$ since the action of $G$ is given on $TM$ by the differential of the action on $M$. Replacing $\xi$ by $\lambda\xi$, we can assume that $\xi$ is in the domain of the Riemannian exponential map $\exp_x : T_x M \to M$. Since the exponential map $\exp_x$ identifies ($G_x$-equivariantly) a small neighborhood $U_x \subset (T_G M)_x$ with a slice $S_x$ at $x \in M$, we get that $G_{(x,\xi)} = G_{\exp_x(\xi)} \cap G_x$. But the slice theorem also implies that $G_y \subset G_x$ for any $y \in S_x$ therefore $G_{(x,\xi)} = G_{\exp_x(\xi)}$. Now since $K$ is minimal for $M$, it follows that there is $g \in G$ such that $g \cdot K \subset G_{\exp_x(\xi)} = G_{(x,\xi)}$. $\square$

Let

$$A_M := \mathscr{C}(S^*_G M; \mathrm{End}(E)).$$

and $A^G_M := \mathscr{C}(S^*_G M; \mathrm{End}(E))^G$.

**Remark 4.** Recall that a two-sided ideal $I \subset A$ of a $C^*$-algebra $A$ is called *primitive* if it is the kernel of a *non-zero,* irreducible $*$-representation of $A$. By $\mathrm{Prim}(A)$ we shall denote the set of primitive ideals of $A$, called the *primitive ideal spectrum* of $A$, see [10] for more details. For any $\sigma \in A_M^G$ and $(\xi,\rho) \in \Omega_M(E)$ (see Equation 2), we define

$$\pi_{(\xi,\rho)}(\sigma) := \pi_\rho(\sigma(\xi)) = \sigma(\xi)|_{E_{\xi\rho}}. \tag{7}$$

Then the map $\chi : \Omega_M(E) \to \mathrm{Prim}(A_M^G)$ given by $\chi(\xi,\rho) = \ker(\pi_{(\xi,\rho)})$ induces a bijection $\chi_0 : \Omega_M(E)/G \to \mathrm{Prim}(A_M^G)$, see [4] for a detailed proof, which applies *mutatis mutandis* by replacing $S^*M$ with $S_G^*M$ and $\Gamma$ with $G$. See also [11]. Let us recall briefly the ideas of the proof. The map $\mathscr{C}(S_G^*M)^G \to A_M^G$ given by $f \mapsto f\,\mathrm{Id}_E$ allows to see $\mathscr{C}(S_G^*M)^G \subset A_M^G$ as a central subalgebra. If $\pi : A_M^G \to \mathscr{L}(H)$ is an irreducible representation of $A_M^G$, then $\pi(\mathscr{C}(S_G^*M)^G) \subset \pi(A_M^G)' := \{T \in \mathscr{L}(\mathscr{H}),\ T\pi(a) = \pi(a)T,\ \forall\ a \in A_M^G\}$. By Schur's lemma, it follows that $\pi(\mathscr{C}(S_G^*M)^G) = \mathbb{C}\,\mathrm{Id}_{\mathscr{H}}$, because $\pi(\mathrm{Id}_E) \neq 0$. Therefore, $\pi|_{\mathscr{C}(S_G^*M)^G}$ is a multiple of a character of $\mathscr{C}(S_G^*M)^G$ which corresponds to an orbit $G\xi$. Let $\mathrm{ev}_{G\xi}$ be the evaluation map at the orbit of $\xi$. Now notice that $\mathrm{End}(E_\xi)^{G_\xi} \cong A_M^G/\ker(\mathrm{ev}_{G\xi})A_M^G$ and that $\mathrm{End}(E_\xi)^{G_\xi} = \bigoplus \mathrm{End}(E_{\xi\rho})^{G_\xi}$, where the direct sum is over $\rho \in \widehat{G}_\xi$ such that $\rho \subset E_\xi$. But now Schur's lemma for group representations implies that $\mathrm{End}(E_{\xi\rho})^{G_\xi}$ is isomorphic to the simple algebra $M_{k_\rho}(\mathbb{C})$, where $k_\rho$ is the multiplicity of $\rho$ in $E_\xi$. It follows that $\pi$ corresponds modulo the previous identifications to the projection onto $M_{k_\rho}(\mathbb{C})$, for some $\rho \in \widehat{G}_\xi$ such that $\rho \subset E_\xi$. In other words, we can associate to $\pi$ an element $(\xi,\rho) \in \Omega_M(E)$. The element $(\xi,\rho) \in \Omega_M(E)$ is not unique (it depends on the point $\xi \in G\xi$), but it is unique modulo the action of $G$. Therefore this associates to $\ker(\pi)$ the element $G(\xi,\rho) \in \Omega_M(E)/G$ and defines our desired bijection (which is even a homeomorphism).

## 3. Primitive ideals and the proof of the main theorem

Let $\overline{\psi^0}(M,E)$ (respectively $\overline{\psi^{-1}}(M,E)$) be the norm closure of the algebra $\psi^0(M;E)$ (respectively, $\psi^{-1}(M;E)$) of classical, order zero (respectively, order $-1$) pseudodifferential operators on $M$, a compact manifold. Let $K$ be our fixed minimal isotropy group, let $\alpha \in \widehat{G}$, and let $\pi_\alpha$ the restriction morphism to the $\alpha$-isotypical component $L^2(M;E)_\alpha$ of $L^2(M;E)$, see Equation (2). As for the discrete case, the most important (and technically difficult) part of the proof is the identification of the quotient $\pi_\alpha(\overline{\psi^0}(M;E)^G)/\pi_\alpha(\overline{\psi^{-1}}(M;E)^G)$.

As in [2], the map $\mathscr{C}(S^*M;\mathrm{End}(E))^G \to \pi_\alpha(\overline{\psi^0}(M;E)^G)/\pi_\alpha(\overline{\psi^{-1}}(M;E)^G)$ descends to a surjective algebra morphism

$$\widetilde{\mathscr{R}}_M^\alpha : A_M^G := \mathscr{C}(S_G^*M;\mathrm{End}(E))^G \to \pi_\alpha(\overline{\psi^0}(M,E)^G)/\pi_\alpha(\overline{\psi^{-1}}(M,E)^G).$$

Since the map $\widetilde{\mathscr{R}}_M^\alpha$ is surjective, the question of determining the quotient algebra $\pi_\alpha(\overline{\psi^0}(M,E)^G/\pi_\alpha(\overline{\psi^{-1}}(M;E)^G)$ is equivalent to the question of determining the ideal $\ker(\widetilde{\mathscr{R}}_M^\alpha) \subset A_M^G$. In turn, this ideal will be determined by solving the following problem:

**Problem 5.** *Let $A_M^G := \mathscr{C}(S_G^*M;\mathrm{End}(E))^G$, as before. Identify the closed subset*

$$\Xi^\alpha(E) := \mathrm{Prim}(A_M^G/\ker(\widetilde{\mathscr{R}}_M^\alpha)) \subset \mathrm{Prim}(A_M^G).$$

We notice that if $P$ is a $G$-invariant pseudodifferential operator then $\pi_\alpha(P)$ is Fredholm if, and only if, the image by $\widetilde{\mathscr{R}}_M^\alpha$ of its principal symbol is invertible, by an equivariant version of Atkinson's theorem [3,4]. We next determine which $\ker(\pi_{(\xi,\rho)})$ belongs to $\Xi^\alpha(E)$, for $(\xi,\rho) \in \Omega_M(E)$.

### 3.1. *Calculation on the principal orbit bundle*

Let us consider first arbitrary $(\xi,\rho) \in \Omega_{M_{(K)}}(E)$. By tensoring with $\alpha^*$, we can assume that $\alpha = \mathbb{1}_G \in \widehat{G}$, the trivial representation. (This reduction is justified in Theorem 13.)

**Remark 6.** One of the advantages of assuming $\alpha = \mathbb{1} = \mathbb{1}_G$ is the following simplification:

$$\Omega^{\mathbb{1}}_{M_{(K)}}(E) = \left\{(\xi, \mathbb{1}_{G_\xi}) \,\middle|\, \xi \in S^*_G M_{(K)},\, E_\xi^{G_\xi} \neq 0\right\}.$$

Indeed, this is because if $\alpha = \mathbb{1}_G$, then, in the Equation (5) defining $\Omega^{\mathbb{1}}_{M_{(K)}}(E)$, we have $K = g \cdot G_\xi$, and hence $\rho = \mathbb{1}_{G_\xi}$ and $E_\xi^{G_\xi} \neq 0$.

Let $x_0 := \pi(\xi) \in M_{(K)}$. There is no loss of generality to assume that $G_{x_0} = K$. Let $U \subset (T_G M)_{x_0} \simeq (T^*_G M)_{x_0}$ be a slice at $x_0$, let $W = G\exp_{x_0}(U) \cong G/K \times U$ be the associated tube around $x_0$, and let

$$\eta \in E^K_{x_0} \text{ and } f \in \mathscr{C}^\infty_c(U),\ f(x_0) = 1. \tag{8}$$

We have $(S^*_K U)_{x_0} = S^*_{x_0} U$, since we have assumed $x_0 := \pi(\xi) \in M_{(K)}$. Hence $\xi \in S^*_{x_0} U$. We define then $s_\eta \in \mathscr{C}^\infty_c(W; E)^G$ and $e_t \in \mathscr{C}(W)^G$ by $s_\eta(g\exp_{x_0}(y)) := f(y) g\eta$ and $e_t(g\exp_{x_0}(y)) := e^{\imath t\langle y,\xi\rangle}$, $t \in \mathbb{R}$. That is, they are the functions on $W$ extending the functions $y \mapsto f(y)\eta$ and $y \mapsto e^{\imath t\langle y,\xi\rangle}$ defined on $U \subset T_{x_0}U = (T_K U)_{x_0}$ by $G$-invariance via $W = G\exp_{x_0}(U)$. Let us notice that, if we let $\Phi^G_K$ denote the Frobenius isomorphism, then $s_\eta := \Phi^G_K(f\eta)$ and $e_t = \Phi^G_K(e^{\imath t\langle \bullet,\xi\rangle})$. Recall the map $\chi : \Omega_M(E) \to \mathrm{Prim}(A^G_M)$ of Remark 4. Using oscillatory testing techniques, see, for instance [13, 17], and Remark (6), we obtain.

**Proposition 7.** *Assume that* $0 \neq \eta \in E^K_{x_0}$ *(recall that* $Kx_0 = x_0$*). Then, for every* $P \in \psi^0(M; E)$*, we have* $\lim_{t\to\infty} P(e_t s_\eta)(x_0) = \sigma_0(P)(\xi)\eta$*. In particular, if* $P \in \psi^0(M; E)^G$*, then*

$$\lim_{t\to\infty} \pi_{\mathbb{1}}(P)(e_t s_\eta)(x_0) = \sigma_0(P)(\xi)\eta =: \pi_{(\xi,\mathbb{1})}\big(\sigma_0(P)\big)\eta.$$

*Consequently,* $\ker\big(\pi_{(\xi,\mathbb{1})}\big) \in \Xi^{\mathbb{1}}(E)$*. Equivalently,* $\chi\big(\Omega^{\mathbb{1}}_{M_{(K)}}(E)\big) \subset \Xi^{\mathbb{1}}(E)$*.*

Above $M_{(K)}$, we also have the opposite inclusion.

**Theorem 8.** *We have* $\chi\big(\Omega^{\mathbb{1}}_{M_{(K)}}(E)\big) = \Xi^{\mathbb{1}}(E) \cap \mathrm{Prim}(A^G_{M_{(K)}})$*, and hence*

$$\Xi^{\mathbb{1}}_0(E) := \Xi^{\mathbb{1}}(E) \cap \mathrm{Prim}(A^G_{M_{(K)}}) = \{\ker\big(\pi_{(\xi,\mathbb{1}_{G_\xi})}\big) \mid \xi \in S^*_G M_{(K)},\, E_\xi^{G_\xi} \neq 0\}.$$

**Proof.** Proposition 7 states that $\chi\big(\Omega^{\mathbb{1}}_{M_{(K)}}(E)\big) \subset \Xi^{\mathbb{1}}(E)$. This gives hence the inclusion $\chi\big(\Omega^{\mathbb{1}}_{M_{(K)}}(E)\big) \subset \Xi^{\mathbb{1}}_0(E)$. To prove our result, we need to prove the opposite inclusion. Let $\chi(\xi,\rho) := \ker(\pi_{(\xi,\rho)}) \in \Xi^{\mathbb{1}}_0(E)$. By definition, this means:

- $(\xi,\rho) \in \Omega_M(E)$ and $\xi \in S^*_G M_{(K)}$ and, most importantly,
- $\pi_{(\xi,\rho)}|_{\ker(\widetilde{\mathscr{R}}^{\mathbb{1}}_M)} = 0$.

We need to prove that $(\xi,\rho) \in \Omega^{\mathbb{1}}_{M_{(K)}}(E)$. As before, by replacing $\xi$ with a suitable conjugate $g\xi$, we can assume that $G_\xi = K$ and therefore we need to prove that $E^K_\xi \neq 0$ and $\rho = \mathbb{1}_K$. Since $x := \pi(\xi) \in M_{(K)}$, we can replace $M$ with a tube $W$ around $x$ with contractible slice $U$ on which $E$ is trivial. We shall prove by contradiction that $E^K_\xi \neq 0$. Indeed, if $0 = E^K_\xi = E^K_x$, then $L^2(W; E)^G = 0$, by Frobenius reciprocity, and hence $\ker(\widetilde{\mathscr{R}}^{\mathbb{1}}_W) = A^G_W$. This is, however, not possible since $\ker(\widetilde{\mathscr{R}}^{\mathbb{1}}_W) \subset \ker(\pi_{(\xi,\rho)})$.

We shall prove, again by contradiction, that $\rho = \mathbb{1}_K$. Let us hence assume that $\mathbb{1}_K \neq \rho \in \widehat{K}$. Let $p_\rho$ be the projection onto the isotypical component corresponding to $\rho$ in $\mathrm{End}(E_\xi)^K \simeq \mathrm{End}(E_x)^K$. We have $p_\rho \neq 0$, since $(\xi,\rho) \in \Omega_M(E)$. Recall that $\Phi^G_K$ denotes the induction map. Let $f \in \mathscr{C}^\infty_c(U)$ be equal to 1 near $x$ and extend $f p_\rho$ to a $G$-invariant element $\widetilde{Q} := \Phi^G_K(f p_\rho) \in \mathscr{C}^\infty(M; \mathrm{End}(E))^G \subset \overline{\psi^0}(M, E)^G$ via $W = G\exp(U)$ with principal symbol $\sigma_0(\widetilde{Q}) =: \widetilde{q} = \widetilde{Q} \circ \pi \in A^G_M := \mathscr{C}^\infty(M; \mathrm{End}(E))^G$. Let $\mathbb{P} \in \mathscr{C}^\infty(M_{(K)}; \mathrm{End}(E))^G \subset \mathscr{C}^\infty(S^*_G M_{(K)}; \mathrm{End}(E))^G$ be the projection defined by

$$\mathbb{P}(y) = p_{\mathbb{1}_{G_y}}, \qquad y \in M_{(K)}, \tag{9}$$

that is, $\mathbb{P}(y) \in \mathrm{End}(E_y)$ is the projection onto the vectors fixed by $G_y$, the stabilizer of $y$. It has the property that $\mathbb{P}$ is the identity on $L^2(M; E)^G$ and $\widetilde{Q}\mathbb{P} = 0 \in \mathscr{C}^\infty(M_{(K)}; \mathrm{End}(E))$. We have

$$\pi_{(\xi,\rho)}(\widetilde{q}) := \pi_{(\xi,\rho)}\big(\sigma_0(\widetilde{Q})\big) = f(\pi(\xi)) p_\rho|_{E_{\xi\rho}} = p_\rho \quad \text{and} \quad \pi_{\mathbb{1}}(\widetilde{Q}) := \widetilde{Q}|_{L^2(M;E)^G} = \widetilde{Q}\mathbb{P}|_{L^2(M;E)^G} = 0.$$

Therefore, $\widetilde{\mathscr{R}}^{\mathbb{1}}_M(\widetilde{q}) = \pi_{\mathbb{1}}(\widetilde{Q}) + \pi_{\mathbb{1}}(\overline{\psi^{-1}}(M;E)) = 0$ but $\pi_{(\xi,\rho)}(\Phi^G_K(f p_\rho)) \neq 0$, which contradicts our assumption that $\pi_{(\xi,\rho)}|_{\ker(\widetilde{\mathscr{R}}^{\mathbb{1}}_M)} = 0$. Hence $\rho = \mathbb{1}_K$. □

## 3.2. *Calculation for singular* $(\xi,\rho)$

We now consider $(\xi,\rho) \in \Omega_M(E)$ such that $x := \pi(\xi) \notin M_{(K)}$. A lot of work in this subsection will be devoted to finding the right definitions of the $\widetilde{q}$ and $\widetilde{Q}$ used in the previous subsection and still satisfying the relations $\pi_{(\xi,\rho)}(\widetilde{q}) = p_\rho$, $\widetilde{q} = \sigma_0(\widetilde{Q})$, but, this time, $\mathbb{P}\widetilde{Q} = 0$.

### 3.2.1. *The symbol* $\widetilde{q}$ *and a distinguished neighborhood of* $(\xi,\rho)$.

We first consider the construction of $\widetilde{q}$. We could construct $\widetilde{q}$ to satisfy $\pi_{(\xi,\rho)}(\widetilde{q}) = p_\rho$ using a tubular neighborhood of $\xi$ in $S^*M$, but we need $\widetilde{q}$ to satisfy a few other properties in order to be able to construct $\widetilde{Q}$ in the next subsection. Assume $K \subset G_\xi$. Let us then define succesively

$$F_\xi := \{\beta \in \widehat{G}_\xi \mid \beta \subset E_\xi \text{ and } \beta^K \neq 0\},\ \ p_K := \sum_{\beta\in F_\xi} p_\beta,\ \text{ and }\ e_{\xi,K} := \mathrm{Id}_{E_\xi} - p_K. \tag{10}$$

(In particular, $p_K \in \mathrm{End}(E_\xi)^{G_\xi}$.) To construct the right $\widetilde{q}$, we restrict to a tubular neighoborhood of $x := \pi(\xi)$. Let then $H$ be a closed subgroup of $G$ and let $V$ be a vector space on which $H$ acts by isometries. Let $W = G\times_H V$ and assume that $E = G\times_H (V\times E')$ for some $H$-module $E'$. Let $(\xi,\rho) \in \Omega_W(E)$ be our fixed element. If $\pi(\xi) =: x = [e,0] \in G\times_H V = W$, then $\xi = [e,0,v^*] \in S^*W \cong G\times_H (V\times S^*_x W)$, with $v \in S^*_x W$. We have that $H$ acts on $S^*_x W$. We construct then $\widetilde{q}$ first on the sphere $S^*_x W$ to be $H$-invariant and to coincide with $e_{\xi,K} \neq 0$ at $\xi$ by using an $H$-tubular neighborhood of $v^*$ in $S^*_x W$ and call this function $p$. The main properties of $p$ are thus that $p \in \mathscr{C}^\infty(S^*_x W)^H$ and $p(v^*) = e_{\xi,K}$. Then we extend $p$ to $V\times S^*_x W$ to be constant in the $V$-direction. Since this extension (still called $p$) is $H$-invariant, we can further extend $p$ to $S^*W \cong G\times_H (V\times S^*_x W)$ to be $G$-invariant. Finally, $\widetilde{q} := fp \in \mathscr{C}^\infty_c\big(S^*W;\mathrm{End}(E)\big)^G$, where $f \in \mathscr{C}^\infty_c(W)^G$ is such that $f(x) = 1$, which defines also an element of $\mathscr{C}^\infty_c\big(S^*M;\mathrm{End}(E)\big)^G$ if the support of $f$ is small. Recall the projection $\mathbb{P} \in \mathscr{C}^\infty\big(M_{(K)};\mathrm{End}(E)\big)^G$, $\mathbb{P}(y) = p_{\mathbb{1}_{G_y}}$, of Equation (9).

**Lemma 9.** *Let* $(\xi,\rho) \in \Omega_M(E)$ *with* $K \subset G_\xi$ *and* $\rho^K = 0$. *Then*

(i) $\pi_{(\xi,\rho)}(\widetilde{q}) = p_\rho \neq 0$.

(ii) $\pi_{(\xi,\mathbb{1}_{G_\xi})}(\widetilde{q}) = 0$, *for all* $\xi \in S^*M_{(K)}$.

(iii) $\mathbb{P}\,\widetilde{q} = \widetilde{q}\,\mathbb{P} = 0$ *on* $S^*M_{(K)}$.

(iv) *Let* $V_{\widetilde{q}} = \{(\zeta,\beta) \in \Omega_M(E) \mid \pi_{(\zeta,\beta)}(\widetilde{q}) \neq 0\}$. *Then* $\chi(V_{\widetilde{q}})$ *is an open neighbourhood of* $\chi(\xi,\rho)$ *in* $\mathrm{Prim}(A^G_M)$ *and* $\chi(V_{\widetilde{q}}) \cap \Xi^{\mathbb{1}}_0(E) = \emptyset$. *In particular,* $\chi(\xi,\rho) \notin \overline{\Xi^{\mathbb{1}}_0(E)}$.

**Proof. (i).** Let us notice first that $p_\rho \in \mathrm{End}(E_\xi)^{G_\xi} = \mathrm{End}(E')^{G_\xi}$ is non zero because $(\xi,\rho) \in \Omega_M(E)$ implies that $\rho \subset E_\xi$, by definition. Let us prove (i). Since $\rho^K = 0$ we get that $\rho \notin F_\xi$ (see Equation 10). Therefore, for any $\rho' \in F_\xi$, $p_{\rho'}p_\rho = 0$. This gives that

$$\pi_{(\xi,\rho)}(\widetilde{q}) := \widetilde{q}(\xi)p_\rho = e_{\xi,K}p_\rho = p_\rho - \sum_{\rho'\in F_\xi} p_{\rho'}p_\rho = p_\rho = \mathrm{Id}_{E_{\xi\rho}} \in \mathrm{End}(E_{\xi\rho}).$$

**(ii).** In view of the support of $\widetilde{q}$ and its $G$-invariance, this relation reduces to $e_{\xi,K}p_{\mathbb{1}_K} = 0$ on $V$ (see Equation 10). The relation $e_{\xi,K}p_{\mathbb{1}_K} = 0$ is valid because $E^K_\xi = \bigoplus_{\rho'\in F_\xi}(E_{\xi\rho'})^K$ and therefore $p_K p_{\mathbb{1}_K} = p_{\mathbb{1}_K}$.

**(iii).** The relation (iii) is just another formulation of (ii) since $\mathbb{P}(y) = p_{\mathbb{1}_K}$ on $V_{(K)}$.

**(iv).** It is standard that the set $\chi(V_{\widetilde{q}})$ is open, see for example [3]. By Theorem 8, $\Xi^{\mathbb{1}}_0(E|_W) = \big\{\ker(\pi_{(\zeta,\mathbb{1}_{G_\zeta})}),\ \zeta \in S^*_G W_{(K)},\ E^{G_\zeta}_\zeta \neq 0\big\}$ and then (ii) implies (iv). □

3.2.2. *Density of* $\Xi_0^{\mathbb{1}}$ *in* $\Xi^{\mathbb{1}}$

**Lemma 10.** *Assume that* $(\xi,\rho)\in\Omega_M(E)$ *is such that* $\chi(\xi,\rho)\notin\overline{\Xi_0^{\mathbb{1}}(E)}$ *and that* $K\subset G_\xi$. *Then*

(i) $\rho^K=0$, *and hence, in particular,* $\chi(\Omega_M^{\mathbb{1}}(E))=\overline{\Xi_0^{\mathbb{1}}(E)}=\overline{\chi(\Omega_{M_{(K)}}^{\mathbb{1}}(E))}$ *and*

(ii) *there is* $\widetilde{Q}\in\overline{\psi}^0(M;E)^G$ *such that* $\sigma_0(\widetilde{Q})=\widetilde{q}$ *and* $\pi_{\mathbb{1}}(\widetilde{Q})=0$ *(i.e.,* $\widetilde{Q}=0$ *on* $L^2(M;E)^G$*).*

**Proof. (i).** We already showed in Theorem 8 that $\Xi_0^{\mathbb{1}}(E)=\chi(\Omega_{M_{(K)}}^{\mathbb{1}}(E))$. Since $\ker(\pi_{(\xi,\rho)})$ is maximal (because $A_M^G/\ker(\pi_{(\xi,\rho)})\simeq\mathrm{End}(E_{\xi\rho})^{G_\xi}\simeq M_k(\mathbb{C})$, due to the fact that $\rho\in\widehat{G}_\xi$), the assumption that $\chi(\xi,\rho)\notin\overline{\Xi_0^{\mathbb{1}}(E)}$ implies that there is $\sigma\in A_M^G$ such that $\pi_{(\xi,\rho)}(\sigma)=\mathrm{Id}_{E_{\xi\rho}}$ and the associated neighbourhood $V_\sigma=\{(\eta,\rho')\in\Omega_M(E)\,|\,\pi_{(\eta,\rho')}(\sigma)\neq0\}$ does not intersect $\Xi_0^{\mathbb{1}}(E)$. Hence $\pi_{(\zeta,\mathbb{1})}(\sigma)=0$ for all $(\zeta,\mathbb{1})\in\Omega_{M_{(K)}}^{\mathbb{1}}(E)$. Let us assume, by contradiction, that $\rho^K\neq0$ and let $\zeta_n\in S_G^*M_{(K)}$ be a sequence converging to $\xi$, which exists by Lemma 3. Let us then write $\sigma(\zeta_n)|_{E_{\xi\rho}}=\sigma(\zeta_n)|_{E_{\xi\rho}^K}\oplus\sigma(\zeta_n)|_{(E_{\xi\rho}^K)^\perp}$, for $n$ large enough. Since $(E_{\xi\rho})^K\subset E_\xi^K$ and $\sigma(\zeta_n)|_{E_\xi^K}=\pi_{(\zeta_n,\mathbb{1})}(\sigma)=0$, we obtain that $\mathrm{Id}_{E_{\xi\rho}}=\pi_{(\xi,\rho)}(\sigma)=\lim\sigma(\zeta_n)|_{E_{\xi\rho}}$ vanishes on $(E_{\xi\rho})^K\neq0$, which is a contradiction. The relation $\chi(\Omega_M^{\mathbb{1}}(E))=\overline{\Xi_0^{\mathbb{1}}(E)}$ follows by combining what we have just proved with Lemma 9 (iv). The rest follows from $\Xi_0^{\mathbb{1}}(E)=\chi\big(\Omega_{M_{(K)}}^{\mathbb{1}}(E)\big)$.

**(ii).** Since the problem is local, we can assume that $M=W:=G\times_H V$, as in the previous subsection. Using the compactness of $G/H$, let us cover it with finitely many contractible local charts $D_i$ and let $(\varphi_i^2)$ be a partition of unity subordinated to this covering $\{D_i\}$. Let $\kappa:W:=G\times_H V\to G/H$ be the projection and $Y_i:=\kappa^{-1}(D_i)$. Then $Y_i$ is a finite covering of $W$ and $E|_{Y_i}\simeq Y_i\times E_i'$ is trivial. Let $\chi_i\in\mathscr{C}_c^\infty(D_i)$ be such that $\chi_i\varphi_i=\varphi_i$. We extend trivially the partition of unity $\phi_i$ and the functions $\chi_i$ to $W$ by replacing, for instance, $\phi_i$ with $\phi_i\circ\kappa$. Each $Y_i$ identifies with an open subset of a vector space $Z_i$. Then let $\psi_i\in\mathscr{C}^\infty(Z_i^*)$ be such that $\psi(0)=0$ if $|\eta|<1/2$ and $\psi(\eta)=1$ whenever $|\eta|\geq1$. For any order zero classical symbol $a\in S_{cl}^0(Y_i\times Y_i\times Z_i^*)$ on $Y_i$ and any $s\in\mathscr{C}_c^\infty(Y_i,E)$, we let

$$Op(a)s(y):=(2\pi)^{-\dim W}\int_{Z_i^*}\int_{Z_i}e^{\imath(y-z)\cdot\eta}\,a(y,z,\eta)s(z)\mathrm{d}z\mathrm{d}\eta.$$

We shall use this construction for $a_i(y,z,\eta):=\chi_i(y)\psi_i(\eta)\widetilde{q}\Big(y,\frac{\eta}{|\eta|}\Big)\chi_i(z)$ to obtain $Q_i:=Op(a_i)$, where we have regarded $(y,\eta)\in Y_i\times Z_i^*=T^*Y_i\subset T^*W$. Next we define $Q:=\sum_i\varphi_iQ_i\varphi_i$ and $\widetilde{Q}:=\mathrm{Av}(Q):=\int_G gQg^{-1}dg$. Then $\widetilde{Q}$ is an order zero, $G$-invariant pseudodifferential operator on $W$ with principal symbols $\widetilde{q}$ because the average map commutes with the principal symbol map.

Let us prove now that $\widetilde{Q}|_{L^2(W,E)^G}=0$. By density, it is sufficient to show that $\widetilde{Q}s(y)=0$ for $y\in W_{(K)}$ and $s\in\mathscr{C}_c^\infty(W,E)^G$. By Lemma 9 (iii), we have for $y\in W_{(K)}\cap Y_i$, $z\in Y_i$, and $\eta\in Z_i^*$ that $\mathbb{P}a_i(y,z,\eta)=0$. Thus for any $s\in\mathscr{C}_c^\infty(Y_i,E)$ and $y\in Y_i\cap W_{(K)}$, we obtain that $\mathbb{P}(y)Q_i(s)(y)=0$. Thus we get that $\mathbb{P}Q(s)=0$ on $L^2(W;E)=L^2(W_{(K)};E)$. Using the average map, we obtain

$$0=\big[\mathrm{Av}\big(\mathbb{P}Q(s)\big)\big](y)=\mathbb{P}(y)\big[\mathrm{Av}(Q)s\big](y)=\mathbb{P}(y)(\widetilde{Q}s)(y),\quad\forall\,y\in W_{(K)}.$$

It follows that $\mathbb{P}\widetilde{Q}=0$ on $L^2(W;E)$. Since $\widetilde{Q}(L^2(W;E)^G)\subset L^2(W;E)^G$ and $\mathbb{P}$ acts as the identity on $L^2(W;E)^G$, we obtain that $\widetilde{Q}(L^2(W;E)^G)=\mathbb{P}\widetilde{Q}(L^2(W;E)^G)=0$. □

**Theorem 11.** *The set* $\Xi^{\mathbb{1}}(E):=\mathrm{Prim}(A_M^G/\ker(\widetilde{\mathscr{R}}_M^{\mathbb{1}}))\subset\mathrm{Prim}(A_M^G)$ *associated to the ideal* $\ker(\widetilde{\mathscr{R}}_M^{\mathbb{1}})$ *of* $A_M^G:=\mathscr{C}(S_G^*M;\mathrm{End}(E))^G$ *is the closure in* $\mathrm{Prim}(A_M^G)$ *of the set* $\Xi_0^{\mathbb{1}}(E):=\Xi^{\mathbb{1}}(E)\cap\mathrm{Prim}(A_{M_{(K)}}^G)$.

**Proof.** Since $\Xi^{\mathbb{1}}(E)$ is closed, it is enough to show that, if $\ker(\pi_{(\xi,\rho)})\notin\overline{\Xi_0^{\mathbb{1}}(E)}$, then $\ker(\pi_{(\xi,\rho)})\notin\Xi^{\mathbb{1}}(E)$. We may assume that $K\subset G_\xi$ and $M=G\times_H V$ as before. From Lemma 10 (i), we know that $\rho^K=0$. Furthermore Lemma 10 (ii) gives that there is $\widetilde{Q}\in\psi^0(M;E)^G$ such that $\sigma_0(\widetilde{Q})=\widetilde{q}$ and $\pi_{\mathbb{1}}(\widetilde{Q})=0$ on $L^2(M;E)^G$. Therefore, $\sigma_0(\widetilde{Q})=\widetilde{q}\in\ker(\widetilde{\mathscr{R}}_M^{\mathbb{1}})$. Since, $\pi_{(\xi,\rho)}(\widetilde{q})\neq0$ by Lemma 9 (i), $\pi_{(\xi,\rho)}$ does not vanish on $\ker(\widetilde{\mathscr{R}}_M^{\mathbb{1}})$, which means that $\ker(\pi_{(\xi,\rho)})\notin\Xi^{\mathbb{1}}(E)$. □

Combining the previous results, we obtain the desired determination of the set $\Xi^{\mathbb{1}}(E)$ and hence of the kernel of the morphism $\widetilde{\mathscr{R}}^{\mathbb{1}}_M$.

**Proposition 12.** *We have* $\Xi^{\mathbb{1}}(E) = \overline{\Xi^{\mathbb{1}}_0(E)} = \chi(\Omega^{\mathbb{1}}_M(E))$, *and hence* $\chi_0 : \Omega^{\mathbb{1}}_M(E)/G \to \Xi^{\mathbb{1}}(E)$ *is a bijection.*

The form of the main result given in the introduction is then obtained from the following sequence of equivalences (which was referred to as "reduction to $\alpha = \mathbb{1}$"). The last equivalence of the following theorem was shown for finite groups in [3, Proposition 5.9].

**Theorem 13.** *The following statements are equivalent:*

(i) $\pi_\alpha(P)$ *is Fredholm;*
(ii) $\pi_{\mathbb{1}}(P \otimes \mathrm{Id}_{\alpha^*})$ *is Fredholm;*
(iii) $P \otimes \mathrm{Id}_{\alpha^*}$ *is transversally* $\mathbb{1}$*-elliptic;*
(iv) $P$ *is transversally* $\alpha$*-elliptic;*
(v) *For all* $\xi \in (S^*_G M)^K$ *the linear map*

$$\sigma_m(P)(x,\xi) \otimes \mathrm{Id}_{\alpha^*} : (E_x \otimes \alpha^*)^K \to (E_x \otimes \alpha^*)^K$$

*is an isomorphism.*

**Sketch of the proof.** The equivalence of (i) and (ii) is an elementary, direct operator theory argument. The equivalence of (ii) and (iii) is the difficult part of the statement and it is what we have proved for the most part of this note. The equivalence of (iii) and (iv) is an elementary, direct representation theoretic argument. Finally, (iii) ⇒ (v) by the definition of $\mathbb{1}$-transverse ellipticity. Let $h \in G_\xi$ be such that $h\cdot K \subset G_\xi$. Then $h^{-1}$ will map $E_\xi$ to $E_{h^{-1}\xi}$ and $E_\xi^{h\cdot K}$ to $E^K_{h^{-1}\xi}$. The invertibility of the $G$-invariant symbol $\sigma_m(P)$ on $E_\xi^{h\cdot K}$ (and hence on each $E_{\xi\beta}$ with $\beta^{h\cdot K} \neq 0$) will follow therefore from its invertibility on $E^K_{h^{-1}\xi}$. Thus (v) ⇒ (iii). □

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