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Kanishka Perera

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Partial differential equations / *Equations aux dérivées partielles*

On the existence of ground state solutions to critical growth problems nonresonant at zero

Kanishka Perera^a

^a Department of Mathematical Sciences, Florida Institute of Technology, Melbourne,

FL 32901, USA

E-mail: kperera@fit.edu

Abstract. We prove the existence of ground state solutions to critical growth p -Laplacian and fractional p -Laplacian problems that are nonresonant at zero.

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Consider the problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u + |u|^{p^*-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where Ω is a bounded domain in \mathbb{R}^N , $1 < p < N$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p -Laplacian of u , $\lambda \in \mathbb{R}$, and $p^* = Np/(N-p)$ is the critical Sobolev exponent. Solutions of this problem coincide with critical points of the C^1 -functional

$$E(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx - \frac{\lambda}{p} \int_{\Omega} |u|^p \, dx - \frac{1}{p^*} \int_{\Omega} |u|^{p^*} \, dx, \quad u \in W_0^{1,p}(\Omega).$$

Let $K = \{u \in W_0^{1,p}(\Omega) \setminus \{0\} : E'(u) = 0\}$ be the set of nontrivial critical points of E and set

$$c = \inf_{u \in K} E(u).$$

Recall that $u_0 \in K$ is called a ground state solution if $E(u_0) = c$. For each $u \in K$,

$$E(u) = E(u) - \frac{1}{p^*} E'(u) u = \frac{1}{N} \int_{\Omega} |u|^{p^*} \, dx > 0,$$

so $c \geq 0$, and $c > 0$ if there is a ground state solution. Let

$$S = \inf_{u \in \mathcal{D}^{1,p}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^p \, dx}{\left(\int_{\mathbb{R}^N} |u|^{p^*} \, dx \right)^{p/p^*}}$$

be the best Sobolev constant. Denote by $\sigma(-\Delta_p)$ the Dirichlet spectrum of $-\Delta_p$ in Ω consisting of those $\lambda \in \mathbb{R}$ for which the eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (2)$$

has a nontrivial solution. We have the following theorem.

Theorem 1. *If problem (1) has a nontrivial solution u with*

$$E(u) < \frac{1}{N} S^{N/p} \tag{3}$$

and $\lambda \notin \sigma(-\Delta_p)$, then it has a ground state solution.

Proof. Let $(u_j) \subset K$ be a minimizing sequence for c . Then (u_j) is a $(PS)_c$ sequence for E . Since problem (1) has a nontrivial solution satisfying (3), $c < S^{N/p}/N$. So E satisfies the $(PS)_c$ condition (see Guedda and Véron [6, Theorem 3.4]). Hence a renamed subsequence of (u_j) converges to a critical point u_0 of E with $E(u_0) = c$. We claim that u_0 is nontrivial and hence a ground state solution of problem (1). To see this, suppose $u_0 = 0$. Then $\rho_j := \|u_j\| \rightarrow 0$. Let $\tilde{u}_j = u_j/\rho_j$. Since $\|\tilde{u}_j\| = 1$, a renamed subsequence of (\tilde{u}_j) converges to some \tilde{u} weakly in $W_0^{1,p}(\Omega)$, strongly in $L^p(\Omega)$, and a.e. in Ω . Since $E'(u_j) = 0$,

$$\int_{\Omega} |\nabla u_j|^{p-2} \nabla u_j \cdot \nabla v \, dx = \lambda \int_{\Omega} |u_j|^{p-2} u_j v \, dx + \int_{\Omega} |u_j|^{p^*-2} u_j v \, dx \quad \forall v \in W_0^{1,p}(\Omega),$$

and dividing this by ρ_j^{p-1} gives

$$\int_{\Omega} |\nabla \tilde{u}_j|^{p-2} \nabla \tilde{u}_j \cdot \nabla v \, dx = \lambda \int_{\Omega} |\tilde{u}_j|^{p-2} \tilde{u}_j v \, dx + o(\|v\|) \quad \forall v \in W_0^{1,p}(\Omega). \tag{4}$$

Passing to the limit in (4) gives

$$\int_{\Omega} |\nabla \tilde{u}|^{p-2} \nabla \tilde{u} \cdot \nabla v \, dx = \lambda \int_{\Omega} |\tilde{u}|^{p-2} \tilde{u} v \, dx \quad \forall v \in W_0^{1,p}(\Omega),$$

so \tilde{u} is a weak solution of (2). Taking $v = \tilde{u}_j$ in (4) and passing to the limit shows that $\lambda \int_{\Omega} |\tilde{u}|^p \, dx = 1$, so \tilde{u} is nontrivial. This contradicts the assumption that $\lambda \notin \sigma(-\Delta_p)$ and completes the proof. \square

Combining this theorem with the existence results in García Azorero and Peral Alonso [5], Egnell [4], Guedda and Véron [6], Arioli and Gazzola [1], and Degiovanni and Lancelotti [3] gives us the following theorem for the case $N \geq p^2$.

Theorem 2. *If $N \geq p^2$ and $\lambda \in (0, \infty) \setminus \sigma(-\Delta_p)$, then problem (1) has a ground state solution.*

For $N < p^2$, combining Theorem 1 with Perera et al. [10, Corollary 1.2] gives the following theorem, where $(\lambda_k) \subset \sigma(-\Delta_p)$ is the sequence of eigenvalues based on the \mathbb{Z}_2 -cohomological index introduced in Perera [8] and $|\cdot|$ denotes the Lebesgue measure in \mathbb{R}^N .

Theorem 3. *If $N < p^2$ and*

$$\lambda \in \bigcup_{k=1}^{\infty} \left(\lambda_k - \frac{S}{|\Omega|^{p/N}}, \lambda_k \right) \setminus \sigma(-\Delta_p),$$

then problem (1) has a ground state solution.

Remark 4. In the semilinear case $p = 2$, Theorem 2 was proved in Szulkin et al. [11] using a Nehari–Pankov manifold approach, and Theorems 1 and 3 were proved in Chen et al. [2] using a more direct approach. Moreover, they allow λ to be an eigenvalue when $N \geq 5$. However, their proofs are strongly dependent on the fact that $H_0^1(\Omega)$ splits into the direct sum of its subspaces spanned by the eigenfunctions of the Laplacian that correspond to eigenvalues that are less than or equal to λ and those that are greater than λ . Those proofs do not extend to the p -Laplacian since it is a nonlinear operator and hence has no linear eigenspaces.

Remark 5. We conjecture that the assumption $\lambda \notin \sigma(-\Delta_p)$ can be removed from Theorems 1 and 2 when $N^2/(N + 1) > p^2$.

Our argument can be easily adapted to obtain ground state solutions of other types of critical growth problems as well. For example, consider the nonlocal problem

$$\begin{cases} (-\Delta)_p^s u = \lambda |u|^{p-2} u + |u|^{p_s^*-2} u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{5}$$

where Ω is a bounded domain in \mathbb{R}^N with Lipschitz boundary, $s \in (0, 1)$, $1 < p < N/s$, $(-\Delta)_p^s$ is the fractional p -Laplacian operator defined on smooth functions by

$$(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))}{|x - y|^{N+sp}} dy, \quad x \in \mathbb{R}^N,$$

$\lambda \in \mathbb{R}$, and $p_s^* = Np/(N - sp)$ is the fractional critical Sobolev exponent. Let $|\cdot|_p$ denote the norm in $L^p(\mathbb{R}^N)$, let

$$[u]_{s,p} = \left(\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy \right)^{1/p}$$

be the Gagliardo seminorm of a measurable function $u : \mathbb{R}^N \rightarrow \mathbb{R}$, and let

$$W^{s,p}(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N) : [u]_{s,p} < \infty\}$$

be the fractional Sobolev space endowed with the norm

$$\|u\|_{s,p} = ([u]_{s,p}^p + |u|_p^p)^{1/p}.$$

We work in the closed linear subspace

$$W_0^{s,p}(\Omega) = \{u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}$$

equivalently renormed by setting $\|\cdot\| = [\cdot]_{s,p}$. Solutions of problem (5) coincide with critical points of the C^1 -functional

$$E_s(u) = \frac{1}{p} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy - \frac{\lambda}{p} \int_{\Omega} |u|^p dx - \frac{1}{p_s^*} \int_{\Omega} |u|^{p_s^*} dx, \quad u \in W_0^{s,p}(\Omega).$$

As before, a ground state is a least energy nontrivial solution. Let

$$\dot{W}^{s,p}(\mathbb{R}^N) = \left\{ u \in L^{p_s^*}(\mathbb{R}^N) : [u]_{s,p} < \infty \right\}$$

endowed with the norm $\|\cdot\|$ and let

$$S = \inf_{u \in \dot{W}^{s,p}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy}{\left(\int_{\mathbb{R}^N} |u|^{p_s^*} dx \right)^{p/p_s^*}}$$

be the best fractional Sobolev constant. Denote by $\sigma((-\Delta)_p^s)$ the Dirichlet spectrum of $(-\Delta)_p^s$ in Ω consisting of those $\lambda \in \mathbb{R}$ for which the eigenvalue problem

$$\begin{cases} (-\Delta)_p^s u = \lambda |u|^{p-2} u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases}$$

has a nontrivial solution. Following theorem can be proved arguing as in the proof of Theorem 1.

Theorem 6. *If problem (5) has a nontrivial solution u with*

$$E_s(u) < \frac{S}{N} S^{N/sp}$$

and $\lambda \notin \sigma((-\Delta)_p^s)$, then it has a ground state solution.

Combining this theorem with the existence results in Mosconi et al. [7] and Perera et al. [9] gives us the following theorem, where $(\lambda_k) \subset \sigma((-\Delta)_p^s)$ is the sequence of eigenvalues based on the \mathbb{Z}_2 -cohomological index.

Theorem 7. *Problem (5) has a ground state solution in each of the following cases:*

- (i) $N > sp^2$ and $\lambda \in (0, \infty) \setminus \sigma((-\Delta)_p^s)$,
- (ii) $N = sp^2$ and $\lambda \in (0, \lambda_1)$,
- (iii) $N \leq sp^2$ and

$$\lambda \in \bigcup_{k=1}^{\infty} \left(\lambda_k - \frac{S}{|\Omega|^{sp/N}}, \lambda_k \right) \setminus \sigma((-\Delta)_p^s).$$

Remark 8. Theorems 6 and 7 are new even in the semilinear case $p = 2$.

Remark 9. We conjecture that problem (5) has a ground state solution for all $\lambda > 0$ when $N^2/(N+s) > sp^2$.

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