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**On the existence of ground state solutions to critical growth problems nonresonant at zero**

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**Abstract.** We prove the existence of ground state solutions to critical growth $p$-Laplacian and fractional $p$-Laplacian problems that are nonresonant at zero.

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Consider the problem

$$
\begin{cases}
-\Delta_p u = \lambda |u|^{p-2} u + |u|^{p^*-2} u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$, $1 < p < N$, $\Delta_p u = \text{div}(\nabla |\nabla u|^{p-2} \nabla u)$ is the $p$-Laplacian of $u$, $\lambda \in \mathbb{R}$, and $p^* = Np/(N-p)$ is the critical Sobolev exponent. Solutions of this problem coincide with critical points of the $C^1$-functional

$$
E(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx - \frac{\lambda}{p} \int_{\Omega} |u|^p \, dx - \frac{1}{p^*} \int_{\Omega} |u|^{p^*} \, dx,
$$

where $u \in W^{1,p}_0(\Omega)$.

Let $K = \{u \in W^{1,p}_0(\Omega) \setminus \{0\} : E'(u) = 0\}$ be the set of nontrivial critical points of $E$ and set

$$
c = \inf_{u \in K} E(u).
$$

Recall that $u_0 \in K$ is called a ground state solution if $E(u_0) = c$. For each $u \in K$,

$$
E(u) = E(u) - \frac{1}{p^*} E'(u) u = \frac{1}{N} \int_{\Omega} |u|^{p^*} \, dx > 0,
$$

so $c \geq 0$, and $c > 0$ if there is a ground state solution. Let

$$
S = \inf_{u \in \mathcal{D}^{1,p}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^p \, dx}{\left( \int_{\mathbb{R}^N} |u|^{p^*} \, dx \right)^{p/p^*}}
$$

be the best Sobolev constant. Denote by $\sigma(-\Delta_p)$ the Dirichlet spectrum of $-\Delta_p$ in $\Omega$ consisting of those $\lambda \in \mathbb{R}$ for which the eigenvalue problem

$$
\begin{cases}
-\Delta_p u = \lambda |u|^{p-2} u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
$$

(2)
has a nontrivial solution. We have the following theorem.

**Theorem 1.** If problem (1) has a nontrivial solution \( u \) with
\[
E(u) < \frac{1}{N} S^{N/p}
\]
and \( \lambda \notin \sigma(-\Delta_p) \), then it has a ground state solution.

**Proof.** Let \( (u_j) \subset K \) be a minimizing sequence for \( c \). Then \( (u_j) \) is a \((PS)_c\) sequence for \( E \). Since problem (1) has a nontrivial solution satisfying (3), \( c < S^{N/p}/N \). So \( E \) satisfies the \((PS)_c\) condition (see Guedda and Véron [6, Theorem 3.4]). Hence a renamed subsequence of \( (u_j) \) converges to a critical point \( u_0 \) of \( E \) with \( E(u_0) = c \). We claim that \( u_0 \) is nontrivial and hence a ground state solution of problem (1). To see this, suppose \( u_0 = 0 \). Then \( \rho_j := \|u_j\| \to 0 \). Let \( \bar{u}_j = u_j/\rho_j \). Since \( \|\bar{u}_j\| = 1 \), a renamed subsequence of \( (\bar{u}_j) \) converges to some \( \bar{u} \) weakly in \( W^{1,p}_0(\Omega) \), strongly in \( L^p(\Omega) \), and a.e. in \( \Omega \). Since \( E'(u_j) = 0 \),
\[
\int_{\Omega} |\nabla u_j|^{p-2} \nabla u_j \cdot \nabla v \, dx = \lambda \int_{\Omega} |u_j|^{p-2} u_j v \, dx + \int_{\Omega} |u_j|^{p^*-2} u_j v \, dx \quad \forall \, v \in W^{1,p}_0(\Omega),
\]
and dividing this by \( \rho_j^{p-1} \) gives
\[
\int_{\Omega} |\nabla \bar{u}_j|^{p-2} \nabla \bar{u}_j \cdot \nabla v \, dx = \lambda \int_{\Omega} |\bar{u}_j|^{p-2} \bar{u}_j v \, dx + o(\|v\|) \quad \forall \, v \in W^{1,p}_0(\Omega). \tag{4}
\]
Passing to the limit in (4) gives
\[
\int_{\Omega} |\nabla \bar{u}|^{p-2} \nabla \bar{u} \cdot \nabla v \, dx = \lambda \int_{\Omega} |\bar{u}|^{p-2} \bar{u} v \, dx \quad \forall \, v \in W^{1,p}_0(\Omega),
\]
so \( \bar{u} \) is a weak solution of (2). Taking \( v = \bar{u}_j \) in (4) and passing to the limit shows that \( \lambda \int_{\Omega} |\bar{u}|^p \, dx = 1 \), so \( \bar{u} \) is nontrivial. This contradicts the assumption that \( \lambda \notin \sigma(-\Delta_p) \) and completes the proof. \( \square \)

Combining this theorem with the existence results in García Azorero and Peral Alonso [5], Egnell [4], Guedda and Véron [6], Arioli and Gazzola [1], and Degiovanni and Lancelotti [3] gives us the following theorem for the case \( N \geq p^2 \).

**Theorem 2.** If \( N \geq p^2 \) and \( \lambda \in (0, \infty) \setminus \sigma(-\Delta_p) \), then problem (1) has a ground state solution.

For \( N < p^2 \), combining Theorem 1 with Perera et al. [10, Corollary 1.2] gives the following theorem, where \( (\lambda_k) \subset \sigma(-\Delta_p) \) is the sequence of eigenvalues based on the \( \mathbb{Z}_2 \)-cohomological index introduced in Perera [8] and \( |\cdot| \) denotes the Lebesgue measure in \( \mathbb{R}^N \).

**Theorem 3.** If \( N < p^2 \) and
\[
\lambda \in \bigcup_{k=1}^{\infty} \left( \lambda_k - \frac{S}{|\Omega|^{p/N}}, \lambda_k \right) \setminus \sigma(-\Delta_p),
\]
then problem (1) has a ground state solution.

**Remark 4.** In the semilinear case \( p = 2 \), Theorem 2 was proved in Szulkin et al. [11] using a Nehari–Pankov manifold approach, and Theorems 1 and 3 were proved in Chen et al. [2] using a more direct approach. Moreover, they allow \( \lambda \) to be an eigenvalue when \( N \geq 5 \). However, their proofs are strongly dependent on the fact that \( H^1_0(\Omega) \) splits into the direct sum of its subspaces spanned by the eigenfunctions of the Laplacian that correspond to eigenvalues that are less than or equal to \( \lambda \) and those that are greater than \( \lambda \). Those proofs do not extend to the \( p \)-Laplacian since it is a nonlinear operator and hence has no linear eigenspaces.

**Remark 5.** We conjecture that the assumption \( \lambda \notin \sigma(-\Delta_p) \) can be removed from Theorems 1 and 2 when \( N^2/(N+1) > p^2 \).
Our argument can be easily adapted to obtain ground state solutions of other types of critical growth problems as well. For example, consider the nonlocal problem

$$\begin{cases}
(-\Delta)^s_p u = \lambda |u|^{p-2} u + |u|^{p_s-2} u & \text{in } \Omega \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary, $s \in (0, 1)$, $1 < p < N/s$, $(-\Delta)^s_p$ is the fractional $p$-Laplacian operator defined on smooth functions by

$$(-\Delta)^s_p u(x) = 2 \lim_{r \to 0} \int_{B_r(x)} \frac{|u(x) - u(y)|^p - |u(x) - u(y)|^{p_s}}{|x-y|^{N+sp}} \, dy, \quad x \in \mathbb{R}^N,$$

$\lambda \in \mathbb{R}$, and $p_s^* = Np/(N-sp)$ is the fractional critical Sobolev exponent. Let $|\cdot|_p$ denote the norm in $L^p(\mathbb{R}^N)$, let

$$|u|_{s,p} = \left( \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dx \, dy \right)^{1/p}$$

be the Gagliardo seminorm of a measurable function $u : \mathbb{R}^N \to \mathbb{R}$, and let

$$W^{s,p}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) : |u|_{s,p} < \infty \right\}$$

be the fractional Sobolev space endowed with the norm

$$\|u\|_{s,p} = (|u|^p + |u|_{s,p}^p)^{1/p}.$$

We work in the closed linear subspace

$$W^{s,p}_0(\Omega) = \left\{ u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \right\}$$

equivalently renormed by setting $\|\cdot\| = [\cdot]_{s,p}$. Solutions of problem (5) coincide with critical points of the $C^1$-functional

$$E_s(u) = \frac{1}{p} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dx \, dy - \frac{\lambda}{p} \int_{\Omega} |u|^p \, dx - \frac{1}{p_s^*} \int_{\Omega} |u|^{p_s^*} \, dx, \quad u \in W^{s,p}_0(\Omega).$$

As before, a ground state is a least energy nontrivial solution. Let

$$\hat{W}^{s,p}(\mathbb{R}^N) = \left\{ u \in L^{p_s^*}(\mathbb{R}^N) : |u|_{s,p} < \infty \right\}$$

endowed with the norm $\|\cdot\|$ and let

$$S = \inf_{u \in \hat{W}^{s,p}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dx \, dy}{\left( \int_{\mathbb{R}^N} |u|^{p_s^*} \, dx \right)^{p/p_s^*}}$$

be the best fractional Sobolev constant. Denote by $\sigma((-\Delta)^s_p)$ the Dirichlet spectrum of $(-\Delta)^s_p$ in $\Omega$ consisting of those $\lambda \in \mathbb{R}$ for which the eigenvalue problem

$$\begin{cases}
(-\Delta)^s_p u = \lambda |u|^{p-2} u & \text{in } \Omega \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega
\end{cases}$$

has a nontrivial solution. Following theorem can be proved arguing as in the proof of Theorem 1.

**Theorem 6.** If problem (5) has a nontrivial solution $u$ with

$$E_s(u) < \frac{s}{N} S^{N/sp}$$

and $\lambda \in \sigma((-\Delta)^s_p)$, then it has a ground state solution.

Combining this theorem with the existence results in Mosconi et al. [7] and Perera et al. [9] gives us the following theorem, where $(\lambda_k) \subset \sigma((-\Delta)^s_p)$ is the sequence of eigenvalues based on the $\mathbb{Z}_2$-cohomological index.
Theorem 7. Problem (5) has a ground state solution in each of the following cases:

(i) $N > sp^2$ and $\lambda \in (0, \infty) \setminus \sigma((-\Delta)^s_p)$,
(ii) $N = sp^2$ and $\lambda \in (0, \lambda_1)$,
(iii) $N \leq sp^2$ and

$$\lambda \in \bigcup_{k=1}^{\infty} \left( \lambda_k - \frac{S}{|\Omega|^{sp/N}} \lambda_k \right) \setminus \sigma((-\Delta)^s_p).$$

Remark 8. Theorems 6 and 7 are new even in the semilinear case $p = 2$.

Remark 9. We conjecture that problem (5) has a ground state solution for all $\lambda > 0$ when $N^2/(N + s) > sp^2$.

References