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Tanner Duality Between the Oldroyd–Maxwell and Grade-two Fluid Models

La dualité de Tanner entre les Modèles de Fluides de Oldroyd–Maxwell et de Grade deux

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Abstract. We prove an asymptotic relationship between the grade-two fluid model and a class of models for non-Newtonian fluids suggested by Oldroyd, including the upper-convected and lower-convected Maxwell models. This confirms an earlier observation of Tanner. We provide a new interpretation of the temporal instability of the grade-two fluid model for negative coefficients. Our techniques allow a simple proof of the convergence of the steady grade-two model to the Navier–Stokes model as $\alpha \to 0$ (under suitable conditions) in three dimensions. They also provide a proof of the convergence of the steady Oldroyd models to the Navier–Stokes model as their parameters tend to zero.

Résumé. On démontre une relation asymptotique entre un modèle de fluides de grade deux et une classe de modèles de fluides non Newtoniens proposés par Oldroyd, comprenant les modèles de Maxwell de convection supérieure et convection inférieure. Ceci confirme une observation faite à l'origine par Tanner. On donne une interprétation nouvelle de l'instabilité en temps du modèle de fluides de grade deux lorsque ses coefficients sont négatifs. Notre approche inclut une démonstration simple de la convergence de la solution du modèle stationnaire de fluides de grade deux vers celle du modèle de Navier–Stokes quand $\alpha \to 0$ (sous des hypothèses convenables) en dimension trois. Elle donne aussi une preuve de la convergence de la solution des modèles stationnaires de Oldroyd, quand ses paramètres tendent vers zéro, vers celle du modèle de Navier–Stokes.

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1. Tanner duality

Tanner [16] noticed a close similarity between two well-known, but not obviously related, rheological models. We have been able to prove this rigorously [11] under some restrictive assumptions. Here we outline these results and suggest possible extensions.

1.1. *Model equations*

In most models of fluids, the equations of fluid motion take the form

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \nabla\cdot\mathbf{T} + \mathbf{f},$$

where $\mathbf{T}$ is called the extra (or deviatoric) stress and $\mathbf{f}$ is externally given data. In the case of the grade-two model, a time derivative of $\mathbf{u}$ appears in the expression for $\mathbf{T}$ as well as explicitly in the convection term. Since the time-dependent models that we consider exhibit an instability for certain parameters, we focus on the time-independent versions of the models, which take the form

$$\mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \nabla\cdot\mathbf{T} + \mathbf{f}, \tag{1}$$

and are known to be well-posed in many cases of interest. The models differ only according to the way the stress $\mathbf{T}$ depends on the velocity $\mathbf{u}$.

For incompressible fluids, the equation (1) is accompanied by the condition

$$\nabla\cdot\mathbf{u} = 0, \tag{2}$$

which we now assume. In addition, we will assume that $\mathbf{u} = \mathbf{0}$ on the boundary $\partial\mathscr{D}$. Our results can be extended to the case where $\mathbf{u} = \mathbf{g}$ on $\partial\mathscr{D}$ provided $\mathbf{g}\cdot\mathbf{n} = 0$. For suitable expressions for $\mathbf{T}$ defined in terms of $\mathbf{u}$, the problem (1) and (2) can be shown to be well-posed, as we indicate in special cases.

1.2. *Notation*

We assume $\mathscr{D}$ is a domain in $\mathbb{R}^d$ for $d = 2$ or 3. We use standard Sobolev spaces on $\mathscr{D}$ with seminorms and norms

$$\begin{aligned} |\mathbf{T}|_{W_q^m(\mathscr{D})} &= \left(\sum_i \sum_{|\alpha|=m} \int_{\mathscr{D}} \left|D^\alpha T_i(x)\right|^q \,\mathrm{d}x\right)^{1/q}, \\ \|\mathbf{T}\|_{W_q^m(\mathscr{D})} &= \left(\sum_{j=0}^m |\mathbf{T}|^q_{W_q^j(\mathscr{D})}\right)^{1/q}, \end{aligned} \tag{3}$$

where the indices i in (3) range over the index set of the tensor $\mathbf{T}$, with the usual modification for $q = \infty$. We also denote by $L^q(\mathscr{D})$ the case when $m = 0$. When $q = 2$, we replace W_q^m by H^m.

1.3. *Regularity assumptions on $\mathscr{D}$*

Reflecting [9], we make the following assumptions. Consider the elliptic equations

$$\begin{aligned} v - \Delta v &= f \ \text{ in } \mathscr{D} \\ \nabla v\cdot\mathbf{n} &= 0 \ \text{ on } \partial\mathscr{D}, \end{aligned} \tag{4}$$

where $\mathbf{n}$ is the unit outer normal to $\partial\mathscr{D}$, and

$$\begin{aligned} -\Delta v &= f \ \text{ in } \mathscr{D} \\ v &= 0 \ \text{ on } \partial\mathscr{D}. \end{aligned} \tag{5}$$

We introduce the following condition: suppose that the domain $\mathscr{D}$ has the property that there is a constant C such that each problem (4) and (5) has a unique solution $v \in H^2(\mathscr{D})$ for all $f \in L^2(\mathscr{D})$ satisfying

$$\|v\|_{H^2(\mathscr{D})} \le C\|f\|_{L^2(\mathscr{D})}. \tag{6}$$

Similarly, we consider a Stokes system,

$$\begin{aligned} -\Delta \mathbf{v} + \nabla p = \mathbf{f} \ \text{in}\, \mathscr{D} \\ \nabla\cdot \mathbf{v} = 0 \ \text{in}\, \mathscr{D}, \quad \mathbf{v} = \mathbf{0} \ \text{on}\, \partial\mathscr{D}. \end{aligned} \tag{7}$$

Let $q_0 = 1$ in two dimensions ($d = 2$) and $q_0 = 6/5$ in three dimensions ($d = 3$). For all $q > q_0$ (and $q \ge q_0$ if $d = 3$), if $f \in L^q(\mathscr{D})$, then $f \in H^{-1}(\mathscr{D})$. So we know in this case that (7) is well-posed in $H^1(\mathscr{D})^d \times L^2(\mathscr{D})/\mathbb{R}$ for all $\mathbf{f} \in L^q(\mathscr{D})^d$. We introduce the following condition: suppose that, for some $q > q_0$ ($q \ge q_0$ if $d = 3$), the domain $\mathscr{D}$ has the property that there is a constant C_q such that for all $\mathbf{f} \in L^q(\mathscr{D})^d$ there is a unique pair $\mathbf{v} \in W^2_q(\mathscr{D})^d$ and $p \in W^1_q(\mathscr{D})/\mathbb{R}$ solving (7) and satisfying

$$\|\mathbf{v}\|_{W^2_q(\mathscr{D})} + \|p\|_{W^1_q(\mathscr{D})/\mathbb{R}} \le C_{q,\mathscr{D}}\|\mathbf{f}\|_{L^q(\mathscr{D})} \ \text{for all}\ \mathbf{f} \in L^q(\mathscr{D})^d. \tag{8}$$

There are many sufficient conditions known that guarantee (6) or (8) [3,5]. Ultimately, in order to use Lemma 1 below, we must also assume further restrictions.

Finally, for certain results we will assume that

$$\|\mathbf{v}\|_{H^3(\mathscr{D})} + \|p\|_{H^2(\mathscr{D})/\mathbb{R}} \le C_3\|\mathbf{f}\|_{H^1(\mathscr{D})}, \tag{9}$$

which requires additional smoothness on $\partial\mathscr{D}$, for one key result.

We will use the Sobolev inequality

$$\|\mathbf{T}\|_{W^s_\infty(\mathscr{D})} \le c_q\|\mathbf{T}\|_{W^{s+1}_q(\mathscr{D})} \tag{10}$$

for tensor functions $\mathbf{T}$ of arity less than 3, in dimension $d \le 3$, and for $0 \le s \le 1$, with $q > d$. We recall from [10] the inequality

$$\|uv\|_{L^2(\mathscr{D})} \le \sigma_q\|u\|_{L^q(\mathscr{D})}\|v\|_{H^1(\mathscr{D})}, \tag{11}$$

valid provided $q > d$ for $d = 2$ or $q \ge d$ for $d \ge 3$, where σ_q is a Sobolev constant. An immediate consequence is

$$\left|\int_{\mathscr{D}} u(x)v(x)w(x)\,\mathrm{d}x\right| \le \sigma_q\|u\|_{L^q(\mathscr{D})}\|v\|_{H^1(\mathscr{D})}\|w\|_{L^2(\mathscr{D})}. \tag{12}$$

2. Grade-two model

The grade-two fluid model [6] has the following constitutive equation for the extra stress tensor $\mathbf{T}_{\mathrm{G}}$ (in the time-independent case)

$$\begin{aligned} \mathbf{T}_{\mathrm{G}} &= \eta\mathbf{A}_1(\mathbf{u}) + \alpha_1\left(\mathbf{u}\cdot\nabla\mathbf{A}_1(\mathbf{u}) + \mathbf{R}(\mathbf{u})\mathbf{A}_1(\mathbf{u}) + \mathbf{A}_1(\mathbf{u})\mathbf{R}(\mathbf{u})^t\right) + (\alpha_1+\alpha_2)\mathbf{A}_1(\mathbf{u})^2 \\ &= \eta\left((\nabla\mathbf{u})^t + \nabla\mathbf{u}\right) + \alpha_1\mathscr{G}(\mathbf{u}, (\nabla\mathbf{u})^t + \nabla\mathbf{u}, 1 + \alpha_2/\alpha_1), \end{aligned} \tag{13}$$

where $\mathbf{A}_1(\mathbf{v}) = (\nabla\mathbf{v})^t + \nabla\mathbf{v}$, $\mathbf{R}(\mathbf{v}) = \frac{1}{2}\left((\nabla\mathbf{v})^t - \nabla\mathbf{v}\right)$, and

$$\mathscr{G}(\mathbf{v},\mathbf{U},\tau) = \mathbf{v}\cdot\nabla\mathbf{U} + \mathbf{R}(\mathbf{v})\mathbf{U} + \mathbf{U}\mathbf{R}(\mathbf{v})^t + \tfrac{1}{2}\tau\left(\mathbf{A}_1(\mathbf{v})\mathbf{U} + \mathbf{U}\mathbf{A}_1(\mathbf{v})\right). \tag{14}$$

The following collects results proved in [1,4,8].

Lemma 1. *Suppose that $\alpha_2 = -\alpha_1$ and that $\mathscr{D}$ is either smooth or a convex polyhedron. For $d = 3$, we further assume that all inner angles are less than $3\pi/4$. Then there are constants $\eta_0 > 0$, $\alpha_0 > 0$, $q_0 > 2$, and $C > 0$ such that, for any $\mathbf{f} \in H(\mathrm{curl},\mathscr{D})$ satisfying $\|\mathbf{f}\|_{H(\mathrm{curl},\mathscr{D})} \le C$, the grade-two system*

$$\begin{aligned} -\eta\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \alpha_1\nabla\cdot\mathscr{G}(\mathbf{u},\mathbf{A}_1(\mathbf{u}),0) = \mathbf{f} \ \text{in}\ \mathscr{D} \\ \nabla\cdot\mathbf{u} = 0 \ \text{in}\,\mathscr{D}, \quad \mathbf{u} = \mathbf{0} \ \text{on}\ \partial\mathscr{D} \end{aligned} \tag{15}$$

has a unique solution with

$$\mathbf{u} \in \begin{cases} W^2_q(\mathscr{D})^2 \text{ for } q_0 > q > 2 & d = 2 \text{ (see [8])} \\ W^1_\infty(\mathscr{D})^3 \cap W^2_3(\mathscr{D})^3 & d = 3 \text{ (see [1])} \end{cases}$$

and $p \in H^1(\mathscr{D})/\mathbb{R}$ *for all* $0 < |\alpha_1| < \alpha_0$ *and* $\eta \geq \eta_0$.

The norm $\|\mathbf{u}\|_{W^2_q(\mathscr{D})}$ may not remain bounded as $\alpha_1 \to 0$, but such a bound is not needed in the subsequent arguments. Indeed, it would be unlikely that this would hold under the assumptions given, since we do not assume that $\mathbf{f} \in L^q(\mathscr{D})^d$ for any $q > 2$. Only the special structure of the grade-two model provides the extra regularity.

3. Oldroyd's models

An important three parameter subset of the eight parameter model of Oldroyd [13] has a constitutive relation of the form

$$\mathbf{T} + \lambda_1\left(\mathbf{u}\cdot\nabla\mathbf{T} + \mathbf{RT} + \mathbf{TR}^t\right) - \mu_1\left(\mathbf{ET} + \mathbf{TE}\right) = 2\eta\mathbf{E} \tag{16}$$

where $\mathbf{E} = \mathbf{E}(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^t\right) = \frac{1}{2}\mathbf{A}_1$ (recall that $\mathbf{R} = \mathbf{R}(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u}^t - \nabla\mathbf{u}\right)$). Note that $\mathbf{E}^t = \mathbf{E}$, $\mathbf{R} + \mathbf{E} = \nabla\mathbf{u}^t$, and $\mathbf{R} - \mathbf{E} = -\nabla\mathbf{u}$.

We can relate the Maxwell models to the Oldroyd scheme as follows. The upper-convected model is the case $\lambda_1 = \mu_1$ and the lower-convected model is the case $\lambda_1 = -\mu_1$.

We can write (16) as

$$\mathbf{T} + \lambda_1(\mathbf{u}\cdot\nabla\mathbf{T} + \mathbf{RT} + \mathbf{TR}^t) - \mu_1(\mathbf{ET} + \mathbf{TE}) = \mathbf{T} + \lambda_1\mathscr{G}(\mathbf{u}, \mathbf{T}, -\mu_1/\lambda_1) = 2\eta\mathbf{E}, \tag{17}$$

where $\mathscr{G}$ is defined in (14).

3.1. *Bounds for Oldroyd*

The following is proved in [9, Theorem 5.8] and [10, Theorem 5.8].

Lemma 2. *Suppose that* $q > d$ *and the conditions* (6) *and* (8) *hold. There are constants* $\eta_0 > 0$, $\mu_0 > 0$, $\lambda_0 > 0$, $C_1 > 0$ *and* $C_2 > 0$ *such that, for any* $\mathbf{f} \in W^1_q(\mathscr{D})^d$ *satisfying* $\|\mathbf{f}\|_{W^1_q(\mathscr{D})} \leq C_2$, *the Oldroyd system* (1), (17) *has a solution for all* $|\lambda_1| < \lambda_0\eta$, $|\mu_1| \leq \mu_0|\lambda_1|$, *and* $\eta \geq \eta_0$, *satisfying*

$$\eta\left(\|\mathbf{u}\|_{W^2_q(\mathscr{D})} + \|\mathbf{T}\|_{W^1_q(\mathscr{D})}\right) + \|p\|_{W^1_q(\mathscr{D})/\mathbb{R}} \leq C_1\|\mathbf{f}\|_{W^1_q(\mathscr{D})}, \tag{18}$$

where C_1 *is independent of* λ_1 *and* μ_1.

3.2. *Oldroyd as a perturbation of Navier–Stokes*

We can characterize the Oldroyd problem as a perturbation of the Navier–Stokes equations via

$$\begin{aligned} -\eta\Delta\mathbf{u}_{\mathrm{O}} + \mathbf{u}_{\mathrm{O}}\cdot\nabla\mathbf{u}_{\mathrm{O}} + \nabla p_{\mathrm{O}} &= \lambda_1\nabla\cdot\mathbf{M}_{\mathrm{O}} + \mathbf{f} \\ \nabla\cdot\mathbf{u}_{\mathrm{O}} &= 0, \end{aligned} \tag{19}$$

where we define the tensor

$$\mathbf{M}_{\mathrm{O}} = \frac{1}{\lambda_1}\left(\mathbf{T}_{\mathrm{O}} - 2\eta\mathbf{E}(\mathbf{u}_{\mathrm{O}})\right) \tag{20}$$

and we denote by $\mathbf{T}_{\mathrm{O}}$ the solution of (17) with $\mathbf{u} = \mathbf{u}_{\mathrm{O}}$.

Let $\mathbf{u}_{\mathrm{N}}$ denote the solution to the Navier–Stokes equations

$$\begin{aligned} -\eta\Delta\mathbf{u}_{\mathrm{N}} + \mathbf{u}_{\mathrm{N}}\cdot\nabla\mathbf{u}_{\mathrm{N}} + \nabla p_{\mathrm{N}} &= \mathbf{f} \\ \nabla\cdot\mathbf{u}_{\mathrm{N}} &= 0, \end{aligned} \tag{21}$$

subject to Dirichlet boundary conditions $\mathbf{u}_{\mathrm{N}} = \mathbf{0}$ on $\partial\mathscr{D}$.

Let σ be the smallest constant such that

$$|(\mathbf{w}\cdot\nabla\mathbf{v},\mathbf{w})| \le \sigma|\mathbf{v}|_{H^1(\mathscr{D})}|\mathbf{w}|^2_{H^1(\mathscr{D})} \quad \forall\ \mathbf{v},\mathbf{w}\in H^1_0(\mathscr{D})^d. \tag{22}$$

Theorem 3. *Suppose that the assumptions of Lemma 2 hold. Let $\mathbf{u}_{\mathrm{O}}$ be a solution of* (1) *and* (17) *guaranteed by Lemma 2. For any solution $\mathbf{u}_{\mathrm{N}}$ of* (21) *such that*

$$|\mathbf{u}_{\mathrm{N}}|_{H^1(\mathscr{D})} \le \eta/2\sigma, \tag{23}$$

where σ is defined in (22)*, we have*

$$\eta|\mathbf{u}_{\mathrm{O}}-\mathbf{u}_{\mathrm{N}}|_{H^1(\mathscr{D})} \le 2C|\lambda_1|\,\|\mathbf{f}\|^2_{W^1_q(\mathscr{D})}, \tag{24}$$

where the constant C is made explicit in [11].

The convergence of the grade-two model to Navier–Stokes as $\alpha\to 0$ was known [2, 8]. Early work on the Oldroyd models was done by Renardy [14, 15]. Additional work was done in [7]. In [9] and [10], complete proofs appeared regarding the wellposedness of the 3-parameter Oldroyd system, which lack explicit dissipation. But this is the first proof of convergence for this Oldroyd model to Navier–Stokes that we are aware of. Estimates in the full $H^1(\mathscr{D})$ norm follow from (23) and (24) from Poincaré's inequality.

Remark 4. The proof of Theorem 3 also shows that, if there is a solution $\mathbf{u}_{\mathrm{N}}$ for the Navier–Stokes fluid model satisfying the condition (23), then it must be unique.

4. Grade two/Oldroyd comparison

Now let us relate the grade-two models in the case of general α_1 and α_2 to the Oldroyd models. From (13), we see that the extra-stress tensor $\mathbf{T}_{\mathrm{G}}$ reads

$$\mathbf{T}_{\mathrm{G}} = 2\eta\mathbf{E}(\mathbf{u}_{\mathrm{G}}) + \alpha_1\mathscr{G}(\mathbf{u}_{\mathrm{G}},\mathbf{A}_1(\mathbf{u}_{\mathrm{G}}),1+\alpha_2/\alpha_1).$$

This should be compared with the corresponding time-independent Oldroyd model (17):

$$\mathbf{T}_{\mathrm{O}} = 2\eta\mathbf{E}(\mathbf{u}_{\mathrm{O}}) - \lambda_1\mathscr{G}(\mathbf{u}_{\mathrm{O}},\mathbf{T}_{\mathrm{O}},-\mu_1/\lambda_1). \tag{25}$$

For λ_1 and μ_1 small, we expect the solution to (17) to be asymptotically

$$\mathbf{T}_{\mathrm{O}} \approx 2\eta\mathbf{E}(\mathbf{u}_{\mathrm{O}}) = \eta\mathbf{A}_1(\mathbf{u}_{\mathrm{O}}),$$

so that the next order of approximation, based on (25) is

$$\mathbf{T}_{\mathrm{O}} \approx 2\eta\mathbf{E}(\mathbf{u}_{\mathrm{O}}) - \lambda_1\eta\mathscr{G}(\mathbf{u}_{\mathrm{O}},\mathbf{A}_1(\mathbf{u}_{\mathrm{O}}),-\mu_1/\lambda_1). \tag{26}$$

Thus the general grade-two model corresponds to the Oldroyd model with $\lambda_1 \iff -\alpha_1/\eta$ and $\mu_1 \iff (\alpha_1+\alpha_2)/\eta$. These relations can be inverted to write

$$\alpha_1 \iff -\eta\lambda_1 \quad\text{and}\quad \alpha_2 \iff \eta(\lambda_1+\mu_1). \tag{27}$$

5. Rigorous verification

We now demonstrate that the Oldroyd models and grade-two models are asymptotically similar for steady solutions, provided that the grade-two model has a smooth solution and a certain smoothness condition holds for the Oldroyd model. Smooth solutions for grade-two are known in a special case, namely $\alpha_2 = -\alpha_1$, and we write α for α_1. (However, we do not assume any particular sign for α, although we will eventually choose $|\alpha|$ to be small.) In this case, we have $\mu_1 = 0$ in the Oldroyd model.

Let κ be the smallest constant such that the Sobolev-type inequality

$$\left|(\mathbf{w}\cdot\nabla\mathbf{v},\mathbf{w})-\alpha\sum_{k=1}^{d}\big((\nabla\mathbf{w})\mathbf{v}_{,k}-(\nabla\mathbf{v})\mathbf{w}_{,k}-(\nabla\mathbf{v}_{,k})\mathbf{w},\mathbf{w}_{,k}\big)\right|\le\kappa\|\mathbf{v}\|_{W^2_q(\mathscr{D})}|\mathbf{w}|^2_{H^1(\mathscr{D})} \tag{28}$$

holds for all $\mathbf{v}\in W^2_q(\mathscr{D})^d$ and $\mathbf{w}\in H^1_0(\mathscr{D})^d$, which is finite in view of Sobolev's inequality.

The following theorem is a consequence of the Lipschitz continuity (37) of the solution operator for the grade-two model from $H^{-1}(\mathscr{D})^d$ to $H^1_0(\mathscr{D})^d$, that will be proved in Theorem 7. All the constants in the following theorem are made explicit in [11].

Theorem 5. *Suppose that the assumptions of Lemmas 1 and 2 hold. There are constants C, C', $\eta_0>0$ and $\Lambda>0$ such that for $\eta\ge\eta_0$ and $\lambda_1\in\mathbb{R}$ with $0<|\lambda_1|\le\Lambda$, if $\mathbf{u}_{\mathrm{O}}$ solves* (1) *and* (17) *with $\mu_1=0$, satisfying*

$$\|\mathbf{u}_{\mathrm{O}}\|_{W^2_q(\mathscr{D})}\le\eta/2\kappa, \tag{29}$$

where κ is defined in (28)*, and $\mathbf{u}_{\mathrm{G}}\in H^2(\mathscr{D})^d$ is a solution of the grade-two problem* (15) *with $\alpha_1=-\eta\lambda_1$ and $\alpha_2=\eta\lambda_1$, then*

$$\eta|\mathbf{u}_{\mathrm{O}}-\mathbf{u}_{\mathrm{G}}|_{H^1(\mathscr{D})}\le 2C\big(\lambda_1^2\|\mathbf{f}\|^3_{W^1_q(\mathscr{D})}+|\lambda_1|\|\mathbf{f}\|^2_{W^1_q(\mathscr{D})}\big). \tag{30}$$

If in addition $\|\mathbf{u}_{\mathrm{O}}\|_{H^3(\mathscr{D})}\le K$ holds as $\lambda_1\to 0$, then

$$\eta|\mathbf{u}_{\mathrm{O}}-\mathbf{u}_{\mathrm{G}}|_{H^1(\mathscr{D})}\le 2C'\lambda_1^2\|\mathbf{f}\|^2_{W^1_q(\mathscr{D})}\big(K+\|\mathbf{f}\|_{W^1_q(\mathscr{D})}\big). \tag{31}$$

In Theorem 3, we show that $\mathbf{u}_{\mathrm{O}}$ tends to the solution of the Navier–Stokes equation $\mathbf{u}_{\mathrm{N}}$ as $\lambda_1\to 0$. In two dimensions, it was also known [8] that $\mathbf{u}_{\mathrm{G}}$ tends to the solution of the Navier–Stokes equation $\mathbf{u}_{\mathrm{N}}$ as $\alpha\to 0$. In the time dependent case, this is also proved in [2] in three dimensions. In view of Theorem 3 and Theorem 5, we obtain an independent verification that this holds as well in three dimensions in the time independent case. We state this result, obtained by combining (24) and (30), as the following.

Theorem 6. *Suppose that the assumptions of Lemmas 1 and 2 hold. Let $\mathbf{u}_{\mathrm{G}}$ be a solution of* (15) *guaranteed by Lemma 1. For any solution $\mathbf{u}_{\mathrm{N}}$ of* (21) *such that* (23) *holds, where σ is defined in* (22)*, we have*

$$\eta|\mathbf{u}_{\mathrm{G}}-\mathbf{u}_{\mathrm{N}}|_{H^1(\mathscr{D})}\le C\big(|\lambda_1|\,\|\mathbf{f}\|^2_{W^1_q(\mathscr{D})}+\lambda_1^2\|\mathbf{f}\|^3_{W^1_q(\mathscr{D})}\big), \tag{32}$$

where the constant C is made explicit in [11]. Note that $\lambda_1=-\alpha=-\alpha_1$ here.

Suppose there exist functions $\mathbf{u}'_{\mathrm{O}}$ and $\mathbf{u}'_{\mathrm{G}}$ in $H^1(\mathscr{D})^d$, independent of λ_1, such that

$$\mathbf{u}_{\mathrm{X}}=\mathbf{u}_{\mathrm{N}}+\lambda_1\mathbf{u}'_{\mathrm{X}}+\mathrm{o}(\lambda_1), \tag{33}$$

where X stands for either O or G. Then, provided $\|\mathbf{u}_{\mathrm{O}}\|_{H^3(\mathscr{D})}\le K$ holds as $\lambda_1\to 0$, Theorem 5 implies that $\mathbf{u}'_{\mathrm{G}}=\mathbf{u}'_{\mathrm{O}}$:

$$\mathbf{u}'_{\mathrm{G}}-\mathbf{u}'_{\mathrm{O}}=\frac{1}{\lambda_1}\Big(\big(\mathbf{u}_{\mathrm{G}}-\mathbf{u}_{\mathrm{N}}+\mathrm{o}(\lambda_1)\big)-\big(\mathbf{u}_{\mathrm{O}}-\mathbf{u}_{\mathrm{N}}+\mathrm{o}(\lambda_1)\big)\Big)=\frac{1}{\lambda_1}\big(\mathbf{u}_{\mathrm{G}}-\mathbf{u}_{\mathrm{O}}\big)+\mathrm{o}(1)\to 0$$

as $\lambda_1\to 0$.

6. Lipschitz estimates for grade-two

To prove inequalities (30) and (31), we demonstrate a Lipschitz continuity estimate of the form (37) below. Define the grade-two fluid operator

$$\mathscr{G}(\mathbf{v},q)=-\eta\Delta\mathbf{v}+\mathbf{v}\cdot\nabla\mathbf{v}+\nabla q-\alpha\nabla\cdot\big(\mathbf{v}\cdot\nabla\mathbf{A}_1(\mathbf{v})+\mathbf{R}(\mathbf{v})\mathbf{A}_1(\mathbf{v})+\mathbf{A}_1(\mathbf{v})\mathbf{R}(\mathbf{v})^t\big), \tag{34}$$

where $\mathbf{R}(\mathbf{v})=\frac{1}{2}(\nabla\mathbf{v}^t-\nabla\mathbf{v})$ and $\mathbf{A}_1(\mathbf{v})=\nabla\mathbf{v}^t+\nabla\mathbf{v}$. A natural domain for $\mathscr{G}$ is the space

$$\mathscr{V}=\Big\{(\mathbf{v},q)\,:\,\mathbf{v}\in H^2(\mathscr{D})^d,\ q\in H^1(\mathscr{D})/\mathbb{R},\ \nabla\cdot\mathbf{v}=0\text{ in }\mathscr{D}\Big\}, \tag{35}$$

and $\mathscr{G}$ maps $\mathscr{V}$ to $H^{-1}(\mathscr{D})^d$. Thus we consider two pairs of functions $(\mathbf{u}^i, p^i) \in \mathscr{V}$ and define $\mathbf{f}^i = \mathscr{G}(\mathbf{u}^i, p^i)$. In one application, $\mathbf{u}^1 = \mathbf{u}_{\mathrm{O}}$ and $\mathbf{u}^2 = \mathbf{u}_{\mathrm{G}}$.

Theorem 7. *Let $q > d$ and assume that $\mathscr{D}$ is Lipschitz. Let $\mathbf{u}^1$ and $\mathbf{u}^2$ be two solutions of the grade-two model with right-hand sides $\mathbf{f}^1$ and $\mathbf{f}^2$. If $\mathbf{u}^1 \in W^2_q(\mathscr{D})^d$ satisfies*

$$\|\mathbf{u}^1\|_{W^2_q(\mathscr{D})} \le \eta/2\kappa, \tag{36}$$

where κ is defined in (28)*, and $\mathbf{u}^2 \in H^2(\mathscr{D})^d$, then*

$$\eta|\mathbf{u}^1 - \mathbf{u}^2|_{H^1(\mathscr{D})} \le 2\|\mathbf{f}^1 - \mathbf{f}^2\|_{H^{-1}(\mathscr{D})}. \tag{37}$$

Remark 8. Theorem 5 shows that, if there is a solution $\mathbf{u}_{\mathrm{G}}$ for the grade-two fluid model satisfying the condition (36), then it must be unique.

The theorem means that the correspondence $\mathbf{f} \to \mathbf{u}$ is locally Lipschitz continuous as a mapping of $H^{-1}(\mathscr{D})^d \to H^1(\mathscr{D})^d$ provided we perturb around $\mathbf{f}^1 \in \mathscr{G}\big(W^2_q(\mathscr{D})^d \times W^1_q(\mathscr{D})\big)$, $q > d$.

7. Regularity for T

In addition to the results in Section 3.1, we recall some additional results that allow us to bootstrap the regularity for the Oldroyd solution, and prove further regularity results, in particular that $\mathbf{u}_{\mathrm{O}} \in H^3(\mathscr{D})^d$.

To fit into the framework of [12], we view a general tensor $\mathbf{W}$ as a function whose values are vectors of dimension m, and we use the Frobenius product " : " as the inner-product on such vectors, with norm $|\mathbf{W}(x)| = \sqrt{\mathbf{W}(x) : \mathbf{W}(x)}$. In particular, [12, (4)] and [12, Theorem 3] can be phrased as follows.

Lemma 9. *Suppose that $2 \le d \le 4$, $q \ge 2$, $\mathscr{D} \subset \mathbb{R}^d$ is a bounded, Lipschitz domain, and $\mathbf{v} \in H^1(\mathscr{D})^d$ with $\nabla \cdot \mathbf{v} = 0$ in $\mathscr{D}$ and $\mathbf{v} \cdot \mathbf{n} = 0$ on $\partial\mathscr{D}$. Suppose further that $\mathbf{C}$ is an $m \times m$ matrix-valued function such that $\mathbf{C} \in L^\infty(\mathscr{D})^{m^2}$ and for some constant $c_0 > 0$*

$$\big(\mathbf{C}(x)\boldsymbol{\xi}\big) \cdot \boldsymbol{\xi} \ge c_0|\boldsymbol{\xi}|^2 \quad \forall\, \boldsymbol{\xi} \in \mathbb{R}^m \tag{38}$$

for almost all $x \in \mathscr{D}$. Then for all $\mathbf{g} \in L^q(\mathscr{D})^m$, there is a unique solution $\mathbf{W} \in L^q(\mathscr{D})^m$ of

$$\mathbf{v} \cdot \nabla\mathbf{W} + \mathbf{C}\mathbf{W} = \mathbf{g}, \tag{39}$$

satisfying

$$\|\mathbf{W}\|_{L^q(\mathscr{D})} \le \frac{1}{c_0}\|\mathbf{g}\|_{L^q(\mathscr{D})}. \tag{40}$$

As observed in [10], the results in [12] were stated for the special case when the size of the vector m was the same as the dimension of the domain d (i.e., $m = d$), but it can be easily checked that the result holds for vectors of arbitrary length $m \ge 1$.

Using the techniques in [10], we can prove the following. We recall from (12) the Sobolev constant σ_q.

Lemma 10. *In addition to the conditions of Lemma 9, suppose that $\mathbf{C} \in W^1_q(\mathscr{D})^{m^2}$ and $\mathbf{v} \in W^1_\infty(\mathscr{D})^d$, satisfying*

$$\|\nabla\mathbf{v}\|_{L^\infty(\mathscr{D})} \le \tfrac{1}{4}c_0, \qquad \|\nabla\mathbf{C}\|_{L^q(\mathscr{D})} \le \tfrac{1}{4}c_0/\sigma_q,\ q > d. \tag{41}$$

Then for each $\mathbf{g} \in H^1(\mathscr{D})^m$, there is a unique solution $\mathbf{W} \in H^1(\mathscr{D})^m$ of (39) *such that*

$$\int_{\mathscr{D}} |\nabla\mathbf{W}|^2\,\mathrm{d}x \le \frac{1}{c_0^2}\int_{\mathscr{D}} |\mathbf{g}|^2\,\mathrm{d}x + \frac{4}{c_0^2}\int_{\mathscr{D}} |\nabla\mathbf{g}|^2\,\mathrm{d}x. \tag{42}$$

7.1. *Higher derivatives*

This process can be iterated, although the form of the transport equation changes at each step. Thus we let $\mathbf{W} = \nabla\mathbf{T}$ and consider the equation that it solves. First, assume that $\mathbf{u}$ is sufficiently smooth and that $\mathbf{T}$ solves

$$\mathbf{T} + \lambda_1\big(\mathbf{u}\cdot\nabla\mathbf{T} + \mathbf{R}\mathbf{T} + \mathbf{T}\mathbf{R}^t\big) - \mu_1(\mathbf{E}\mathbf{T} + \mathbf{T}\mathbf{E}) = \eta(\nabla\mathbf{u} + \nabla\mathbf{u}^t), \tag{43}$$

where $\mathbf{R} = \frac{1}{2}\left(\nabla\mathbf{u}^t - \nabla\mathbf{u}\right)$.

Now we derive a bound for $\nabla^2\mathbf{T}$. Recall that $\mathbf{u}\cdot\nabla\mathbf{T} = (\nabla\mathbf{T})\mathbf{u}$. Then

$$\nabla(\mathbf{u}\cdot\nabla\mathbf{T}) = \mathbf{u}\cdot\nabla(\nabla\mathbf{T}) + \nabla\mathbf{T}\nabla\mathbf{u}$$

and $\mathbf{W} = \nabla\mathbf{T}$ solves

$$\mathbf{W} + \lambda_1\big(\mathbf{u}\cdot\nabla\mathbf{W} + \mathbf{W}\nabla\mathbf{u} + \mathbf{R}\mathbf{W} + \mathbf{W}\mathbf{R}^t\big) - \mu_1(\mathbf{E}\mathbf{W} + \mathbf{W}\mathbf{E}) = \mathbf{g} \tag{44}$$

where

$$\mathbf{g} = \eta(\nabla^2\mathbf{u} + \nabla^2\mathbf{u}^t) - \lambda_1\big((\nabla\mathbf{R})\mathbf{T} + \mathbf{T}\nabla\mathbf{R}^t\big) + \mu_1\big((\nabla\mathbf{E})\mathbf{T} + \mathbf{T}\nabla\mathbf{E}^t\big). \tag{45}$$

Thus the operator $\mathbf{C}$ in this case is defined by

$$\mathbf{C}\boldsymbol{\xi} = \boldsymbol{\xi} + \lambda_1\big(\boldsymbol{\xi}\nabla\mathbf{u} + \mathbf{R}\boldsymbol{\xi} + \boldsymbol{\xi}\mathbf{R}^t\big) - \mu_1(\mathbf{E}\boldsymbol{\xi} + \boldsymbol{\xi}\mathbf{E}),$$

and $\mathbf{v} = \lambda_1\mathbf{u}$.

Lemma 11. *Suppose that the assumptions of Lemmas 9 and 10 hold. Suppose that* $\mathbf{u} \in H^3(\mathscr{D})^d$ *satisfies*

$$\|\nabla\mathbf{u}\|_{L^\infty(\mathscr{D})} \le \frac{1}{4 + 2(|\lambda_1| + |\mu_1|)}, \qquad \|\mathbf{u}\|_{W_q^2(\mathscr{D})} \le \frac{1}{(4\sigma_q + c_q)2(|\lambda_1| + |\mu_1|)}, \tag{46}$$

and $\mathbf{T} \in W_q^1(\mathscr{D})^{d^2}$ *solves* (43). *Then* $\mathbf{T} \in H^2(\mathscr{D})^{d^2}$ *and satisfies the bound*

$$\|\nabla^2\mathbf{T}\|_{L^2(\mathscr{D})} \le 5\big(1 + \tfrac{1}{2}(|\lambda_1| + |\mu_1|)\big)\big(\eta + (\sigma_q + c_q)(|\lambda_1| + |\mu_1|)\|\mathbf{T}\|_{W_q^1(\mathscr{D})}\big)\|\mathbf{u}\|_{H^3(\mathscr{D})}. \tag{47}$$

This says that, if $\mathbf{u}$ is smooth, then $\mathbf{T}$ is smooth. But there is not a simple relationship that bounds $\mathbf{u}$ in terms of $\mathbf{T}$ for the 3-parameter Oldroyd model. Thus to get a bound on $\|\mathbf{u}\|_{H^3(\mathscr{D})}$ in terms of $\|\mathbf{T}\|_{W_q^1(\mathscr{D})}$, we must reach back into the arguments in [10]. But we are able to prove in [11] the following main result.

Theorem 12. *Suppose that the assumptions of Lemma 2 hold, and assume that* (9) *holds. Then there is a constant* $\phi > 0$ *such that, if* $\|\mathbf{f}\|_{W_q^1(\mathscr{D})} \le \phi$ *and* $\mathbf{f} \in H^2(\mathscr{D})^d$, *then the solution* $\mathbf{u}_{\mathrm{O}}$ *of* (1) *and* (17) *guaranteed by Lemma 2 is in* $H^3(\mathscr{D})^d$. *Moreover,* $\mathbf{u}_{\mathrm{O}}$ *satisfies the bound*

$$\|\mathbf{u}_{\mathrm{O}}\|_{H^3(\mathscr{D})} \le K, \tag{48}$$

where K *depends on* $\|\mathbf{f}\|_{H^2(\mathscr{D})}$ *and the constant* C_3 *in* (9) *and is given explicitly in [11]. Thus*

$$\|\mathbf{u}_{\mathrm{O}} - \mathbf{u}_{\mathrm{G}}\|_{H^1(\mathscr{D})} \le C\lambda_1^2$$

as $\lambda_1 \to 0$.

8. Further results

Extensions to the case $\mu_1 \neq 0$ ($\alpha_1 + \alpha_2 \neq 0$) appear to require a different approach to the Lipschitz continuity, since a straightforward approach to Theorem 7 appears to require that $\mathbf{u}^2 \in W_q^2(\mathscr{D})^d$ for $q > d$. Such a result is known in two dimensions [8], but the bounds on the norm degenerate like α_1^{-1}, and thus are not sufficient to yield a useful result. Indeed, although we know that $\mathbf{u}_{\mathrm{G}} \to \mathbf{u}_{\mathrm{N}}$ in $H^1(\mathscr{D})^d$ as $\alpha \to 0$, it is not known if $\|\mathbf{u}_{\mathrm{G}}\|_{W_\infty^1(\mathscr{D})}$ remains bounded in this limit, even in two dimensions ($d = 2$).

Thus the case $\mu_1 \neq 0$ ($\alpha_1 + \alpha_2 \neq 0$) is completely open. Relaxing the smoothness conditions is more challenging and would likely require a substantially different technique of proof.

Extensions to more complicated boundary conditions also appear to require substantial restructuring. To begin with, allowing inflow boundary conditions requires a more complete understanding of such boundary conditions for transport equations.

Time dependent problems may be less difficult, depending on the boundary conditions chosen. The steady-state case considered here corresponds to long-time behavior of the time-dependent flows, but earlier transients require a separate consideration.

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