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Statistics / Statistiques

# On the nonparametric estimation of the functional expectile regression

## *Sur l'estimation non-paramétrique dans un modèle de régression expectile fonctionnelle*

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**Abstract.** In this note, we investigate the kernel-type estimator of the nonparametric expectile regression model for functional data. More precisely, we establish the almost complete convergence rate of this estimator under some mild conditions. Finally, the usefulness of the expectile regression is discussed, in particular, the connection with the regression function.

**Résumé.** Dans cette note, nous nous intéressons au problème d'estimation non-paramétrique de la fonction de régression expectile lorsqu'on régresse une variable réelle sur une variable fonctionnelle. Plus précisément, nous obtenons la convergence presque complète de l'estimateur à noyau de la fonction de régression expectile sous des conditions générales. Nous discutons brièvement notre résultat et mettons en évidence le lien avec la fonction de régression.

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## 1. Introduction

The mean regression analysis has proved to be a flexible tool which provides a powerful statistical modeling framework in a variety of applied and theoretical contexts, where one intends to model the predictive relationship between related responses and predictors. Mean regression, however, only captures the conditional mean of the response and is not sufficient to capture a complete picture of the relationship between the response variable and predictors, in particular, when dealing with heterogeneous data. This motivates the introduction of quantile regression by [13], that

plays a fundamental role in various statistical applications. In the literature in risk management and, more generally, in mathematical economics and mathematical finance modelling, it is commonly known as Value-at-Risk (VaR). It complements the classical regression on the conditional mean by offering a more useful tool for examining how the regressors influence the entire distribution of a response variable. Expectiles as defined by [2], [18], can be introduced in two ways, either as the generalization of the ordinary mean or as an alternative to quantiles. Indeed, from the classical regression, we get the high sensitivity to the extreme value which permits for more reactive risk management. On the other hand, from quantile regression we inherit the possibility to acquire an exhaustive information on the impact of explanatory variable on the response one by exploring its conditional distribution. In this work, we consider the expectile regression to analyze the effect of a functional covariate on the scalar response variable. The expectile regression has been widely studied in applied areas such as econometrics, finance and actuarial science, see for instance [14] or [10] and the reference therein for more details. However, the literature on the theoretical properties of this model is still limited. We cite, for instance, [6] who generalized the mean regression to the expectile regression by means of the minimisation of an asymmetric quadratic loss function and presented their main properties. The theoretical and numerical results of the comparison study for these risk measures are given in [5] and indicate that the expectiles are perfectly reasonable alternatives to the Value-at-Risk (VaR) and expected shortfall (ES) risk measures. In [20] authors proposed an estimation of the VaR and ES or conditional VaR by using the expectiles. While, these cited works consider the finite dimensional case, the functional case problems form a basically unsolved open problem in the literature. We aim at filling this gap in the literature by deriving the almost complete convergence rate kernel estimator of the expectile regression. It is worth noticing that questions of functional data analysis are particularly interesting in many applied areas. For good sources of references to research literature in this area along with statistical applications consult the monographs of [11] and [19] or the special issue [4]. It should be noted that the recent technological development of the measuring instruments, allows the data recording over a thinner discretization grid, which consists the principal motivation of the functional statistics. In this context, the economic or financial data, which are the principal applied areas of the expectile model, constitute a natural source of functional data. Moreover, one of crucial issue in these areas is to find the best strategy for managing the volatility of the portfolios. Therefore, exploring the functional path of this data in risk analysis has a great impact in practice. At this stage the expectile regression plays an important role. It allows to control the behavior of the data in the center as well as in the tails. Its main features are the subadditivity and the sensitivity to the magnitude of the extreme losses. In particular it is well documented that the expectile model is the only elicitable coherent risk measure (see, [6]). As far as we know, the problem that we consider is open up to now, which motivated us to provide a further study. Nevertheless, other nonparametric functional regression models such as the conditional expectation and the conditional quantiles, are also widely applied. Among the wide literature concerning the nonparametric study of this functional model we only refer to [3], [11], [15], [16] and the references therein.

This note is structured as follows. In the following section, we introduce the kernel estimator of the functional expectile regression. In Section 3, we will establish our main result that is the almost complete convergence of the kernel estimator. Some comments on the importance of the studied model are given in Section 4. The details of the proof can be obtained upon request.

## 2. Model and estimator

Let  $(X_i, Y_i)$  for  $i = 1, \dots, n$ , be a sample of independent and identically distributed pairs as  $(X, Y)$  which is a random vector valued in  $\mathcal{F} \times \mathbb{R}$ , where  $\mathcal{F}$  is a semi-metric space. In the sequel,  $d$

denotes a semi-metric on  $\mathcal{F}$ ,  $x$  is a fixed point in  $\mathcal{F}$ ,  $N_x$  is a fixed neighborhood of  $x$  and the closed ball centered at  $x$  and of radius  $\alpha$  is denoted by

$$B(x, \alpha) = \{y \in \mathcal{F} \text{ such that } d(y, x) \leq \alpha\}.$$

The aim of this note is to estimate the  $p^{th}$  conditional expectile, of  $Y$  given  $X = x$ , denoted by  $\theta(p; x)$  which is defined by

$$\theta(p; x) = \operatorname{argmin}_{t \in \mathbb{R}} [\mathbb{E}[p(Y - t)^2 \mathbb{1}_{(Y-t) > 0} | X = x] + \mathbb{E}[(1 - p)(Y - t)^2 \mathbb{1}_{(Y-t) \leq 0} | X = x]], \quad (1)$$

where  $\mathbb{1}_A$  is the indicator function of the set  $A$ , see primarily [18] and also [9] and [1] for further references. It is worth noticing that (1) generalizes the conditional *expectation* of  $Y$  given  $X = x$ , which coincides with  $\theta(p; x)$  when specifically  $p = 1/2$ . On the other hand, (1) is similar to the conditional  $p$ -quantile of  $Y$  given  $X = x$ , which can be obtained by replacing  $(Y - t)^2$  by  $|Y - t|$  in (1). By a simple manipulation, we show that  $\theta(p; x)$  is the unique solution with respect to  $t$  of

$$\zeta(p; x) = G(t; x) := \frac{G_1(t; x)}{G_2(t; x)},$$

where

$$\begin{cases} \zeta(p; x) = \frac{p}{1-p}, \\ G_1(t; x) = -\mathbb{E}[(Y - t)\mathbb{1}_{(Y-t) \leq 0} | X = x], \\ G_2(t; x) = \mathbb{E}[(Y - t)\mathbb{1}_{(Y-t) > 0} | X = x]. \end{cases}$$

Thus, due to the monotony of the function  $G(\cdot; x)$ , we derive that

$$\theta(p; x) = \inf\{t \in \mathbb{R} : G(t; x) \geq \zeta(p; x)\}.$$

Now, let  $K(\cdot)$  be a kernel function and  $h := h_n$  be a sequence of positive real numbers tending to zero as  $n$  tends to infinity. The kernel estimator of the function  $G(\cdot; x)$  is given by

$$\widehat{G}_{n, h_n}(t; x) = \frac{-\sum_{i=1}^n K(h^{-1}d(x, X_i))(Y_i - t)\mathbb{1}_{(Y_i-t) \leq 0}}{\sum_{i=1}^n K(h^{-1}d(x, X_i))(Y_i - t)\mathbb{1}_{(Y_i-t) > 0}}, \quad \text{for } t \in \mathbb{R}.$$

It follows that the kernel estimator of the conditional expectile  $\theta(p; x)$ , denoted by  $\widehat{\theta}_{n, h_n}(p; x)$  is explicitly defined by

$$\widehat{\theta}_{n, h_n}(p; x) = \inf\{t \in \mathbb{R} : \widehat{G}_{n, h_n}(t; x) \geq \zeta(p; x)\}.$$

### 3. Asymptotic properties of the estimator

In order to establish the almost complete convergence of the estimator  $\widehat{\theta}_{n, h_n}(p; x)$  for a fixed point  $x$  in  $\mathcal{F}$ , we consider the following assumptions.

- (A1)  $\mathbb{P}(X \in B(x, h_n)) = \phi_x(h_n) > 0$ ,
- (A2) For  $i = 1, 2$ , the functions  $G_i(\cdot; x)$  are continuously differentiable functions in  $\mathbb{R}$  and satisfy the following Lipschitz's condition:  $\forall (t_1, t_2) \in \mathbb{R}, \forall x_1, x_2 \in N_x$ ,

$$|G_i(t_1; x_1) - G_i(t_2; x_2)| \leq C(d^{a_i}(x_1, x_2) + |t_1 - t_2|^{b_i}) \text{ for } a_i, b_i > 0.$$

- (A3) For each  $m \geq 2$  and for all  $x' \in N_x$ ,

$$\mathbb{E}[|Y|^m | X = x'] \leq \sigma_m(x') < \infty.$$

- (A4)  $K(\cdot)$  is a measurable function with support  $[0, 1]$  and satisfies: there exist  $C_2, C_3 > 0$  such that

$$0 < C_2 < K(\cdot) < C_3 < \infty.$$

- (A5) The smoothing parameter fulfills  $n\phi_x(h_n)/\log n \rightarrow \infty$  as  $n \rightarrow \infty$ .

The main result of this note is stated in the following theorem.

**Theorem 1.** *Under the Assumptions (A1)–(A5) and if  $\min\left(G_2(\theta(p; x); x), \frac{\partial G(\theta(p; x); x)}{\partial t}\right) > 0$ , we have, almost completely, as  $n \rightarrow \infty$ ,*

$$|\hat{\theta}_{n, h_n}(p; x) - \theta(p; x)| = O\left(h_n^{b_1}\right) + O\left(\sqrt{\frac{\log n}{n\phi_x(h_n)}}\right).$$

**Remark 2.** Theorem 1 can be considered as a generalization of the result concerning the almost complete of the kernel estimator of the regression operator investigated in [11]. Indeed, it is easy to prove that

$$m(x) := \theta(0.5; x) = \mathbb{E}[Y|X = x] \text{ and } \hat{m}(x) := \hat{\theta}_{n, h_n}(0.5; x).$$

Hence, with a simple modification of the condition (A2), considering the operator  $m(\cdot)$ , we obtain the same convergence rate as in [11]

#### 4. Some comments

##### *Some remarks on the hypotheses*

It is clear that our main result is stated under standard conditions of the almost complete consistency in nonparametric functional statistics. In particular the structure of the considered assumptions match to those given in the monograph by [11]. They cover the three structural axes of this study (data, model and estimator). More precisely, assumptions (A1) and (A2) allow to explore the dimensionality of the data and the model, respectively. Whilst (A4) and (A5) concern the main parameters of the proposed estimator, i.e., that are the kernel  $K(\cdot)$  and the smoothing parameter  $h_n$ . Recall that the integrability condition (A3) is a technical assumption allowing the application of the Bernstein's inequality to obtain the almost complete convergence. Of course this condition can be weakened if we limit our study to the convergence in probability.

##### *Some remarks on the expectile regression*

Although they present differences in their construction, both quantiles and expectiles share similar properties. The main reason for this, as shown in [12], is the fact that expectiles are precisely quantiles but for a transformation of the original distribution. [1] established an important feature is that quantiles and expectiles of the same distribution coincide under the hypothesis of weighted symmetry and pointed out that inference on expectiles is much easier than inference on quantiles. Notice that quantiles are not always satisfactory and can be criticized for being somewhat difficult to compute as the corresponding loss function is not continuously differentiable. The key advantage of the expectile over the quantile is its efficiency and computing expedience, although it does not have a direct interpretation as the quantile in terms of the relative frequency, see [7]. Another substantial difference is that expectiles rely on the distance to observations, whereas quantiles only use the information whether an observation is below or above the predictor. From an empirical point of view the use of the expectation is more informative than the probability distribution (characterized by the frequency of the data), because the expectation is based on both (values of the data and their frequencies). On the other hand, the expectile regressions are used to construct alternative estimators for both known risk measures such as Conditional VaR (CVAR) or the Conditional Expected Shortfall (CES) (see [20]). Thus, the main contribution of this research is the preliminary study of the nonparametric estimation of CVAR and the CES in functional statistics. From a practical point of view, the expectile regression, as a risk measure, allows to overcome the drawbacks of the CVAR or CES such as the non-coherence

and the non-elicitability. For further details about the use of the expectile we refer to the excellent study of [8]. For example, [2] construct expectiles to estimate production frontiers and provides an additional argument for using expectiles by stating that expectile regression is a way to treat asymmetric consequences as it places different weights on positive and negative residuals. For a recent comparison between quantile and expectile regression and references see [21]. In conclusion we can say that the expectile regression is of potential interest in theory as well as in practice.

### *Some perspectives*

As a new model in nonparametric functional statistics, the expectile regression model, opens the way for further research and applications in the future. The natural perspective of this work is the implementation of this estimator in practice. Of course the applicability of this estimator is related to the choice of the different parameters involved in the estimator. Some preliminary simulation studies show that the cross validation procedure used by [11] on classical regression provides satisfactory results. However, it would be more important to introduce a specific selection procedure of the expectile regression. The second prospect of this work is an extension to the multidimensional framework where  $\mathbf{Y} \in \mathbb{R}^d$ . On the basis of the multidimensional expectile in the paper of [17], let us introduce the conditional multidimensional expectile : Let  $\|\cdot\|$  be a norm on  $\mathbb{R}^d$ . We denote by  $(\mathbf{Y}_1)_+$  the vector  $(\mathbf{Y}_1)_+ = ((Y_1)_+, \dots, (Y_d)_+)^T$  and by  $(\mathbf{Y})_-$  the vector  $(\mathbf{Y})_- = (Y_1)_-, \dots, (Y_d)_-)^T$ . We define the following scoring function, for all  $\mathbf{y} \in \mathbb{R}^d$ ,

$$s_\alpha(\mathbf{Y}, \mathbf{y}) = \alpha \|\mathbf{Y} - \mathbf{y}\|_+^2 + (1 - \alpha) \|\mathbf{y} - \mathbf{Y}\|_+^2.$$

We call a *multivariate expectile regression* any minimizer

$$\mathbf{y}^* \in \underset{\mathbf{y} \in \mathbb{R}^d}{\operatorname{argmin}} \mathbb{E}[s_\alpha(\mathbf{Y}, \mathbf{y}) \mid X = x].$$

It will be of interest to consider this extension in a future investigation. In the present work we have considered the properties of the nonparametric conditional expectile in the complete data and in the independent framework. A challenging task would be to consider an extension of our results to the censored data and dependent observations, which requires nontrivial mathematics, that goes well beyond the scope of the present paper.

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