INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus
Mathématique

Nguyen Bin

Some examples of algebraic surfaces with canonical map of degree 20

Volume 359, issue 9 (2021), p. 1145-1153

<https://doi.org/10.5802/crmath.267>

© Académie des sciences, Paris and the authors, 2021.
Some rights reserved.

This article is licensed under the
Creative Commons Attribution 4.0 International License.
http://creativecommons.org/licenses/by/4.0/

Les Comptes Rendus. Mathématique *sont membres du*
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org

Comptes Rendus
Mathématique
2021, 359, nº 9, p. 1145-1153
https://doi.org/10.5802/crmath.267

Algebraic geometry / *Géométrie algébrique*

Some examples of algebraic surfaces with canonical map of degree 20

Nguyen Bin [a]

[a] Mathematics Division, National Center for Theoretical Sciences, Taiwan
E-mail: nguyenbin@ncts.ntu.edu.tw (Nguyen Bin)

Dedicated to Margarida Mendes Lopes on the occasion of her sixty-fifth birthday

Abstract. In this note, we construct two minimal surfaces of general type with geometric genus $p_g = 3$, irregularity $q = 0$, self-intersection of the canonical divisor $K^2 = 20, 24$ such that their canonical map is of degree 20. In one of these surfaces, the canonical linear system has a non-trivial fixed part. These surfaces, to our knowledge, are the first examples of minimal surfaces of general type with canonical map of degree 20.

2020 Mathematics Subject Classification. 14J29.

Funding. The author was partially supported by Fundação para a Ciência e Tecnologia (FCT), Portugal through the program Lisbon Mathematics PhD (LisMath) of the University of Lisbon, scholarship FCT - PD/BD/113632/2015 and project UID/MAT/04459/2019 of CAMGSD. This paper was finished during the author's postdoctoral fellowship at the National Center for Theoretical Sciences (NCTS), Taiwan, under the grant number MOST 109-2119-M-002-014. The author would like to thank NCTS for the financial support and kind hospitality.

Manuscript received 4th January 2021, revised 5th May 2021 and 4th September 2021, accepted 5th September 2021.

1. Introduction

If X is a minimal smooth complex projective surface, we denote by $\varphi_{|K_X|} : X \dashrightarrow \mathbb{P}^{p_g(X)-1}$ the canonical map of X, where K_X is the canonical divisor of X and $p_g(X) = \dim H^0(X, K_X)$ is the geometric genus. It is interesting to know which positive integers d occur as the degree of such canonical maps for surfaces of general type. This problem is motivated by the work of A. Beauville [1]. One knows that, for surfaces of general type, the degree d of the canonical map is at most 36 [9, Proposition 5.7]. While surfaces with $d = 1, 2, 3, \ldots, 8$ are easy to construct, only few surfaces with $d > 8$ have been known so far. The first example was found by U. Persson [9] in 1977; in this example, the canonical map has degree 16. Then, a surface with $d = 9$ was constructed by S. L. Tan [14] in 1992. In the last decade, some surfaces with $d = 12, 16, 24, 27, 32, 36$ were constructed by C. Rito [10–13], C. Gleissner, R. Pignatelli and C. Rito [4], Ching-Jui Lai and Sai-Kee Yeung [5], and the author [2]. In this paper, we present a way to construct surfaces with $d = 20$ as $\mathbb{Z}_2^4$-covers of the Del Pezzo surface Y_4 of degree 5.

Throughout this paper all surfaces are projective algebraic over the complex numbers. The linear equivalence of divisors is denoted by $\equiv$. We call a surface X no non-trivial 2-torsion if the only 2-torsion in Pic(X) is $\mathcal{O}_X$. A character χ of the group $\mathbb{Z}_2^4$ is a homomorphism from $\mathbb{Z}_2^4$ to $\mathbb{C}^*$, the multiplicative group of the non-zero complex numbers. We also use the following notations for Del Pezzo surfaces of degree 5:

Notation 1. We denote by Y_4 the blow-up of $\mathbb{P}^2$ at four points in general position P_1, P_2, P_3, P_4. Let us denote by l the pull-back of a general line in $\mathbb{P}^2$, by e_1, e_2, e_3, e_4 the exceptional divisors corresponding to P_1, P_2, P_3, P_4, respectively, by f_1, f_2, f_3, f_4 the strict transforms of a general line through P_1, P_2, P_3, P_4, respectively and by h_{ij} the strict transforms of the line P_iP_j, for all $i \neq j$ in $\{1,2,3,4\}$, respectively. The anti-canonical class

$$-K_{Y_4} \equiv f_1 + f_2 + f_3 - e_4 \equiv f_1 + f_2 + f_4 - e_3 \equiv f_1 + f_3 + f_4 - e_2 \equiv f_2 + f_3 + f_4 - e_1$$

is very ample and the linear system $|-K_{Y_4}|$ embeds Y_4 as a smooth Del Pezzo surface of degree 5 in $\mathbb{P}^5$.

The construction of abelian covers was studied by R. Pardini in [7]. For details about the building data of abelian covers and their notations, we refer the reader to Section 1 and Section 2 of R. Pardini's work ([7]). For the sake of completeness, we recall some facts on $\mathbb{Z}_2^4$-covers, in a form which is convenient for our later constructions. We will denote by $\chi_{j_1j_2j_3j_4}$ the character of $\mathbb{Z}_2^4$ defined by

$$\chi_{j_1j_2j_3j_4}(a_1,a_2,a_3,a_4) := e^{(\pi a_1 j_1)\sqrt{-1}} e^{(\pi a_2 j_2)\sqrt{-1}} e^{(\pi a_3 j_3)\sqrt{-1}} e^{(\pi a_4 j_4)\sqrt{-1}}$$

for all $j_1, j_2, j_3, j_4, a_1, a_2, a_3, a_4 \in \mathbb{Z}_2$. A $\mathbb{Z}_2^4$-cover $X \to Y$ can be determined by a collection of non-trivial divisors L_χ labelled by characters of $\mathbb{Z}_2^4$ and effective divisors D_σ labelled by elements of $\mathbb{Z}_2^4$ of the surface Y. More precisely, from [7, Theorem 2.1] we can define $\mathbb{Z}_2^4$-covers as follows:

Proposition 2. *Given Y a smooth projective surface with no non-trivial 2-torsion, let L_χ be divisors of Y such that $L_\chi \not\equiv \mathcal{O}_Y$ for all non-trivial characters χ of $\mathbb{Z}_2^4$ and let D_σ be effective divisors of Y for all $\sigma \in \mathbb{Z}_2^4 \setminus \{(0,0,0,0)\}$ such that the total branch divisor $B := \sum_{\sigma\neq 0} D_\sigma$ is reduced. Then $\{L_\chi, D_\sigma\}_{\chi,\sigma}$ is the building data of a $\mathbb{Z}_2^4$-cover $f : X \to Y$ if and only if*

$$2L_\chi \equiv \sum_{\chi(\sigma)=-1} D_\sigma \tag{1}$$

for all non-trivial characters χ of $\mathbb{Z}_2^4$.

The following theorem is a result of this note:

Theorem 3. *Let $f : X \to Y_4$ be a $\mathbb{Z}_2^4$-cover with the building data $\{L_\chi, D_\sigma\}_{\chi,\sigma}$ such that the following hold:*

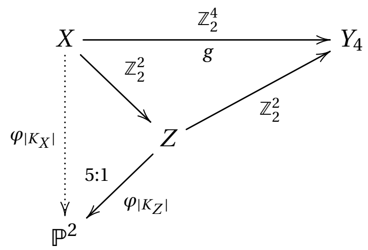
(a) *Each branch component D_σ is smooth, the total branch locus B is a simple normal crossings divisor and no more than two of these divisors D_σ go through the same point;*
(b) $D_{0100}+D_{0101}+D_{0110}+D_{0111}$, $D_{1000}+D_{1001}+D_{1010}+D_{1011}$, $D_{1100}+D_{1101}+D_{1110}+D_{1111} \in |-K_{Y_4}|$;
(c) $h^0(K_{Y_4}+L_\chi) = 0$ *for all* $\chi \notin \{\chi_{1000}, \chi_{0100}, \chi_{1100}\}$;
(d) *The divisor* $D_{0001}+D_{0010}+D_{0011}-K_{Y_4}$ *is nef and big.*

Then X is a minimal surface of general type with canonical map of degree 20 satisfying the following:

$$p_g(X) = 3, \quad K_X^2 = 4\left(D_{0001}+D_{0010}+D_{0011}-K_{Y_4}\right)^2.$$

Moreover, the reduced divisor supported on $f^*(D_{0001}+D_{0010}+D_{0011})$ is the fixed part of the canonical system $|K_X|$.

Let us summarize the proof of Theorem 3. Assumptions (a), (b) and (d) show that the surface X is a minimal surface of general type. Assumption (c) implies that the following diagram commutes (see Remark 6 for the proof):



In the above diagram, the intermediate surface $Z := X/\Gamma$ is the quotient surface of X, where $\Gamma := \langle (0,0,0,1), (0,0,1,0) \rangle$ is the subgroup of $\mathbb{Z}_2^4$. The surface Z is the bidouble cover of Y_4 ramified on

$$(D_{0100} + D_{0101} + D_{0110} + D_{0111}) + (D_{1000} + D_{1001} + D_{1010} + D_{1011}) + (D_{1100} + D_{1101} + D_{1110} + D_{1111}).$$

Assumption (b) shows that the canonical map of Z is of degree 5 (see Remark 6 for the proof). Therefore, the canonical map of X is of degree 20. As application of Theorem 3, we construct two surfaces with $d = 20$ described as follows:

Theorem 4. *There exist minimal surfaces of general type X satisfying the following*

d	K_X^2	$p_g(X)$	$q(X)$	$\lvert K_X \rvert$
20	20	3	0	*base point free*
20	24	3	0	*has a non-trivial fixed part*

2. $\mathbb{Z}_2^4$-coverings

For the convenience of the reader, we leave here the relations (1) of the building data of $\mathbb{Z}_2^4$-covers:

B	$= D_{0001}$	$+D_{0010}$	$+D_{0011}$	$+D_{0100}$	$+D_{0101}$	$+D_{0110}$	$+D_{0111}$	$+D_{1000}$	$+D_{1001}$	$+D_{1010}$	$+D_{1011}$	$+D_{1100}$	$+D_{1101}$	$+D_{1110}$	$+D_{1111}$
$2L_{0001}$	$\equiv D_{0001}$		$+D_{0011}$		$+D_{0101}$		$+D_{0111}$		$+D_{1001}$		$+D_{1011}$		$+D_{1101}$		$+D_{1111}$
$2L_{0010}$	$\equiv$	D_{0010}	$+D_{0011}$			$+D_{0110}$	$+D_{0111}$			$+D_{1010}$	$+D_{1011}$			$+D_{1110}$	$+D_{1111}$
$2L_{0100}$	$\equiv$			D_{0100}	$+D_{0101}$	$+D_{0110}$	$+D_{0111}$					$+D_{1100}$	$+D_{1101}$	$+D_{1110}$	$+D_{1111}$
$2L_{1000}$	$\equiv$							D_{1000}	$+D_{1001}$	$+D_{1010}$	$+D_{1011}$	$+D_{1100}$	$+D_{1101}$	$+D_{1110}$	$+D_{1111}$
$2L_{0011}$	$\equiv D_{0001}$	$+D_{0010}$			$+D_{0101}$	$+D_{0110}$			$+D_{1001}$	$+D_{1010}$			$+D_{1101}$	$+D_{1110}$	
$2L_{0101}$	$\equiv D_{0001}$		$+D_{0011}$	$+D_{0100}$		$+D_{0110}$			$+D_{1001}$		$+D_{1011}$	$+D_{1100}$		$+D_{1110}$	
$2L_{0110}$	$\equiv$	D_{0010}	$+D_{0011}$	$+D_{0100}$	$+D_{0101}$					$+D_{1010}$	$+D_{1011}$	$+D_{1100}$	$+D_{1101}$		
$2L_{0111}$	$\equiv D_{0001}$	$+D_{0010}$		$+D_{0100}$			$+D_{0111}$		$+D_{1001}$	$+D_{1010}$		$+D_{1100}$			$+D_{1111}$
$2L_{1001}$	$\equiv D_{0001}$		$+D_{0011}$		$+D_{0101}$		$+D_{0111}$	$+D_{1000}$		$+D_{1010}$		$+D_{1100}$		$+D_{1110}$	
$2L_{1010}$	$\equiv$	D_{0010}	$+D_{0011}$			$+D_{0110}$	$+D_{0111}$	$+D_{1000}$	$+D_{1001}$			$+D_{1100}$	$+D_{1101}$		
$2L_{1011}$	$\equiv D_{0001}$	$+D_{0010}$			$+D_{0101}$	$+D_{0110}$		$+D_{1000}$			$+D_{1011}$	$+D_{1100}$			$+D_{1111}$
$2L_{1100}$	$\equiv$			D_{0100}	$+D_{0101}$	$+D_{0110}$	$+D_{0111}$	$+D_{1000}$	$+D_{1001}$	$+D_{1010}$	$+D_{1011}$				
$2L_{1101}$	$\equiv D_{0001}$		$+D_{0011}$	$+D_{0100}$		$+D_{0110}$		$+D_{1000}$		$+D_{1010}$			$+D_{1101}$		$+D_{1111}$
$2L_{1110}$	$\equiv$	D_{0010}	$+D_{0011}$	$+D_{0100}$	$+D_{0101}$			$+D_{1000}$	$+D_{1001}$					$+D_{1110}$	$+D_{1111}$
$2L_{1111}$	$\equiv D_{0001}$	$+D_{0010}$		$+D_{0100}$			$+D_{0111}$	$+D_{1000}$			$+D_{1011}$		$+D_{1101}$	$+D_{1110}$	

By [7, Theorem 3.1] if each branch component D_σ is smooth and the total branch locus B is a simple normal crossings divisor, the surface X is smooth.

Also from [7, Lemma 4.2, Proposition 4.2] we have:

Proposition 5. *If Y is a smooth surface and $f : X \to Y$ is a smooth $\mathbb{Z}_2^4$-cover with the building data $\{L_\chi, D_\sigma\}_{\chi,\sigma}$, the surface X satisfies the following:*

$$2K_X \equiv f^*\left(2K_Y + \sum_{\sigma\neq 0} D_\sigma\right); \tag{2}$$

$$f_*\mathcal{O}_X = \mathcal{O}_Y \oplus \bigoplus_{\chi\neq\chi_{0000}} L_\chi^{-1}; \tag{3}$$

$$H^0(X, K_X) = H^0(Y, K_Y) \oplus \bigoplus_{\chi\neq\chi_{0000}} H^0\left(Y, K_Y + L_\chi\right); \tag{4}$$

$$K_X^2 = 4\left(2K_Y + \sum_{\sigma\neq 0} D_\sigma\right)^2; \tag{5}$$

$$p_g(X) = p_g(Y) + \sum_{\chi\neq\chi_{0000}} h^0\left(L_\chi + K_Y\right); \tag{6}$$

$$\chi(\mathcal{O}_X) = 16\chi(\mathcal{O}_Y) + \sum_{\chi\neq\chi_{0000}} \frac{1}{2} L_\chi\left(L_\chi + K_Y\right). \tag{7}$$

Moreover, the canonical linear system $|K_X|$ is generated by

$$f^*\left(|K_Y + L_\chi|\right) + \sum_{\chi(\sigma)=1} R_\sigma, \quad \forall\, \chi \in J \tag{8}$$

where $J := \{\chi' : |K_Y + L_{\chi'}| \neq \emptyset\}$ and R_σ is the reduced divisor supported on $f^*(D_\sigma)$.

For the proof of the last statement of Proposition 5, we refer the reader to [4, p. 3].

3. Surfaces with $d = 20$ as $\mathbb{Z}_2^4$-covers

3.1. *Proof of Theorem 3*

The surface X is smooth because each branch component D_σ is smooth, the total branch locus B is a normal crossings divisor and no more than two of these divisors D_σ go through the same point. Moreover, by Proposition 5, the surface X satisfies the following:

$$\begin{aligned} 2K_X &\equiv f^*\left(2K_{Y_4} + \sum_\sigma D_\sigma\right) \\ &\equiv f^*\left(D_{0001} + D_{0010} + D_{0011} - K_{Y_4}\right). \end{aligned}$$

We notice that a surface is of general type and minimal if the canonical divisor is big and nef (see e.g. [6, Section 2]). We remark that the divisor $D_{0001} + D_{0010} + D_{0011} - K_{Y_4}$ is nef and big by Assumption (d). Since the divisor $2K_X$ is the pull-back of a nef and big divisor, the canonical divisor K_X is nef and big. Thus, the surface X is of general type and minimal. Furthermore, from Proposition 5, the surface X possesses the following invariants:

$$p_g(X) = 3, \quad K_X^2 = 4\left(D_{0001} + D_{0010} + D_{0011} - K_{Y_4}\right)^2.$$

We show that the canonical map $\varphi_{|K_X|}$ has degree 20. By Assumptions (b) and (c), we have

$$\begin{aligned} &L_{1000} + K_{Y_4} \equiv L_{0100} + K_{Y_4} \equiv L_{1100} + K_{Y_4} \equiv \mathcal{O}_{Y_4}, \\ &h^0\left(L_\chi + K_{Y_4}\right) = 0, \qquad \forall\, \chi \notin \{\chi_{1000}, \chi_{0100}, \chi_{1100}\}. \end{aligned}$$

By (8), the linear system $|K_X|$ is generated by the three following divisors:

$$\begin{aligned} &\overline{D}_{0001} + \overline{D}_{0010} + \overline{D}_{0011} + \overline{D}_{0100} + \overline{D}_{0101} + \overline{D}_{0110} + \overline{D}_{0111}, \\ &\overline{D}_{0001} + \overline{D}_{0010} + \overline{D}_{0011} + \overline{D}_{1000} + \overline{D}_{1001} + \overline{D}_{1010} + \overline{D}_{1011}, \\ &\overline{D}_{0001} + \overline{D}_{0010} + \overline{D}_{0011} + \overline{D}_{1100} + \overline{D}_{1101} + \overline{D}_{1110} + \overline{D}_{1111}, \end{aligned}$$

where $\overline{D}_\sigma$ are the reduced divisors supported $f^*(D_\sigma)$, for all σ. Because the divisors $\overline{D}_{0001}$, $\overline{D}_{0010}$, $\overline{D}_{0011}$ are common components of the three above divisors, these divisors $\overline{D}_{0001}$, $\overline{D}_{0010}$, $\overline{D}_{0011}$ are fixed components of $|K_X|$.

On the other hand, by Assumption (a) the three divisors $\overline{D}_{0100}+\overline{D}_{0101}+\overline{D}_{0110}+\overline{D}_{0111}$, $\overline{D}_{1000}+\overline{D}_{1001}+\overline{D}_{1010}+\overline{D}_{1011}$, $\overline{D}_{1100}+\overline{D}_{1101}+\overline{D}_{1110}+\overline{D}_{1111}$ have no common intersection. So the linear system $|M|$ is base point free, where $M := \overline{D}_{0100}+\overline{D}_{0101}+\overline{D}_{0110}+\overline{D}_{0111}$. This together with $M^2 = 4(3l-e_1-e_2-e_3-e_4)^2 = 20 > 0$ implies that the linear system $|K_X|$ is not composed with a pencil. Thus, the canonical image is $\mathbb{P}^2$, the canonical map is of degree 20, and the divisor $\overline{D}_{0001}+\overline{D}_{0010}+\overline{D}_{0011}$ is the fixed part of $|K_X|$.

Remark 6. The canonical map $\varphi_{|K_X|}$ of X is the composition of the quotient map $X \to Z := X/\Gamma$ with the canonical map $\varphi_{|K_Z|}$ of Z. Moreover, the canonical map of Z is of degree 5.

In fact, by (4), we have the following decomposition:

$$H^0(X,K_X) = H^0\left(Y_4,K_{Y_4}\right) \oplus \bigoplus_{\chi\neq\chi_{0000}} H^0\left(Y_4,K_{Y_4}+L_\chi\right).$$

The group $\Gamma := \langle(0,0,0,1),(0,0,1,0)\rangle$ is the subgroup of $\mathbb{Z}_2^4$. Let $\Gamma^\perp$ denote the kernel of the restriction map $\left(\mathbb{Z}_2^4\right)^* \to \Gamma^*$, where Γ^* is the character group of Γ. We have $\Gamma^\perp = \left\langle \chi_{1000}, \chi_{0100}, \chi_{1100}\right\rangle$. The subgroup Γ acts trivially on $H^0(X,K_X)$ since $h^0\left(L_\chi+K_{Y_4}\right)=0$ for all $\chi \notin \Gamma^\perp$ by Assumption (c). So the canonical map $\varphi_{|K_X|}$ is the composition of the quotient map $X \to Z := X/\Gamma$ with the canonical map $\varphi_{|K_Z|}$ of Z (see e.g. [8, Example 2.1]).

The intermediate surface Z is the bidouble cover of Y_4 with the building data $\{D_1,D_2,D_3,L_1,L_2,L_3\}$ determined as follows:

$$\begin{aligned}
D_1 &:= D_{0100}+D_{0101}+D_{0110}+D_{0111} \equiv -K_{Y_4}, & L_1 &:= L_{1000} \equiv -K_{Y_4},\\
D_2 &:= D_{1000}+D_{1001}+D_{1010}+D_{1011} \equiv -K_{Y_4}, & L_2 &:= L_{0100} \equiv -K_{Y_4},\\
D_3 &:= D_{1100}+D_{1101}+D_{1110}+D_{1111} \equiv -K_{Y_4}, & L_3 &:= L_{1100} \equiv -K_{Y_4}.
\end{aligned}$$

Assumption (a) shows that the singularities of Z are nodes and the canonical map of Z is of degree $(3l-e_1-e_2-e_3-e_4)^2 = 5$.

3.2. *Constructions of the surfaces in Theorem 4*

3.2.1. *A surface with $d = 20$, $p_g = 3$, $q = 0$, $K^2 = 20$*

In this section, we construct the surface described in the first row of Theorem 4. Let Y_4 be a Del Pezzo surface of degree 5 (see Notation 1). We consider the following smooth divisors of Y_4:

$$\begin{aligned}
D_{0101} &:= h_{14} & D_{0110} &:= f_{31}+e_1 & D_{0111} &:= h_{12}\\
D_{1001} &:= f_{11}+e_2 & D_{1010} &:= h_{23} & D_{1011} &:= h_{24}\\
D_{1101} &:= h_{13} & D_{1110} &:= h_{34} & D_{1111} &:= f_{21}+e_3
\end{aligned}$$

and $D_\sigma = 0$ for the other σ, where $f_{11} \in |f_1|$, $f_{21} \in |f_2|$ and $f_{31} \in |f_3|$ such that no more than two of these divisors D_σ go through the same point. We consider the following non-trivial divisors of Y_4:

$$\begin{array}{llllll}
L_{0001} := & 2f_1 & +f_2 & & -e_4 \\
L_{0010} := & & 2f_2 & +f_3 & -e_4 \\
L_{0100} := & f_1 & +f_2 & +f_3 & -e_4 \\
L_{1000} := & f_1 & +f_2 & +f_3 & -e_4 \\
L_{0011} := & f_1 & & +2f_3 & -e_4 \\
L_{0101} := & & & f_3 & +f_4 \\
L_{0110} := & h_{12} & +h_{34} & & \\
L_{0111} := & f_1 & +f_2 & & \\
L_{1001} := & h_{12} & +h_{34} & & \\
L_{1010} := & f_1 & & +f_3 & \\
L_{1011} := & & f_2 & & +f_4 \\
L_{1100} := & f_1 & +f_2 & +f_3 & -e_4 \\
L_{1101} := & & f_2 & +f_3 & \\
L_{1110} := & f_1 & & & +f_4 \\
L_{1111} := & h_{12} & +h_{34}. & &
\end{array}$$

These divisors D_σ, L_χ satisfy the following relations:

$$\begin{array}{llllllllllllllll}
2L_{0001} & \equiv & D_{0101} & & +D_{0111} & +D_{1001} & & +D_{1011} & +D_{1101} & & +D_{1111} & \equiv & 4f_1 & +2f_2 & & -2e_4 \\
2L_{0010} & \equiv & & D_{0110} & +D_{0111} & & +D_{1010} & +D_{1011} & & +D_{1110} & +D_{1111} & \equiv & & 4f_2 & +2f_3 & -2e_4 \\
2L_{0100} & \equiv & D_{0101} & +D_{0110} & +D_{0111} & & & & +D_{1101} & +D_{1110} & +D_{1111} & \equiv & 2f_1 & +2f_2 & +2f_3 & -2e_4 \\
2L_{1000} & \equiv & & & & D_{1001} & +D_{1010} & +D_{1011} & +D_{1101} & +D_{1110} & +D_{1111} & \equiv & 2f_1 & +2f_2 & +2f_3 & -2e_4 \\
2L_{0011} & \equiv & D_{0101} & +D_{0110} & & +D_{1001} & +D_{1010} & & +D_{1101} & +D_{1110} & & \equiv & 2f_1 & & +4f_3 & -2e_4 \\
2L_{0101} & \equiv & & D_{0110} & & +D_{1001} & & +D_{1011} & & +D_{1110} & & \equiv & & & 2f_3 & +2f_4 \\
2L_{0110} & \equiv & D_{0101} & & & & +D_{1010} & +D_{1011} & +D_{1101} & & & \equiv & 2h_{12} & +2h_{34} & & \\
2L_{0111} & \equiv & & & D_{0111} & +D_{1001} & +D_{1010} & & & & +D_{1111} & \equiv & 2f_1 & +2f_2 & & \\
2L_{1001} & \equiv & D_{0101} & & +D_{0111} & & +D_{1010} & & & +D_{1110} & & \equiv & 2h_{12} & +2h_{34} & & \\
2L_{1010} & \equiv & & D_{0110} & +D_{0111} & +D_{1001} & & & +D_{1101} & & & \equiv & 2f_1 & & +2f_3 & \\
2L_{1011} & \equiv & D_{0101} & +D_{0110} & & & & +D_{1011} & & & +D_{1111} & \equiv & & 2f_2 & & +2f_4 \\
2L_{1100} & \equiv & D_{0101} & +D_{0110} & +D_{0111} & +D_{1001} & +D_{1010} & +D_{1011} & & & & \equiv & 2f_1 & +2f_2 & +2f_3 & -2e_4 \\
2L_{1101} & \equiv & & D_{0110} & & & +D_{1010} & & +D_{1101} & & +D_{1111} & \equiv & & 2f_2 & +2f_3 & \\
2L_{1110} & \equiv & D_{0101} & & & +D_{1001} & & & & +D_{1110} & +D_{1111} & \equiv & 2f_1 & & & +2f_4 \\
2L_{1111} & \equiv & & & D_{0111} & & & +D_{1011} & +D_{1101} & +D_{1110} & & \equiv & 2h_{12} & +2h_{34}. & &
\end{array}$$

Thus by Proposition 2, the divisors D_σ, L_χ define a $\mathbb{Z}_2^4$-cover $g : X \to Y_4$. Moreover, this $\mathbb{Z}_2^4$-cover fulfils the hypotheses of Theorem 3. In fact, we have that

$$D_{0100} + D_{0101} + D_{0110} + D_{0111} = h_{14} + f_{31} + e_1 + h_{12} \equiv 3l - e_1 - e_2 - e_3 - e_4$$
$$D_{1000} + D_{1001} + D_{1010} + D_{1011} = f_{11} + e_2 + h_{23} + h_{24} \equiv 3l - e_1 - e_2 - e_3 - e_4$$
$$D_{1100} + D_{1101} + D_{1110} + D_{1111} = h_{13} + h_{34} + f_{21} + e_3 \equiv 3l - e_1 - e_2 - e_3 - e_4,$$

$h^0\left(K_{Y_4} + L_\chi\right) = 0$ for all $\chi \notin \{\chi_{1000}, \chi_{0100}, \chi_{1100}\}$, and the divisor $D_{0001} + D_{0010} + D_{0011} - K_{Y_4} \equiv 3l - e_1 - e_2 - e_3 - e_4$ is nef and big. Thus by Theorem 3 and Proposition 5, the surface X is a minimal surface of general type and possesses the following invariants:

$$K_X^2 = 20, p_g(X) = 3, \chi(\mathscr{O}_X) = 4, q(X) = 0.$$

Moreover, the canonical map $\varphi_{|K_X|}$ is of degree 20 and the linear system $|K_X|$ is base point free.

Remark 7. The surface X has four pencils of genus 9 corresponding to the fibres f_1, f_2, f_3, f_4.

In the above construction, for each choice of $f_{11} \in |f_1|$, $f_{21} \in |f_2|$ and $f_{31} \in |f_3|$, we obtain a natural deformation of the surface X (we refer [7, Definition 5.1] for the definition of natural deformations of an abelian cover). It is worth pointing out that a natural deformation of an abelian cover $X \to Y$ is a deformation of the map $X \to Y$ by [7, Proposition 5.1].

Remark 8. The surface X admits natural deformations. Moreover, all the natural deformations of X are Galois.

In fact, by [7, Definition 5.1] the natural deformations of the $\mathbb{Z}_2^4$-cover $g: X \to Y_4$ are parametrized by the direct sum of the vector spaces

$$\bigoplus_{\sigma\neq 0} H^0(Y_4, D_\sigma) \bigoplus \bigoplus_{\substack{\sigma\neq 0\\ \chi\neq\chi_{0000}\\ \chi(\sigma)=1}} H^0\left(Y_4, D_\sigma - L_\chi\right).$$

Moreover, all the natural deformations of X are Galois if the second summand $\bigoplus_{\substack{\sigma\neq 0\\ \chi\neq\chi_{0000}\\ \chi(\sigma)=1}} H^0\left(Y_4, D_\sigma - L_\chi\right)$ is zero (see [3, Definition 3.2]). We have that

$$H^0(Y_4, D_{0110}) = H^0\left(Y_4, f_{31}\right) \cong \mathbb{C}^2$$
$$H^0(Y_4, D_{1001}) = H^0\left(Y_4, f_{11}\right) \cong \mathbb{C}^2$$
$$H^0(Y_4, D_{1111}) = H^0\left(Y_4, f_{21}\right) \cong \mathbb{C}^2$$

and $H^0(Y_4, D_\sigma) \cong \mathbb{C}$ for the other non-trivial D_σ. So the famify of natural deformations of $g: X \to Y_4$ is parametrized by the base space $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Furthermore, all natural deformations of X are Galois since $\bigoplus_{\substack{\sigma\neq 0\\ \chi\neq\chi_{0000}\\ \chi(\sigma)=1}} H^0\left(Y_4, D_\sigma - L_\chi\right) = 0$.

3.2.2. *A surface with* $d = 20$, $p_g = 3$, $q = 0$, $K^2 = 24$

In this section, we construct the surface described in the second row of Theorem 4. We consider the following smooth divisors of a del Pezzo surface Y_4 of degree 5:

$$\begin{array}{llll}
D_{0011} := e_4 & & & \\
D_{0101} := h_{14} & D_{0110} := f_{21} & D_{0111} := f_{31} & \\
D_{1000} := e_2 & D_{1001} := h_{23} & D_{1010} := h_{24} & D_{1011} := f_{11} \\
D_{1100} := h_{34} & D_{1101} := h_{12} + h_{13} & D_{1110} := e_1 & D_{1111} := e_3
\end{array}$$

and the other $D_\sigma = 0$, where $f_{11} \in |f_1|$, $f_{21} \in |f_2|$ and $f_{31} \in |f_3|$ such that no more than two of these divisors D_σ go through the same point. We consider the following non-trivial divisors of Y_4:

$$\begin{array}{llllllll}
L_{0001} := & 2f_1 & +f_2 & & & -e_3 & \\
L_{0010} := & & f_2 & & & +l & \\
L_{0100} := & f_1 & +f_2 & +f_3 & & & -e_4 \\
L_{1000} := & f_1 & +f_2 & +f_3 & & & -e_4 \\
L_{0011} := & f_1 & +2f_2 & & & -e_3 & -e_4 \\
L_{0101} := & & f_2 & +f_3 & & & \\
L_{0110} := & 2f_1 & +f_2 & & & -e_3 & -e_4 \\
L_{0111} := & & f_2 & +f_3 & & & -e_4 \\
L_{1001} := & & & f_3 & +f_4 & & \\
L_{1010} := & f_1 & +f_2 & +f_3 & -e_3 & & \\
L_{1011} := & f_1 & & & +f_4 & & \\
L_{1100} := & f_1 & +f_2 & +f_3 & & & -e_4 \\
L_{1101} := & f_1 & +f_2 & & & & \\
L_{1110} := & & & & & l & \\
L_{1111} := & f_1 & & +f_3. & & &
\end{array}$$

These divisors D_σ, L_χ satisfy the following relations:

$$\begin{array}{llllllllllllllllll}
2L_{0001} & \equiv & D_{0011} & +D_{0101} & & +D_{0111} & +D_{1001} & & +D_{1011} & +D_{1101} & & +D_{1111} & \equiv & 4f_1 & +2f_2 & & -2e_3 & \\
2L_{0010} & \equiv & D_{0011} & & +D_{0110} & +D_{0111} & & +D_{1010} & +D_{1011} & & +D_{1110} & +D_{1111} & \equiv & & 2f_2 & & +2l & \\
2L_{0100} & \equiv & & D_{0101} & +D_{0110} & +D_{0111} & & & & +D_{1101} & +D_{1110} & +D_{1111} & \equiv & 2f_1 & +2f_2 & +2f_3 & & -2e_4 \\
2L_{1000} & \equiv & & & & & D_{1001} & +D_{1010} & +D_{1011} & +D_{1101} & +D_{1110} & +D_{1111} & \equiv & 2f_1 & +2f_2 & +2f_3 & & -2e_4 \\
2L_{0011} & \equiv & & D_{0101} & +D_{0110} & & +D_{1001} & +D_{1010} & & +D_{1101} & +D_{1110} & & \equiv & 2f_1 & +4f_2 & & -2e_3 & -2e_4 \\
2L_{0101} & \equiv & D_{0011} & & +D_{0110} & & +D_{1001} & & +D_{1011} & & +D_{1110} & & \equiv & & 2f_2 & +2f_3 & & \\
2L_{0110} & \equiv & D_{0011} & +D_{0101} & & & & +D_{1010} & +D_{1011} & +D_{1101} & & & \equiv & 4f_1 & +2f_2 & & -2e_3 & -2e_4 \\
2L_{0111} & \equiv & & & & D_{0111} & +D_{1001} & +D_{1010} & & & & +D_{1111} & \equiv & & 2f_2 & +2f_3 & & -2e_4 \\
2L_{1001} & \equiv & D_{0011} & +D_{0101} & & +D_{0111} & & +D_{1010} & & & +D_{1110} & & \equiv & & & 2f_3 & +2f_4 & \\
2L_{1010} & \equiv & D_{0011} & & +D_{0110} & +D_{0111} & +D_{1001} & & & +D_{1101} & & & \equiv & 2f_1 & +2f_2 & +2f_3 & -2e_3 & \\
2L_{1011} & \equiv & & D_{0101} & +D_{0110} & & & & +D_{1011} & & & +D_{1111} & \equiv & 2f_1 & & & +2f_4 & \\
2L_{1100} & \equiv & & D_{0101} & +D_{0110} & +D_{0111} & +D_{1001} & +D_{1010} & +D_{1011} & & & & \equiv & 2f_1 & +2f_2 & +2f_3 & & -2e_4 \\
2L_{1101} & \equiv & D_{0011} & & +D_{0110} & & & +D_{1010} & & +D_{1101} & & +D_{1111} & \equiv & 2f_1 & +2f_2 & & & \\
2L_{1110} & \equiv & D_{0011} & +D_{0101} & & & +D_{1001} & & & & +D_{1110} & +D_{1111} & \equiv & & & & 2l & \\
2L_{1111} & \equiv & & & & D_{0111} & & & +D_{1011} & +D_{1101} & +D_{1110} & & \equiv & 2f_1 & & +2f_3. & &
\end{array}$$

Thus by Proposition 2, the divisors D_σ, L_χ define a $\mathbb{Z}_2^4$-cover $g : X \to Y_4$. Moreover, this $\mathbb{Z}_2^4$-cover fulfils the hypotheses of Theorem 3. In fact, we have

$$\begin{aligned}
D_{0100} + D_{0101} + D_{0110} + D_{0111} &= h_{14} + f_{21} + f_{31} \equiv 3l - e_1 - e_2 - e_3 - e_4 \\
D_{1000} + D_{1001} + D_{1010} + D_{1011} &= e_2 + h_{23} + h_{24} + f_{11} \equiv 3l - e_1 - e_2 - e_3 - e_4 \\
D_{1100} + D_{1101} + D_{1110} + D_{1111} &= h_{34} + h_{12} + h_{13} + e_1 + e_3 \equiv 3l - e_1 - e_2 - e_3 - e_4,
\end{aligned}$$

$h^0\left(K_{Y_4} + L_\chi\right) = 0$ for all $\chi \notin \left\{\chi_{1000}, \chi_{0100}, \chi_{1100}\right\}$, and the divisor $D_{0001} + D_{0010} + D_{0011} - K_{Y_4} \equiv 3l - e_1 - e_2 - e_3$ is nef and big. Thus by Theorem 3 and Proposition 5, the surface X is a minimal surface of general type and possesses the following invariants:

$$K_S^2 = 24, p_g(S) = 3, \chi(\mathcal{O}_S) = 4, q(S) = 0.$$

Moreover, the canonical map $\varphi_{|K_X|}$ is of degree 20 and the two (-2)-curves coming from $\bar{e}_4$ are the fixed part of $|K_X|$. Therefore, we obtain the surface in the second row of Theorem 4.

Remark 9. The surface X has three pencils of genus 9 corresponding the fibres f_1, f_2, f_3 and a pencil of genus 13 corresponding to the fibre f_4.

Remark 10. The surface X admits natural deformations. Moreover, all the natural deformations of X are Galois.

Similarly to Remark 8, we have that $H^0(Y_4, D_{0110}) \cong H^0(Y_4, D_{0111}) \cong H^0(Y_4, D_{1011}) \cong \mathbb{C}^2$ and $H^0(Y_4, D_\sigma) \cong \mathbb{C}$ for the other non-trivial D_σ. This implies that the famify of natural deformations of $g : X \to Y_4$ is parametrized by the base space $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Furthermore, all natural deformations of X are Galois since $\bigoplus_{\substack{\sigma \neq 0 \\ \chi \neq \chi_{0000} \\ \chi(\sigma)=1}} H^0\left(Y_4, D_\sigma - L_\chi\right) = 0$.

Acknowledgments

The author is deeply indebted to Margarida Mendes Lopes for all her help. Thanks are also due to Jungkai Alfred Chen for the suggestion on the existence of deformations. The author would like to express his gratitude to the anonymous referee for their thorough reading of the paper and valuable suggestion.

References

[1] A. Beauville, "L'application canonique pour les surfaces de type général", *Invent. Math.* **55** (1979), no. 2, p. 121-140.

[2] N. Bin, "A new example of an algebraic surface with canonical map of degree 16", *Arch. Math.* **113** (2019), no. 4, p. 385-390.

[3] B. Fantechi, R. Pardini, "Automorphisms and moduli spaces of varieties with ample canonical class via deformations of abelian covers", *Commun. Algebra* **25** (1997), no. 5, p. 1413-1441.

[4] C. Gleissner, R. Pignatelli, C. Rito, "New surfaces with canonical map of high degree", https://arxiv.org/abs/1807.11854, 2018.
[5] C.-J. Lai, S.-K. Yeung, "Examples of surfaces with canonical maps of maximal degree", *Taiwanese J. Math.* **25** (2021), no. 4, p. 699-716.
[6] M. Mendes Lopes, R. Pardini, "The geography of irregular surfaces", in *Current developments in algebraic geometry*, Mathematical Sciences Research Institute Publications, vol. 59, Cambridge University Press, 2012, p. 349-378.
[7] R. Pardini, "Abelian covers of algebraic varieties", *J. Reine Angew. Math.* **417** (1991), p. 191-213.
[8] ———, "Canonical images of surfaces", *J. Reine Angew. Math.* **417** (1991), p. 215-219.
[9] U. Persson, "Double coverings and surfaces of general type", in *Algebraic geometry (Proc. Sympos., Univ. Tromsø, Tromsø, 1977)*, Lecture Notes in Mathematics, vol. 687, Springer, 1977, p. 168-195.
[10] C. Rito, "New canonical triple covers of surfaces", *Proc. Am. Math. Soc.* **143** (2015), no. 11, p. 4647-4653.
[11] ———, "A surface with canonical map of degree 24", *Int. J. Math.* **28** (2017), no. 6, article no. 1750041 (10 pages).
[12] ———, "A surface with $q = 2$ and canonical map of degree 16", *Mich. Math. J.* **66** (2017), no. 1, p. 99-105.
[13] ———, "Surfaces with canonical map of maximum degree", https://arxiv.org/abs/1903.03017, 2019.
[14] S. L. Tan, "Surfaces whose canonical maps are of odd degrees", *Math. Ann.* **292** (1992), no. 1, p. 13-29.