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# *Comptes Rendus*

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# *Mathématique*

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Volume 359, issue 9 (2021), p. 1161-1164

Published online: 3 November 2021

<https://doi.org/10.5802/crmath.270>



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e-ISSN : 1778-3569



Partial differential equations / *Equations aux dérivées partielles*

# On the existence of ground state solutions to critical growth problems nonresonant at zero

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**Abstract.** We prove the existence of ground state solutions to critical growth  $p$ -Laplacian and fractional  $p$ -Laplacian problems that are nonresonant at zero.

**2020 Mathematics Subject Classification.** 35B33, 35J92, 35R11.

*Manuscript received 23rd June 2021, accepted 8th September 2021.*

Consider the problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u + |u|^{p^*-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where  $\Omega$  is a bounded domain in  $\mathbb{R}^N$ ,  $1 < p < N$ ,  $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$  is the  $p$ -Laplacian of  $u$ ,  $\lambda \in \mathbb{R}$ , and  $p^* = Np/(N-p)$  is the critical Sobolev exponent. Solutions of this problem coincide with critical points of the  $C^1$ -functional

$$E(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \frac{\lambda}{p} \int_{\Omega} |u|^p dx - \frac{1}{p^*} \int_{\Omega} |u|^{p^*} dx, \quad u \in W_0^{1,p}(\Omega).$$

Let  $K = \{u \in W_0^{1,p}(\Omega) \setminus \{0\} : E'(u) = 0\}$  be the set of nontrivial critical points of  $E$  and set

$$c = \inf_{u \in K} E(u).$$

Recall that  $u_0 \in K$  is called a ground state solution if  $E(u_0) = c$ . For each  $u \in K$ ,

$$E(u) = E(u) - \frac{1}{p^*} E'(u) u = \frac{1}{N} \int_{\Omega} |u|^{p^*} dx > 0,$$

so  $c \geq 0$ , and  $c > 0$  if there is a ground state solution. Let

$$S = \inf_{u \in \mathcal{D}^{1,p}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^p dx}{\left( \int_{\mathbb{R}^N} |u|^{p^*} dx \right)^{p/p^*}}$$

be the best Sobolev constant. Denote by  $\sigma(-\Delta_p)$  the Dirichlet spectrum of  $-\Delta_p$  in  $\Omega$  consisting of those  $\lambda \in \mathbb{R}$  for which the eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (2)$$

has a nontrivial solution. We have the following theorem.

**Theorem 1.** *If problem (1) has a nontrivial solution  $u$  with*

$$E(u) < \frac{1}{N} S^{N/p} \tag{3}$$

*and  $\lambda \notin \sigma(-\Delta_p)$ , then it has a ground state solution.*

**Proof.** Let  $(u_j) \subset K$  be a minimizing sequence for  $c$ . Then  $(u_j)$  is a  $(PS)_c$  sequence for  $E$ . Since problem (1) has a nontrivial solution satisfying (3),  $c < S^{N/p}/N$ . So  $E$  satisfies the  $(PS)_c$  condition (see Guedda and Véron [6, Theorem 3.4]). Hence a renamed subsequence of  $(u_j)$  converges to a critical point  $u_0$  of  $E$  with  $E(u_0) = c$ . We claim that  $u_0$  is nontrivial and hence a ground state solution of problem (1). To see this, suppose  $u_0 = 0$ . Then  $\rho_j := \|u_j\| \rightarrow 0$ . Let  $\tilde{u}_j = u_j/\rho_j$ . Since  $\|\tilde{u}_j\| = 1$ , a renamed subsequence of  $(\tilde{u}_j)$  converges to some  $\tilde{u}$  weakly in  $W_0^{1,p}(\Omega)$ , strongly in  $L^p(\Omega)$ , and a.e. in  $\Omega$ . Since  $E'(u_j) = 0$ ,

$$\int_{\Omega} |\nabla u_j|^{p-2} \nabla u_j \cdot \nabla v \, dx = \lambda \int_{\Omega} |u_j|^{p-2} u_j v \, dx + \int_{\Omega} |u_j|^{p^*-2} u_j v \, dx \quad \forall v \in W_0^{1,p}(\Omega),$$

and dividing this by  $\rho_j^{p-1}$  gives

$$\int_{\Omega} |\nabla \tilde{u}_j|^{p-2} \nabla \tilde{u}_j \cdot \nabla v \, dx = \lambda \int_{\Omega} |\tilde{u}_j|^{p-2} \tilde{u}_j v \, dx + o(\|v\|) \quad \forall v \in W_0^{1,p}(\Omega). \tag{4}$$

Passing to the limit in (4) gives

$$\int_{\Omega} |\nabla \tilde{u}|^{p-2} \nabla \tilde{u} \cdot \nabla v \, dx = \lambda \int_{\Omega} |\tilde{u}|^{p-2} \tilde{u} v \, dx \quad \forall v \in W_0^{1,p}(\Omega),$$

so  $\tilde{u}$  is a weak solution of (2). Taking  $v = \tilde{u}_j$  in (4) and passing to the limit shows that  $\lambda \int_{\Omega} |\tilde{u}|^p \, dx = 1$ , so  $\tilde{u}$  is nontrivial. This contradicts the assumption that  $\lambda \notin \sigma(-\Delta_p)$  and completes the proof.  $\square$

Combining this theorem with the existence results in García Azorero and Peral Alonso [5], Egnell [4], Guedda and Véron [6], Arioli and Gazzola [1], and Degiovanni and Lancelotti [3] gives us the following theorem for the case  $N \geq p^2$ .

**Theorem 2.** *If  $N \geq p^2$  and  $\lambda \in (0, \infty) \setminus \sigma(-\Delta_p)$ , then problem (1) has a ground state solution.*

For  $N < p^2$ , combining Theorem 1 with Perera et al. [10, Corollary 1.2] gives the following theorem, where  $(\lambda_k) \subset \sigma(-\Delta_p)$  is the sequence of eigenvalues based on the  $\mathbb{Z}_2$ -cohomological index introduced in Perera [8] and  $|\cdot|$  denotes the Lebesgue measure in  $\mathbb{R}^N$ .

**Theorem 3.** *If  $N < p^2$  and*

$$\lambda \in \bigcup_{k=1}^{\infty} \left( \lambda_k - \frac{S}{|\Omega|^{p/N}}, \lambda_k \right) \setminus \sigma(-\Delta_p),$$

*then problem (1) has a ground state solution.*

**Remark 4.** In the semilinear case  $p = 2$ , Theorem 2 was proved in Szulkin et al. [11] using a Nehari–Pankov manifold approach, and Theorems 1 and 3 were proved in Chen et al. [2] using a more direct approach. Moreover, they allow  $\lambda$  to be an eigenvalue when  $N \geq 5$ . However, their proofs are strongly dependent on the fact that  $H_0^1(\Omega)$  splits into the direct sum of its subspaces spanned by the eigenfunctions of the Laplacian that correspond to eigenvalues that are less than or equal to  $\lambda$  and those that are greater than  $\lambda$ . Those proofs do not extend to the  $p$ -Laplacian since it is a nonlinear operator and hence has no linear eigenspaces.

**Remark 5.** We conjecture that the assumption  $\lambda \notin \sigma(-\Delta_p)$  can be removed from Theorems 1 and 2 when  $N^2/(N + 1) > p^2$ .

Our argument can be easily adapted to obtain ground state solutions of other types of critical growth problems as well. For example, consider the nonlocal problem

$$\begin{cases} (-\Delta)_p^s u = \lambda |u|^{p-2} u + |u|^{p_s^*-2} u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{5}$$

where  $\Omega$  is a bounded domain in  $\mathbb{R}^N$  with Lipschitz boundary,  $s \in (0, 1)$ ,  $1 < p < N/s$ ,  $(-\Delta)_p^s$  is the fractional  $p$ -Laplacian operator defined on smooth functions by

$$(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))}{|x - y|^{N+sp}} dy, \quad x \in \mathbb{R}^N,$$

$\lambda \in \mathbb{R}$ , and  $p_s^* = Np/(N - sp)$  is the fractional critical Sobolev exponent. Let  $|\cdot|_p$  denote the norm in  $L^p(\mathbb{R}^N)$ , let

$$[u]_{s,p} = \left( \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy \right)^{1/p}$$

be the Gagliardo seminorm of a measurable function  $u : \mathbb{R}^N \rightarrow \mathbb{R}$ , and let

$$W^{s,p}(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N) : [u]_{s,p} < \infty\}$$

be the fractional Sobolev space endowed with the norm

$$\|u\|_{s,p} = ([u]_{s,p}^p + |u|_p^p)^{1/p}.$$

We work in the closed linear subspace

$$W_0^{s,p}(\Omega) = \{u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}$$

equivalently renormed by setting  $\|\cdot\| = [\cdot]_{s,p}$ . Solutions of problem (5) coincide with critical points of the  $C^1$ -functional

$$E_s(u) = \frac{1}{p} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy - \frac{\lambda}{p} \int_{\Omega} |u|^p dx - \frac{1}{p_s^*} \int_{\Omega} |u|^{p_s^*} dx, \quad u \in W_0^{s,p}(\Omega).$$

As before, a ground state is a least energy nontrivial solution. Let

$$\dot{W}^{s,p}(\mathbb{R}^N) = \left\{ u \in L^{p_s^*}(\mathbb{R}^N) : [u]_{s,p} < \infty \right\}$$

endowed with the norm  $\|\cdot\|$  and let

$$S = \inf_{u \in \dot{W}^{s,p}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy}{\left( \int_{\mathbb{R}^N} |u|^{p_s^*} dx \right)^{p/p_s^*}}$$

be the best fractional Sobolev constant. Denote by  $\sigma((-\Delta)_p^s)$  the Dirichlet spectrum of  $(-\Delta)_p^s$  in  $\Omega$  consisting of those  $\lambda \in \mathbb{R}$  for which the eigenvalue problem

$$\begin{cases} (-\Delta)_p^s u = \lambda |u|^{p-2} u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases}$$

has a nontrivial solution. Following theorem can be proved arguing as in the proof of Theorem 1.

**Theorem 6.** *If problem (5) has a nontrivial solution  $u$  with*

$$E_s(u) < \frac{S}{N} S^{N/sp}$$

*and  $\lambda \notin \sigma((-\Delta)_p^s)$ , then it has a ground state solution.*

Combining this theorem with the existence results in Mosconi et al. [7] and Perera et al. [9] gives us the following theorem, where  $(\lambda_k) \subset \sigma((-\Delta)_p^s)$  is the sequence of eigenvalues based on the  $\mathbb{Z}_2$ -cohomological index.

**Theorem 7.** *Problem (5) has a ground state solution in each of the following cases:*

- (i)  $N > sp^2$  and  $\lambda \in (0, \infty) \setminus \sigma((-\Delta)_p^s)$ ,
- (ii)  $N = sp^2$  and  $\lambda \in (0, \lambda_1)$ ,
- (iii)  $N \leq sp^2$  and

$$\lambda \in \bigcup_{k=1}^{\infty} \left( \lambda_k - \frac{S}{|\Omega|^{sp/N}}, \lambda_k \right) \setminus \sigma((-\Delta)_p^s).$$

**Remark 8.** Theorems 6 and 7 are new even in the semilinear case  $p = 2$ .

**Remark 9.** We conjecture that problem (5) has a ground state solution for all  $\lambda > 0$  when  $N^2/(N+s) > sp^2$ .

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