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Abstract. In this note, we construct two minimal surfaces of general type with geometric genus $p_g = 3$, irregularity $q = 0$, self-intersection of the canonical divisor $K^2 = 20, 24$ such that their canonical map is of degree 20. In one of these surfaces, the canonical linear system has a non-trivial fixed part. These surfaces, to our knowledge, are the first examples of minimal surfaces of general type with canonical map of degree 20.

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1. Introduction

If $X$ is a minimal smooth complex projective surface, we denote by $\varphi_{|K_X|}: X \rightarrow \mathbb{P}^{p_g(X)-1}$ the canonical map of $X$, where $K_X$ is the canonical divisor of $X$ and $p_g(X) = \dim H^0(X,K_X)$ is the geometric genus. It is interesting to know which positive integers $d$ occur as the degree of such canonical maps for surfaces of general type. This problem is motivated by the work of A. Beauville [1]. One knows that, for surfaces of general type, the degree $d$ of the canonical map is at most 36 [9, Proposition 5.7]. While surfaces with $d = 1, 2, 3, \ldots, 8$ are easy to construct, only few surfaces with $d > 8$ have been known so far. The first example was found by U. Persson [9] in 1977; in this example, the canonical map has degree 16. Then, a surface with $d = 9$ was constructed by S. L. Tan [14] in 1992. In the last decade, some surfaces with $d = 12, 16, 24, 27, 32, 36$ were constructed by C. Rito [10–13], C. Gleissner, R. Pignatelli and C. Rito [4], Ching-Jui Lai and Sai-Kee Yeung [5], and the author [2]. In this paper, we present a way to construct surfaces with $d = 20$ as $\mathbb{Z}_2^4$-covers of the Del Pezzo surface $Y_4$ of degree 5.

Throughout this paper all surfaces are projective algebraic over the complex numbers. The linear equivalence of divisors is denoted by $\equiv$. We call a surface $X$ no non-trivial 2-torsion if the
only 2-torsion in \( \text{Pic}(X) \) is \( \mathcal{O}_X \). A character \( \chi \) of the group \( \mathbb{Z}_2^4 \) is a homomorphism from \( \mathbb{Z}_2^4 \) to \( \mathbb{C}^* \), the multiplicative group of the non-zero complex numbers. We also use the following notations for Del Pezzo surfaces of degree 5:

**Notation 1.** We denote by \( Y_4 \) the blow-up of \( \mathbb{P}^2 \) at four points in general position \( P_1, P_2, P_3, P_4 \). Let us denote by \( l \) the pull-back of a general line in \( \mathbb{P}^2 \), by \( e_1, e_2, e_3, e_4 \) the exceptional divisors corresponding to \( P_1, P_2, P_3, P_4 \), respectively, by \( f_1, f_2, f_3, f_4 \) the strict transforms of a general line through \( P_1, P_2, P_3, P_4 \), respectively and by \( h_{ij} \) the strict transforms of the line \( P_i P_j \), for all \( i \neq j \) in \( \{1, 2, 3, 4\} \), respectively. The anti-canonical class

\[
-K_{Y_4} \equiv f_1 + f_2 + f_3 - e_4 \equiv f_1 + f_2 + f_4 - e_3 \equiv f_1 + f_3 + f_4 - e_2 \equiv f_2 + f_3 + f_4 - e_1
\]

is very ample and the linear system \( |-K_{Y_4}| \) embeds \( Y_4 \) as a smooth Del Pezzo surface of degree 5 in \( \mathbb{P}^5 \).

The construction of abelian covers was studied by R. Pardini in [7]. For details about the building data of abelian covers and their notations, we refer the reader to Section 1 and Section 2 of R. Pardini’s work ([7]). For the sake of completeness, we recall some facts on \( \mathbb{Z}_2^4 \) covers, in a form which is convenient for our later constructions. We will denote by \( \chi_{j_1 j_2 j_3 j_4} \) the character of \( \mathbb{Z}_2^4 \) defined by

\[
\chi_{j_1 j_2 j_3 j_4}(a_1, a_2, a_3, a_4) := e^{(\pi a_1 j_1)\sqrt{-1}} e^{(\pi a_2 j_2)\sqrt{-1}} e^{(\pi a_3 j_3)\sqrt{-1}} e^{(\pi a_4 j_4)\sqrt{-1}}
\]

for all \( j_1, j_2, j_3, j_4, a_1, a_2, a_3, a_4 \in \mathbb{Z}_2 \). A \( \mathbb{Z}_2^4 \) cover \( X \rightarrow Y \) can be determined by a collection of non-trivial divisors \( L_\chi \) labelled by characters of \( \mathbb{Z}_2^4 \) and effective divisors \( D_\sigma \) labelled by elements of \( \mathbb{Z}_2 \) of the surface \( Y \). More precisely, from [7, Theorem 2.1] we can define \( \mathbb{Z}_2^4 \) covers as follows:

**Proposition 2.** Given \( Y \) a smooth projective surface with no non-trivial 2-torsion, let \( L_\chi \) be divisors of \( Y \) such that \( L_\chi \neq \mathcal{O}_Y \) for all non-trivial characters \( \chi \) of \( \mathbb{Z}_2^4 \) and let \( D_\sigma \) be effective divisors of \( Y \) for all \( \sigma \in \mathbb{Z}_2^4 \setminus \{(0,0,0,0)\} \) such that the total branch divisor \( B := \sum_{\sigma \neq 0} D_\sigma \) is reduced. Then \( \{L_\chi, D_\sigma\}_{\chi, \sigma} \) is the building data of a \( \mathbb{Z}_2^4 \) cover \( f : X \rightarrow Y \) if and only if

\[
2L_\chi \equiv \sum_{\chi(\sigma)=-1} D_\sigma
\]

for all non-trivial characters \( \chi \) of \( \mathbb{Z}_2^4 \).

The following theorem is a result of this note:

**Theorem 3.** Let \( f : X \rightarrow Y_4 \) be a \( \mathbb{Z}_2^4 \) cover with the building data \( \{L_\chi, D_\sigma\}_{\chi, \sigma} \) such that the following hold:

(a) Each branch component \( D_\sigma \) is smooth, the total branch locus \( B \) is a simple normal crossings divisor and no more than two of these divisors \( D_\sigma \) go through the same point;

(b) \( D_{0100} + D_{0101} + D_{0111} + D_{1000} + D_{1001} + D_{1010} + D_{1011} + D_{1100} + D_{1101} + D_{1110} + D_{1111} \in |-K_{Y_4}| \);

(c) \( h^0(K_{Y_4} + L_\chi) = 0 \) for all \( \chi \notin \{\chi_{1000}, \chi_{0100}, \chi_{1100}\} \);

(d) The divisor \( D_{0001} + D_{0010} + D_{0011} - K_{Y_4} \) is nef and big.

Then \( X \) is a minimal surface of general type with canonical map of degree 20 satisfying the following:

\[
p_g(X) = 3, \quad K_X^2 = 4\cdot(D_{0001} + D_{0010} + D_{0011} - K_{Y_4})^2.
\]

Moreover, the reduced divisor supported on \( f^*(D_{0001} + D_{0010} + D_{0011}) \) is the fixed part of the canonical system \( |K_X| \).
Let us summarize the proof of Theorem 3. Assumptions (a), (b) and (d) show that the surface $X$ is a minimal surface of general type. Assumption (c) implies that the following diagram commutes (see Remark 6 for the proof):

In the above diagram, the intermediate surface $Z := X/\Gamma$ is the quotient surface of $X$, where $\Gamma := \langle (0,0,0,1), (0,1,0,0) \rangle$ is the subgroup of $Z_2^4$. The surface $Z$ is the bidouble cover of $Y_4$ ramified on

\[(D_{0100} + D_{0101} + D_{0110} + D_{0111}) + (D_{1000} + D_{1001} + D_{1010} + D_{1011}) + (D_{1100} + D_{1101} + D_{1110} + D_{1111}).\]

Assumption (b) shows that the canonical map of $Z$ is of degree 5 (see Remark 6 for the proof). Therefore, the canonical map of $Z$ is of degree 20. As application of Theorem 3, we construct two surfaces with $d = 20$ described as follows:

**Theorem 4.** There exist minimal surfaces of general type $X$ satisfying the following

| $d$ | $K_X^2$ | $P_g(X)$ | $q(X)$ | $|K_X|$ |
|-----|--------|----------|--------|--------|
| 20  | 20     | 3        | 0      | base point free |
| 20  | 24     | 3        | 0      | has a non-trivial fixed part |

2. $Z_2^4$-coverings

For the convenience of the reader, we leave here the relations (1) of the building data of $Z_2^4$-covers:

By [7, Theorem 3.1] if each branch component $D_0$ is smooth and the total branch locus $B$ is a simple normal crossings divisor, the surface $X$ is smooth.

Also from [7, Lemma 4.2, Proposition 4.2] we have:
Proposition 5. If $Y$ is a smooth surface and $f : X \to Y$ is a smooth $\mathbb{Z}^4_2$-cover with the building data $\{L_X, D_\sigma\}_{X, \sigma}$, the surface $X$ satisfies the following:

\begin{align*}
2K_X &= f^* \left( 2K_Y + \sum_{\sigma \neq 0} D_\sigma \right); \\
f^* \mathcal{O}_X &= \mathcal{O}_Y \oplus \bigoplus_{\chi \neq \chi_{0000}} L_{\chi}^{-1}; \\
H^0 (X, K_X) &= H^0 (Y, K_Y) \oplus \bigoplus_{\chi \neq \chi_{0000}} H^0 (Y, K_Y + L_{\chi}); \\
K_X^2 &= 4 \left( 2K_Y + \sum_{\sigma \neq 0} D_\sigma \right)^2; \\
p_g (X) &= p_g (Y) + \sum_{\chi \neq \chi_{0000}} h^0 (L_X + K_Y); \\
\chi (\mathcal{O}_X) &= 16 \chi (\mathcal{O}_Y) + \sum_{\chi \neq \chi_{0000}} \frac{1}{2} L_{\chi} (L_X + K_Y).
\end{align*}

Moreover, the canonical linear system $|K_X|$ is generated by

\begin{equation}
\begin{aligned}
f^* (|K_Y + L_X|) + \sum_{\chi (\sigma) = 1} R_\sigma, \quad \forall \chi \in J
\end{aligned}
\end{equation}

where $J : = \{ \chi' : |K_Y + L_{\chi'}| \neq \emptyset \}$ and $R_\sigma$ is the reduced divisor supported on $f^* (D_\sigma)$.

For the proof of the last statement of Proposition 5, we refer the reader to [4, p. 3].

3. Surfaces with $d = 20$ as $\mathbb{Z}^4_2$-covers

3.1. Proof of Theorem 3

The surface $X$ is smooth because each branch component $D_\sigma$ is smooth, the total branch locus $B$ is a normal crossings divisor and no more than two of these divisors $D_\sigma$ go through the same point. Moreover, by Proposition 5, the surface $X$ satisfies the following:

\begin{align*}
2K_X &= f^* \left( 2K_Y + \sum_{\sigma} D_\sigma \right) \\
&= f^* \left( D_{0001} + D_{0010} + D_{0011} - K_Y \right).
\end{align*}

We notice that a surface is of general type and minimal if the canonical divisor is big and nef (see e.g. [6, Section 2]). We remark that the divisor $D_{0001} + D_{0010} + D_{0011} - K_Y$ is nef and big by Assumption (d). Since the divisor $2K_X$ is the pull-back of a nef and big divisor, the canonical divisor $K_X$ is nef and big. Thus, the surface $X$ is of general type and minimal. Furthermore, from Proposition 5, the surface $X$ possesses the following invariants:

\begin{align*}
p_g (X) &= 3, \quad K_X^2 = 4 \left( D_{0001} + D_{0010} + D_{0011} - K_Y \right)^2.
\end{align*}

We show that the canonical map $\varphi_{|K_X|}$ has degree 20. By Assumptions (b) and (c), we have

\begin{align*}
L_{1000} + K_Y &\equiv L_{0100} + K_Y \equiv L_{1100} + K_Y \equiv \mathcal{O}_Y, \\
h^0 (L_X + K_Y) &= 0, \quad \forall \chi \in \{ \chi_{1000}, \chi_{0100}, \chi_{1100} \}.
\end{align*}

By (8), the linear system $|K_X|$ is generated by the three following divisors:

\begin{align*}
D_{0001} + D_{0010} + D_{0011} + D_{0100} + D_{0110} + D_{0111}, \\
D_{0001} + D_{0010} + D_{0011} + D_{1000} + D_{1010} + D_{1011}, \\
D_{0001} + D_{0010} + D_{0011} + D_{1100} + D_{1110} + D_{1111},
\end{align*}
where $D_\sigma$ are the reduced divisors supported $f^*(D_\sigma)$, for all $\sigma$. Because the divisors $D_{0001}, D_{0010}, D_{0011}$ are common components of the three above divisors, these divisors $D_{0001}, D_{0010}, D_{0011}$ are fixed components of $|K_X|$.

On the other hand, by Assumption (a) the three divisors $D_{0100} + D_{0101} + D_{0110} + D_{0111}, D_{1000} + D_{1010} + D_{1011}, D_{1100} + D_{1101} + D_{1110} + D_{1111}$ have no common intersection. So the linear system $|M|$ is base point free, where $M := D_{0100} + D_{0101} + D_{0110} + D_{0111}$. This together with $M^2 = 4(3I - e_1 - e_2 - e_3 - e_4)^2 = 20 > 0$ implies that the linear system $|K_X|$ is not composed with a pencil. Thus, the canonical image is $\mathbb{P}^2$, the canonical map is of degree 20, and the divisor $D_{0001} + D_{0010} + D_{0011}$ is the fixed part of $|K_X|$.

**Remark 6.** The canonical map $\varphi_{|K_X|}$ of $X$ is the composition of the quotient map $X \to Z := X/\Gamma$ with the canonical map $\varphi_{|K_Z|}$ of $Z$. Moreover, the canonical map of $Z$ is of degree 5.

In fact, by (4), we have the following decomposition:

$$ H^0(X, K_X) = H^0(Y_4, K_{Y_4}) \oplus \bigoplus_{\chi \neq \chi_{1000}} H^0(Y_4, K_{Y_4} + L_X). $$

The group $\Gamma := \langle (0,0,0,1), (0,0,1,0) \rangle$ is the subgroup of $\mathbb{Z}^4_2$. Let $\Gamma^\perp$ denote the kernel of the restriction map $(\mathbb{Z}_2^4)^* \to \Gamma^*$, where $\Gamma^*$ is the character group of $\Gamma$. We have $\Gamma^\perp = \langle \chi_{1000}, \chi_{0100}, \chi_{1100} \rangle$. The subgroup $\Gamma$ acts trivially on $H^0(X, K_X)$ since $h^0(L_X + K_{Y_4}) = 0$ for all $\chi \notin \Gamma^\perp$ by Assumption (c). So the canonical map $\varphi_{|K_X|}$ is the composition of the quotient map $X \to Z := X/\Gamma$ with the canonical map $\varphi_{|K_Z|}$ of $Z$ (see e.g. [8, Example 2.1]).

The intermediate surface $Z$ is the bidouble cover of $Y_4$ with the building data $\{D_1, D_2, D_3, L_1, L_2, L_3\}$ determined as follows:

$$ D_1 := D_{0100} + D_{0101} + D_{0110} + D_{0111} \equiv -K_{Y_4}, $$
$$ D_2 := D_{1000} + D_{1001} + D_{1010} + D_{1011} \equiv -K_{Y_4}, $$
$$ D_3 := D_{1100} + D_{1101} + D_{1110} + D_{1111} \equiv -K_{Y_4}, $$

$$ L_1 := L_{1000} \equiv -K_{Y_4}, $$
$$ L_2 := L_{0100} \equiv -K_{Y_4}, $$
$$ L_3 := L_{1100} \equiv -K_{Y_4}. $$

Assumption (a) shows that the singularities of $Z$ are nodes and the canonical map of $Z$ is of degree $(3l - e_1 - e_2 - e_3 - e_4)^2 = 5$.

### 3.2. Constructions of the surfaces in Theorem 4

#### 3.2.1. A surface with $d = 20, p_g = 3, q = 0, K^2 = 20$

In this section, we construct the surface described in the first row of Theorem 4. Let $Y_4$ be a Del Pezzo surface of degree 5 (see Notation 1). We consider the following smooth divisors of $Y_4$:

$$ D_{0101} := h_{14}, $$
$$ D_{1001} := f_{11} + e_2, $$
$$ D_{1101} := h_{13}, $$
$$ D_{0110} := f_{31} + e_1, $$
$$ D_{1010} := h_{23}, $$
$$ D_{1110} := h_{34}, $$
$$ D_{0111} := h_{12}, $$
$$ D_{1011} := h_{24}, $$
$$ D_{1111} := f_{21} + e_3. $$
and $D_{\sigma} = 0$ for the other $\sigma$, where $f_{11} \in |f_1|$, $f_{21} \in |f_2|$ and $f_{31} \in |f_3|$ such that no more than two of these divisors $D_{\sigma}$ go through the same point. We consider the following non-trivial divisors of $Y_4$:

\[
\begin{align*}
L_{0001} &:= \ f_1 \ + \ f_2 \ + \ f_3 \ - \ e_4 \\
L_{0010} &:= \ f_1 \ + \ f_2 \ + \ f_3 \ - \ e_4 \\
L_{1000} &:= \ f_1 \ + \ f_2 \ + \ f_3 \ - \ e_4 \\
L_{0011} &:= \ f_1 \ + \ f_2 \ + \ f_3 \ - \ e_4 \\
L_{0101} &:= \ f_1 \ + \ f_2 \ + \ f_3 \ - \ e_4 \\
L_{0110} &:= \ h_{12} \ + \ h_{34} \\
L_{0111} &:= \ h_{12} \ + \ h_{34} \\
L_{1010} &:= \ f_1 \ + \ f_3 \\
L_{1011} &:= \ f_2 \ + \ f_4 \\
L_{1100} &:= \ f_1 \ + \ f_2 \ + \ f_3 \ - \ e_4 \\
L_{1101} &:= \ f_2 \ + \ f_3 \\
L_{1110} &:= \ f_1 \ + \ f_4 \\
L_{1111} &:= \ h_{12} \ + \ h_{34} \\
\end{align*}
\]

These divisors $D_{\sigma}, L_X$ satisfy the following relations:

\[
\begin{align*}
2L_{0001} &\equiv D_{0001} + D_{0111} + D_{1000} + D_{1111} = 4f_1 + 2f_2 - 2e_4 \\
2L_{0010} &\equiv D_{0001} + D_{0011} + D_{0101} + D_{1111} = 4f_2 + 2f_3 - 2e_4 \\
2L_{1000} &\equiv D_{0001} + D_{0101} + D_{0111} + D_{1111} = 2f_1 + 2f_3 - 2e_4 \\
2L_{1001} &\equiv D_{0001} + D_{0011} + D_{0101} + D_{1111} = 2f_1 + 2f_3 - 2e_4 \\
2L_{1100} &\equiv D_{0001} + D_{0101} + D_{0111} + D_{1111} = 2f_1 + 2f_3 + 2f_4 - 2e_4 \\
2L_{1101} &\equiv D_{0001} + D_{0101} + D_{0111} + D_{1111} = 2f_1 + 2f_3 + 2f_4 - 2e_4 \\
2L_{1110} &\equiv D_{0001} + D_{0011} + D_{0101} + D_{1111} = 2h_{12} + 2h_{34} \\
2L_{1111} &\equiv D_{0001} + D_{0011} + D_{0101} + D_{1111} = 2h_{12} + 2h_{34} \\
2L_{0100} &\equiv D_{0001} + D_{0111} + D_{1000} + D_{1111} = 2h_{12} + 2h_{34} \\
2L_{0101} &\equiv D_{0001} + D_{0111} + D_{1000} + D_{1111} = 2f_1 + 2f_3 + 2f_4 - 2e_4 \\
2L_{0110} &\equiv D_{0001} + D_{0111} + D_{1000} + D_{1111} = 2f_1 + 2f_3 + 2f_4 - 2e_4 \\
2L_{0111} &\equiv D_{0011} + D_{0101} + D_{0111} + D_{1111} = 2f_1 + 2f_3 + 2f_4 - 2e_4 \\
\end{align*}
\]

Thus by Proposition 2, the divisors $D_\sigma, L_X$ define a $\mathbb{Z}_2^4$-cover $g : X \rightarrow Y_4$. Moreover, this $\mathbb{Z}_2^4$-cover fulfils the hypotheses of Theorem 3. In fact, we have that

\[
\begin{align*}
D_{0100} + D_{0110} + D_{0111} &\equiv h_{14} + f_{31} + e_4 + h_{12} \equiv 3l - e_1 - e_2 - e_3 - e_4 \\
D_{1000} + D_{1001} + D_{1010} + D_{1011} &\equiv f_{11} + e_2 + h_{23} + h_{24} \equiv 3l - e_1 - e_2 - e_3 - e_4 \\
D_{1100} + D_{1101} + D_{1110} + D_{1111} &\equiv h_{13} + h_{34} + f_{21} + e_3 \equiv 3l - e_1 - e_2 - e_3 - e_4,
\end{align*}
\]

$h^0(K_{Y_4} + L_X) = 0$ for all $\chi \notin \{\chi_{1000}, \chi_{0100}, \chi_{1100}\}$, and the divisor $D_{0001} + D_{0010} + D_{0011} - K_{Y_4} \equiv 3l - e_1 - e_2 - e_3 - e_4$ is nef and big. Thus by Theorem 3 and Proposition 5, the surface $X$ is a minimal surface of general type and possesses the following invariants:

\[
K_X^2 = 20, p_g(X) = 3, \chi(\Omega^1_X) = 4, q(X) = 0.
\]

Moreover, the canonical map $\varphi_{|K_X|}$ is of degree 20 and the linear system $|K_X|$ is base point free.

**Remark 7.** The surface $X$ has four pencils of genus 9 corresponding to the fibres $f_1, f_2, f_3, f_4$.

In the above construction, for each choice of $f_{11} \in |f_1|$, $f_{21} \in |f_2|$ and $f_{31} \in |f_3|$, we obtain a natural deformation of the surface $X$ (we refer [7, Definition 5.1] for the definition of natural deformations of an abelian cover). It is worth pointing out that a natural deformation of an abelian cover $X \rightarrow Y$ is a deformation of the map $X \rightarrow Y$ by [7, Proposition 5.1].
Remark 8. The surface $X$ admits natural deformations. Moreover, all the natural deformations of $X$ are Galois.

In fact, by [7, Definition 5.1] the natural deformations of the $\mathbb{Z}_2^4$-cover $g : X \to Y_4$ are parametrized by the direct sum of the vector spaces

$$\bigoplus_{\sigma \neq 0} H^0(Y_4, D_{\sigma}) \bigoplus_{L \neq 0} H^0(Y_4, D_{\sigma} - L_X).$$

Moreover, all the natural deformations of $X$ are Galois if the second summand $\bigoplus_{L \neq 0} H^0(Y_4, D_{\sigma} - L_X)$ is zero (see [3, Definition 3.2]). We have that

$$H^0(Y_4, D_{0110}) = H^0(Y_4, f_{31}) \cong \mathbb{C}^2$$
$$H^0(Y_4, D_{1001}) = H^0(Y_4, f_{11}) \cong \mathbb{C}^2$$
$$H^0(Y_4, D_{1111}) = H^0(Y_4, f_{21}) \cong \mathbb{C}^2$$

and $H^0(Y_4, D_{\sigma}) \cong \mathbb{C}$ for the other non-trivial $D_{\sigma}$. So the family of natural deformations of $g : X \to Y_4$ is parametrized by the base space $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Furthermore, all natural deformations of $X$ are Galois since $\bigoplus_{L \neq 0} H^0(Y_4, D_{\sigma} - L_X) = 0$.

3.2.2. A surface with $d = 20$, $p_g = 3$, $q = 0$, $K^2 = 24$

In this section, we construct the surface described in the second row of Theorem 4. We consider the following smooth divisors of a del Pezzo surface $Y_4$ of degree 5:

$$D_{0011} := e_4$$
$$D_{0101} := h_{14}$$
$$D_{1000} := e_2$$
$$D_{1100} := h_{34}$$

and the other $D_{\sigma} = 0$, where $f_1 \in |f_1|$, $f_2 \in |f_2|$ and $f_3 \in |f_3|$ such that no more than two of these divisors $D_{\sigma}$ go through the same point. We consider the following non-trivial divisors of $Y_4$:

$$L_{0001} := 2f_1 + f_2 - e_3$$
$$L_{0010} := f_2 + l$$
$$L_{0100} := f_1 + f_2 + f_3 - e_4$$
$$L_{1000} := f_1 + f_2 + f_3 - e_4$$
$$L_{0011} := f_1 + 2f_2 - e_3 - e_4$$
$$L_{0110} := f_2 + f_3$$
$$L_{0111} := 2f_1 + f_2 - e_3 - e_4$$
$$L_{0111} := f_2 + f_3 - e_4$$
$$L_{1001} := f_3 + f_4$$
$$L_{1010} := f_1 + f_2 + f_3 - e_3$$
$$L_{1011} := f_1 + f_2 + f_3 - e_4$$
$$L_{1100} := f_1 + f_2 + f_3 - e_4$$
$$L_{1101} := f_1 + f_2 + f_4$$
$$L_{1110} := f_1 + f_2 + f_3 - e_4$$
$$L_{1111} := f_1 + f_3.$$
These divisors $D_\sigma, L_X$ satisfy the following relations:

\[
24_001 = D_{001} + D_{010} + D_{100} + D_{111} = 4f_1 + 2f_2 + 2f_3 - 2e_1
\]
\[
24_201 = D_{001} + D_{010} + D_{100} + D_{111} = 4f_1 + 2f_2 + 2f_3 - 2e_1
\]
\[
24_210 = D_{001} + D_{010} + D_{100} + D_{111} = 4f_1 + 2f_2 + 2f_3 - 2e_1
\]
\[
24_110 = D_{001} + D_{010} + D_{100} + D_{111} = 4f_1 + 2f_2 + 2f_3 - 2e_1
\]
\[
24_111 = D_{001} + D_{010} + D_{100} + D_{111} = 4f_1 + 2f_2 + 2f_3 - 2e_1
\]
\[
24_011 = D_{001} + D_{010} + D_{100} + D_{111} = 4f_1 + 2f_2 + 2f_3 - 2e_1
\]
\[
24_010 = D_{001} + D_{010} + D_{100} + D_{111} = 4f_1 + 2f_2 + 2f_3 - 2e_1
\]
\[
24_000 = D_{001} + D_{010} + D_{100} + D_{111} = 4f_1 + 2f_2 + 2f_3 - 2e_1
\]

Thus by Proposition 2, the divisors $D_\sigma, L_X$ define a $\mathbb{Z}_2^4$-cover $g : X \to Y_4$. Moreover, this $\mathbb{Z}_2^4$-cover fulfills the hypotheses of Theorem 3. In fact, we have

\[
D_{0100} + D_{0101} + D_{0110} + D_{0111} = h_{14} + f_{21} + f_{31} \equiv 3l - e_1 - e_2 - e_3 - e_4
\]
\[
D_{1000} + D_{1001} + D_{1010} + D_{1011} = e_2 + h_{23} + h_{24} + f_{11} \equiv 3l - e_1 - e_2 - e_3 - e_4
\]
\[
D_{1100} + D_{1101} + D_{1110} + D_{1111} = h_{34} + h_{12} + h_{13} + e_1 + e_3 \equiv 3l - e_1 - e_2 - e_3 - e_4.
\]

$h^0(K_{X_1} + L_1) = 0$ for all $\chi \in \{\chi_{1000}, \chi_{0100}, \chi_{1100}\}$, and the divisor $D_{0001} + D_{0101} + D_{0111} - K_{X_1} \equiv 3l - e_1 - e_2 - e_3$ is nef and big. Thus by Theorem 3 and Proposition 5, the surface $X$ is a minimal surface of general type and possesses the following invariants:

\[
K_X^2 = 24, p_g(S) = 3, \chi(\sigma) = 4, q(S) = 0.
\]

Moreover, the canonical map $\varphi_{|K_X|}$ is of degree 20 and the two $(-2)$-curves coming from $\mathbb{Z}_2$ are the fixed part of $|K_X|$. Therefore, we obtain the surface in the second row of Theorem 4.

**Remark 9.** The surface $X$ has three pencils of genus 9 corresponding the fibres $f_1, f_2, f_3$ and a pencil of genus 13 corresponding to the fibre $f_4$.

**Remark 10.** The surface $X$ admits natural deformations. Moreover, all the natural deformations of $X$ are Galois.

Similarly to Remark 8, we have that $H^0(Y_4, D_{0100}) \cong H^0(Y_4, D_{0111}) \cong H^0(Y_4, D_{1011}) \cong \mathbb{C}_2$ and $H^0(Y_4, D_\sigma) \cong \mathbb{C}$ for the other non-trivial $D_\sigma$. This implies that the family of natural deformations of $g : X \to Y_4$ is parametrized by the base space $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Furthermore, all natural deformations of $X$ are Galois since $\bigoplus_{\sigma \neq 0} H^0(Y_4, D_\sigma - L_X) = 0$.

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**References**


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