Henry Fallet

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Cherednik algebra for the normalizer

Henry Fallet

33 Rue St Leu, 80000 Amiens, LAMFA, UMR 7352 CNRS-UPJV, France
E-mail: henry.fallet@u-picardie.fr

Abstract. Ginzburg, Guay, Opdam and Rouquier established an equivalence of categories between a quotient category of the category $O$ for the rational Cherednik algebra and the category of finite dimension modules of the Hecke algebra of a complex reflection group $W$. We announce a generalization of this result to the extension of the Hecke algebra associated to the normalizer of a reflection subgroup.

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1. Introduction

Let $V$ be a $C$-vector space of finite dimension $n$. Let $W < GL(V)$ be a finite complex reflection group. Let $W_0 < W$ be a reflection subgroup of $W$. According to [3], we can associate to $W$ a braid group $B(W)$ and a Hecke algebra $H(W)$. In [14] is introduced an extension of $H(W)$ as an algebra associated to the normalizer $N_W(W_0)$, called its Hecke algebra and denoted $H(W,W_0)$, for more results see [10, 11].

We refer to [3] for the general definitions used below. We have a surjection $\pi : B(W) \to W$ sending a braided reflection of hyperplane $H$ to the distinguished reflection $s_H$ of hyperplane $H$. We denote $\mathcal{A}$ the hyperplane arrangement of $W$. Let $\bar{B}_0 := \pi^{-1}(N_W(W_0))$ and $J$ be the two-sided ideal of $C\bar{B}_0$ generated by $\langle a_H^{m_H} = 1, H \in \mathcal{A} \setminus \mathcal{A}_0 \rangle$ and $a_H^{m_H} = \sum_{k=0}^{m_H-1} a_{H,k}^{m_H} H \in \mathcal{A}_0$ where $m_H$ is the order of the pointwise stabilizer of $H$ in $W$, denoted $W_H$ and the scalars $(a_{H,k})_{k \in \{0, \ldots, m_H-1\}}$ are complex numbers invariant under the action of $N_W(W_0)$, $\forall w \in N_W(W_0)$, $a_{w(H),k} = a_{H,k}$ for all $k \in \{1, \ldots, m_H - 1\}$. As in [14] we define the Hecke algebra of the normalizer as the quotient algebra of $C\bar{B}_0$ by the ideal $J$. 

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There is a second equivalent definition. Let $K := \text{Ker}(\pi_1(X/W) \to \pi_1(X_0/W_0))$ and $\tilde{\mathcal{B}}_0 := \frac{\mathcal{B}_0}{K}$ where $X := V \setminus \bigcup_{H \in \mathcal{O}_d} H$ and $X_0 = V \setminus \bigcup_{H \in \mathcal{O}_d} H$. Then

$$H(W, W_0) = \frac{\mathbb{C}\tilde{\mathcal{B}}_0}{\langle a_H^{m_H} = \sum_{k=0}^{m_H-1} a_{H,k} a_H^k, \quad H \in \mathcal{O}_d \rangle}$$

We introduce Cherednik algebras in this new context, and we prove

**Theorem 1.** There exists an equivalence of categories between the quotient category $\mathcal{O} \mathcal{O}_d$ and the category of $H(W, W_0)$-modules of finite dimension, where $\mathcal{O}_d$ is a highest weight category associated to the Cherednik algebra of the pair $(W_0, W)$.

## 2. Construction of the $KZ_0$-functor

### 2.1. The Cherednik algebra of the pair $(W_0, W)$

We denote $A(W_0, W)$ this algebra, and we define it as an algebra admitting a triangular decomposition in the sense of [13]. As a vector space, $A(W_0, W)$ is $\mathbb{C}[V] \otimes \mathbb{C}N_W(W_0) \otimes \mathbb{C}[V^*]$ and we add the following relations on the generators of $\mathbb{C}[V]$, $\mathbb{C}[V^*]$ and $\mathbb{C}N_W(W_0)$,

$$[x', x] = 0 \text{ for all } (x, x') \in V^* \times V^*$$

$$[y, y'] = 0 \text{ for all } (y, y') \in V \times V$$

$$[y, x] = tx(y) + \sum_{H \in \mathcal{O}_d} \frac{\alpha_{H,y} x(\nu_H)}{\alpha_{H,y}} \sum_{j=0}^{m_H-1} m_H(k_{H,j+1} - k_{H,j}) \epsilon_{H,j}$$

where $\epsilon_{H,j} = \frac{1}{m_H} \sum_{w \in \mathcal{W}_H \setminus \{\text{id}\}} \det(w)^j w$ is a primitive orthogonal idempotent of $\mathbb{C}W_H$, $\alpha_H \in V^*$ such that $\text{Ker}(\alpha_H) = H$. The vector $\nu_H \in V$ is such that $\nu_H$ is a $W_H$-stable complement of $H$.

The set $(k_{H,j})_{j \in \{0, ..., m_H-1\}}$ is a set of complex number such that $k_{w(H),j} = k_{H,j}$ and $t \in \mathbb{C}$. In order to define a $KZ$ functor, we need to assume $t \neq 0$. Therefore, up to renormalization we can assume $t = 1$ which we do from now on.

As noticed by the referee, this algebra is a special case of a symplectic reflection algebra as in [6], for $N_W(W_0)$ acting on $V \otimes V^*$ in natural way.

### 2.2. Dunkl–Opdam operators

We denote by $\mathcal{D}(X)$ the algebra of differential operators over $X$. In [11] is introduced a differential 1-form, $N_W(W_0)$-equivariant and integrable,

$$\omega_0 = \sum_{H \in \mathcal{O}_d} a_H \frac{d\alpha_H}{\alpha_H} \in \Omega^1(X) \otimes \mathbb{C}W_0$$

where $a_H = \sum_{j=0}^{m_H-1} m_H k_{H,j} \epsilon_{H,j}$. We build a connection on a trivial vector bundle over $X$, by

$$\nabla := d + \omega_0.$$  

This connection is flat and $N_W(W_0)$-equivariant. The covariant derivative of this connection in the direction of $y \in V$ is a differential operator called Dunkl–Opdam operator, notated $T_y$.

**Proposition 2.** For all $y \in V$, $T_y := \partial_y + \sum_{H \in \mathcal{O}_d} \frac{a_H(y)}{a_H} a_H \in \mathcal{D}(X) \otimes N_W(W_0)$. This family of differential operators satisfies two properties $\forall (y, y') \in V \times V$,

$$[T_y, T_{y'}] = 0$$

and $\forall y \in V, \forall w \in N_W(W_0), w.T_y.w^{-1} = T_{w(y)}$.

We introduce the algebra $A(W, W_0)_{\text{reg}} = \mathbb{C}[X] \otimes_{\mathbb{C}[V]} A(W, W_0)$. We can define a faithful representation of $A(W, W_0)$.
Theorem 3 (Dunkl embedding).

(1) \[ \Phi : A(W_0, W) \longrightarrow \mathcal{D}(X) \rtimes N_W(W_0) \]
\[ x \in V^* \longrightarrow x \]
\[ w \in N_W(W_0) \longrightarrow w \]
\[ y \in V \longrightarrow T_y \]
is an injective morphism of algebras.

(2) By localization, the morphism \( \Phi \) becomes an isomorphism of algebra. We note \( \Phi_{\text{reg}} \) the isomorphism between \( A(W_0, W)_{\text{reg}} \) and \( \mathcal{D}(X) \rtimes N_W(W_0) \).

2.3. The category \( \mathcal{O} \)

Let \( e u_0 = \sum_{y \in \mathcal{B}} y^* y - \sum_{H \in \mathcal{A}_0} a_H \), where \( \mathcal{B} \) is a basis of \( V \). This operator is called the Euler element. It induces an inner graduation on \( A(W_0, W) \). \( A(W_0, W) \) is a mere finitely generated, locally nilpotent \( \mathcal{O} \)-module, for the action of \( \mathcal{O} \)-torsion, i.e. \( e u_0, x = x, [e u_0, y] = -y, [e u_0, w] = 0 \).

For every simple \( \mathbb{C}_N(W_0) \) module \( E, \sum_{H \in \mathcal{A}_0} a_H \in \mathcal{Z}(\mathbb{C}_N(W_0)) \) acts on \( E \) by multiplication by a scalar \( c_E \). We define a partial ordering on \( \text{Irr}(\mathbb{N}_W(W_0)) \): \( E < E' \) if \( c_E - c_{E'} \in \mathbb{Z}_{>0} \).

For each \( E \in \text{Irr}(\mathbb{N}_W(W_0)) \) we define a \( A(W_0, W) \) module called standard object or Verma module,

\[ \Delta(E) = \text{Ind}_{\mathcal{C}[V^*] \rtimes \mathbb{C}_N(W_0)}^{A(W_0, W)} E \]

The category \( \mathcal{O} \) is a full sub category of the category of \( A(W_0, W) \) modules, where the modules are finitely generated, locally nilpotent for the action of \( \mathbb{C}[V^*] \) and isomorphic to the direct sum of the generalized \( e u_0 \)-eigenspaces. According to \([1, 2, 9] \), the category \( \mathcal{O} \) is Abelian, Artinian. The object \( \Delta(E) \) is indecomposable. The category \( \mathcal{O} \) is highest weight with \( \{ \Delta(E) \}_{E \in \text{Irr}(\mathbb{N}_W(W_0))} \) as the set of standard object. Every standard object \( \Delta(E) \) admits a simple head \( L(E) \). Every simple object in \( \mathcal{O} \) is isomorphic to some \( L(E) \) and \( L(E) \) admits a projective cover. Every object \( M \) of \( \mathcal{O} \) admits a finite composition series. The B.G.G reciprocity law is satisfied inside \( \mathcal{O} \).

2.4. Functor \( KZ_0 \)

Let \( \delta := \prod_{H \in \mathcal{A}_0} a_H \in \mathbb{C}[V] \). Let \( (A(W_0, W)_{\text{mod}})_{\text{tor}} \) be the subcategory of \( A(W_0, W)_{\text{mod}} \)-modules with \( \delta \)-torsion, i.e. \( M \in (A(W_0, W)_{\text{mod}}), M_{\text{tor}} := \{ m \in M \mid \exists n \geq 0 \delta^n m = 0 \} \), then \( M \in (A(W_0, W)_{\text{mod}})_{\text{tor}} \).

We have a localization functor,

\[ \text{Loc} : A(W_0, W)_{\text{mod}} \longrightarrow A(W_0, W)_{\text{reg}}_{\text{mod}} \]
\[ M \longrightarrow (A(W_0, W)_{\text{reg}} \otimes A(W_0, W)) M \]

This functor induces a fully faithful functor \( \mathcal{O} \longrightarrow (A(W_0, W)_{\text{reg}})_{\text{mod}} \).

The Dunkl embedding gives an equivalence of categories between \( A(W_0, W)_{\text{reg}} \)-modules and \( \mathcal{D}(X) \rtimes N_W(W_0) \)-modules. We also have the following equivalence of categories between \( \mathcal{D}(X) \rtimes N_W(W_0) \)-modules and \( e, (\mathcal{D}(X) \rtimes N_W(W_0))_{\text{-mod}} \)-modules and with \( \mathcal{D}(X) \)-modules where \( e = \frac{1}{|N_W(W_0)|} \sum_{g \in N_W(W_0)} g \) is an idempotent of \( \mathbb{C}_N(W_0) \). From the results of \([4] \) we get an isomorphism of algebras \( \mathcal{D}(X)_{\text{reg}} \cong \mathcal{D}(X / N_W(W_0)) \), thanks to the fact that \( N_W(W_0) \) acts without fixed points on \( X \).

Let us examine the structure of \( \mathcal{D}(X) \rtimes N_W(W_0) \)-modules for the case of a localized standard object. The localized Verma module \( \Delta(E)_{\text{reg}} \) is a free \( \mathbb{C}[X] \)-module of dimension \( \text{dim}(E) \), so it corresponds to an algebraic vector bundle over \( X \). We endow this vector bundle with a connection by considering the action of \( T_y \) on an element of \( \Delta(E)_{\text{reg}} \). This leads to the formula

\[ \nabla_y (P \otimes v) := \partial_y (P \otimes v) = \partial_y (P) \otimes v + \sum_{H \in \mathcal{A}_0} \frac{a_H(y)}{a_H} . (P \otimes a_H v) \]
Proposition 4. \( \nabla_y \) is a flat, \( N_W(W_0) \)-equivariant connection with regular singularities over \( \nu \).

Since this property is true for every standard object, it is also true for every object in \( \mathcal{O} \). Applying the Riemann–Hilbert–Deligne correspondence, we get a horizontal sections functor \( \mathcal{O}_{\text{reg}} \rightarrow \mathcal{C}(X/N_W(W_0))\text{-mod} \), \( M \rightarrow (M_{\text{reg}}^W(W_0))^{\text{an}} \). According to [11, Proposition 2.6], this action by monodromy factorizes through \( H(W,W_0) \). So we get a functor \( KZ_0: \mathcal{O}_{\text{reg}} \rightarrow H(W,W_0)\text{-mod} \) which is exact and fully-faithful. From classical results (see [15]), we get that \( KZ_0 \) is representable by a projective object noted \( P_{KZ_0} \). We prove the following

Theorem 5. \( KZ_0 \) is fully faithful and essentially surjective from the category \( \mathcal{O}_{\text{reg}} \) to the category of \( H(W,W_0)\text{-mod} \).

3. Forgetting \( W \)

In this section we provide a related result involving only \( W_0 \), and not the ambient group \( W \). The general setting is as follows. Let \( G \) be a finite subgroup of \( GL(V) \). Let \( G_0 \) be a normal subgroup of \( G \) generated by reflections. Let \( \mathcal{H}_0 \) be the set of reflections of \( G_0 \) and \( \mathcal{A}_0 \) the arrangement of reflecting hyperplanes of \( G_0 \). The first goal is to build up a Hecke algebra for \( G \) from the Hecke algebra of \( G_0 \) generalizing \( H(W_0,W) \) for \( G = N_W(W_0) \).

Let \( X^+ \) be the subspace of \( V \) on which \( G \) acts freely and let \( X_0 \) be the subspace of \( V \) on which \( G_0 \) acts freely. The manifold \( X_0 \setminus X^+ \) is of codimension \( >2 \) then \( \pi_1(X^+) \approx \pi_1(X_0) \) [12, Theorem 2.3]. Since \( G_0 \) acts freely on \( X_0 \), it also acts freely on \( X^+ \) therefore the projection maps \( X_0 \rightarrow X_0/G_0 \) and \( X^+ \rightarrow X^+/G_0 \) are covering maps and we get two short exact sequences.

\[
\begin{array}{ccccccc}
1 & \rightarrow & \pi_1(X^+) & \rightarrow & \pi_1(X^+/G_0) & \rightarrow & G_0 & \rightarrow & 1 \\
| & & \approx & & | & & | & & |
\end{array}
\]

\[
\begin{array}{ccccccc}
1 & \rightarrow & \pi_1(X_0) & \rightarrow & \pi_1(X_0/G_0) & \rightarrow & G_0 & \rightarrow & 1 \\
| & & \approx & & | & & | & & |
\end{array}
\]

The exactness and the commutativity of the diagram together imply \( \pi_1(X^+/G_0) \approx \pi_1(X_0/G_0) \).

The braid group \( B_0 \) of \( G_0 \) is a normal subgroup of \( B := \pi_1(X^+/G) \), we get a short exact sequence

\[
\begin{array}{ccccccc}
1 & \rightarrow & B_0 := \pi_1(X_0/G_0) & \rightarrow & \pi_1(X^+/G) & \rightarrow & G/G_0 & \rightarrow & 1
\end{array}
\]

Let \( I \) be the ideal of \( \mathcal{C} B_0 \) generated by the relations \( \sigma^{m_{ij}}_H = \sum_{k=0}^{m_{ij}-1} a_{H,k} \sigma^j_H \) for \( \sigma_H \) a braided reflection associated to \( H \in \mathcal{A}_0 \). Then the Hecke algebra of \( G_0 \) is the quotient \( H_0 := \mathcal{C} B_0/I \).

According to the now proven BMR freeness conjecture (see the references of [11] or its weaker version in Characteristic \( 0 \) [5]) it is an algebra finitely generated of dimension \( |G_0| \). Let \( I^+ = \mathcal{C} B \otimes_{\mathcal{C} B_0} I \) be the ideal which define the Hecke algebra of \( G \), \( H(G) := \mathcal{C} B_0/I^+ = \mathcal{C} B \otimes_{\mathcal{C} B_0} H_0 \) is of dimension \( |G| \).

Let us make a link between this new algebra and the algebra \( H(W_0,W) \). We defined \( H(W_0,W) \) as a quotient of the algebra \( \mathcal{C} B_0 \). We defined \( B_0 \) as the quotient of \( \pi_1(X/N_W(W_0)) \) by \( K := \text{Ker}(\pi_1(X) \rightarrow \pi_1(X_0)) \). Since \( X_0 \setminus X^+ \) has codimension \( >2 \)

\[
K = \text{Ker}(\pi_1(X) \rightarrow \pi_1(X_0)) = \text{Ker}(\pi_1(X/N_W(W_0)) \rightarrow \pi_1(X^+/N_W(W_0)))
\]

And \( B_0 = \pi_1(X^+/N_W(W_0)) \) is our group \( \pi_1(X^+/G) =: B \). As a result, the algebra \( H(W_0,W) \) is the same as \( H(G) \).

Let us consider the category \( \mathcal{O}_{\text{reg}}^0 \) the full subcategory of \( \mathcal{O} \) of module annihilated by a power of \( \delta_0 := \prod_{H \in \mathcal{A}_0} \alpha_H \). We have

Theorem 6. \( KZ_0 \) is fully faithful and essentially surjective from the category \( \mathcal{O}_{\text{reg}}^0 \) to the category of finite dimension \( H(G)\text{-mod} \).
A priori $\mathcal{O}_{\text{tor}}$ and $\mathcal{O}^0_{\text{tor}}$ are different. Actually, we can prove that these two categories are the same. Let $M \in \mathcal{O}^0_{\text{tor}}$ then $\text{Loc}(M) = \mathbb{C}[X] \otimes_{\mathbb{C}[X_0]} (\mathbb{C}[X_0] \otimes_{\mathbb{C}[V]} M)$, so $M \in \mathcal{O}_{\text{tor}}$.

Conversely, let $M$ be a module inside $\mathcal{O}_{\text{tor}}$, we would like to prove $M_{\text{reg}}^0 := \mathbb{C}[X_0] \otimes_{\mathcal{O}[V]} M = 0$.

Let $i : X^+ \rightarrow X_0$ be a continuous injection of the open set $X^+$ inside $X_0$. We denote by $\mathcal{O}_{X^+}$ the structural sheaf of $X^+$ and $\mathcal{O}_{X_0}$ the structural sheaf of $\mathcal{O}_{X_0}$. We denote by $\mathcal{D}_{X^+}$ the sheaf of algebraic differential operators over $X_0$ and $\mathcal{D}_{X^+}$ the sheaf of algebraic differential operators over $X^+$. [8, definitions 2.1.5 and 2.1.12].

We have a morphism of ringed space $(i, i^\sharp) : (X^+, \mathcal{O}_{X^+}) \rightarrow (X_0, \mathcal{O}_{X_0})$ where $i^\sharp : i^{-1} \mathcal{O}_{X_0} \rightarrow \mathcal{O}_{X^+}$ is the identity map, then $i^\sharp : \mathcal{O}_{X_0} \rightarrow \mathcal{O}_{X^+, x}$ is the identity too. The pull back functor is

$$i^* : \mathcal{D}_{X_0} \rightarrow \mathcal{D}_{X^+}$$

We have two functors $A(W_0)_{\text{reg}}^0 \rightarrow \mathcal{D}_{X_0} \rightarrow \mathcal{D}_{X^+}$.

We need to prove $i^* M_{\text{reg}}^0 = 0$. We have for all $x \in X \subset X_0$, $M_{\text{reg}, x} = 0$ it is due to $M \in \mathcal{O}_{\text{tor}}$. Since $M_{\text{reg}}^0$ and $M_{\text{reg}}$ are locally free $\mathcal{O}_{X_0}$-module, respectively $\mathcal{O}_{X_0}$-module, $(i^* M_{\text{reg}}^0)_x = M_{\text{reg}, x}$. Therefore, $(i^* M_{\text{reg}})_x = M_{\text{reg}, x} \approx \mathcal{O}_{X_0}^n$, so $n = 0$.

Since $i^* M_{\text{reg}}^0$ is a locally free $\mathcal{O}_{X_0}$ module, there exists an open affine covering $(U_i)_{i \in I}$ of $X$ such that $(i^* M_{\text{reg}}^0)_{U_i} \approx (\mathcal{O}_{X_0}^n)_{U_i} = 0$, thus $i^* M_{\text{reg}}^0 = 0$ so $M_{\text{reg}}^0 = 0$ then $M \in \mathcal{O}_0^0$. The categories $\mathcal{O}_{\text{tor}}$ and $\mathcal{O}^0_{\text{tor}}$ are equals. The proof of the equivalence of categories induced by $KZ_0$ uses the same arguments as for 5.

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References

