Translated sums of primitive sets

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Abstract. The Erdős primitive set conjecture states that the sum \( f(A) = \sum_{a \in A} \frac{1}{a \log a} \) ranging over any primitive set \( A \) of positive integers, is maximized by the set of prime numbers. Recently Laib, Derbal, and Mechik proved that the translated Erdős conjecture for the sum \( f(A, h) = \sum_{a \in A} \frac{1}{a \log a + h} \) is false starting at \( h = 81 \), by comparison with semiprimes. In this note we prove that such falsehood occurs already at \( h = 1.04 \cdots \), and show this translate is best possible for semiprimes. We also obtain results for translated sums of \( k \)-almost primes with larger \( k \).


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1. Introduction

A set \( A \subseteq \mathbb{Z}_{>1} \) of positive integers is called primitive if no member divides another (we trivially exclude the singleton \( \{1\} \)). An important family of examples is the set \( \mathbb{N}_k \) of \( k \)-almost primes, that is, numbers with exactly \( k \) prime factors counted with multiplicity. For example, \( k = 1, 2 \) correspond to the sets of primes and semiprimes, respectively.

In 1935 Erdős [3] proved that \( f(A) = \sum_{a \in A} \frac{1}{a \log a} \) converges uniformly for any primitive \( A \). In 1988 he further conjectured that the maximum of \( f(A) \) is attained by the primes \( A = \mathbb{N}_1 \). One may directly compute \( f(\mathbb{N}_1) = 1.636 \ldots \), whereas the best known bound is \( f(A) < e^\gamma = 1.781 \ldots \) for any primitive \( A \) [8]. As a special case, Zhang [9] proved that the primes maximize \( f(\mathbb{N}_k) \), that is, \( f(\mathbb{N}_1) > f(\mathbb{N}_k) \) for all \( k > 1 \).

One may pose a translated analogue of the Erdős conjecture, namely, the maximum of \( f(A, h) = \sum_{a \in A} \frac{1}{a \log a + h} \) is attained by the primes \( A = \mathbb{N}_1 \). Recently Laib, Derbal, and Mechik [5] proved that this translated conjecture is false, by showing \( f(\mathbb{N}_1, h) < f(\mathbb{N}_2, h) \) for all \( h \geq 81 \). Their proof method is direct, by studying partial sum truncations of \( f(A, h) \). (Laib [4] very recently announced a bound \( h \geq 60 \), as a refinement of the same method.)

By different methods, we extend the range of such falsehood down to \( h > 1.04 \cdots \), and show this translate is best possible for semiprimes.
\textbf{Theorem 1.} Let $P(s) = \sum_p p^{-s}$ denote the prime zeta function. We have $f(N_1, h) < f(N_2, h)$ if and only if $h > h_2$, where $t = h_2 = 1.04 \cdots$ is the unique real solution to

$$\int_{h}^{\infty} \left[ P(s) - \frac{1}{2} (P(s)^2 + P(2s)) \right] e^{(1-s)h} \, ds = 0,$$

Moreover $f(N_1, h) < f(N_k, h)$ for all $k$ sufficiently large, if and only if $h > 0.277 \cdots$.

This suggests that the Erdős conjecture, if true, is only barely so. Moreover, for the same value $h = h_2 = 1.04 \cdots$ we show the primes minimize $f(N_k, h)$, which may be viewed as the inverse analogue of Zhang’s maximization result.

\textbf{Theorem 2.} For $h_2 = 1.04 \cdots$, we have $f(N_1, h_2) < f(N_k, h_2)$ for all $k > 1$.

\section{Proof of Theorem 1}

We introduce the zeta function for $k$-almost primes $P_k(s) = \sum_{n \in \mathbb{N}_k} n^{-s}$, whose relevance to us arises from the identity,

$$f(N_k, h) = \sum_{n \in \mathbb{N}_k} \frac{1}{n} \sum_{h=1}^{n} \frac{1}{n \log(n+h)} = \sum_{n \in \mathbb{N}_k} \frac{1}{n \log(ne^h)} = \sum_{n \in \mathbb{N}_k} e^{h} \int_{1}^{\infty} \left( ne^h \right)^{-s} \, ds = \int_{1}^{\infty} P_k(s) e^{(1-s)h} \, ds. \quad (1)$$

Here the interchange of sum and integral holds by Tonelli’s theorem, since $f(N_k, h) \leq f(N_k)$ converges uniformly after Erdős. The significance of the identity (1) was first observed when $k = 1$, $h = 0$ by H. Cohen [2, p. 6], who rapidly computed $f(N_1) = 1.636616 \cdots$ to 50 digits accuracy. By comparison, the direct approach by partial sums $\sum_{p \leq x} 1/(p \log p)$ converge too slowly, i.e. $O(1/\log x)$. Similarly for $k > 1$, we shall see (1) leads to sharper results.

Note one has $P_1(s) = P(s)$ and $P_2(s) = \frac{1}{2} P(s)^2 + \frac{1}{2} P(2s)$, as well as

$$P_3(s) = \frac{1}{6} P(s)^3 + \frac{1}{2} P(s)P(2s) + \frac{1}{3} P(3s),$$

$$P_4(s) = \frac{1}{24} P(s)^4 + \frac{1}{4} P(s)^2P(2s) + \frac{1}{8} P(2s)^2 + \frac{1}{3} P(s)P(3s) + \frac{1}{4} P(4s).$$

In general for $k \geq 1$, [6, Proposition 3.1] gives an explicit formula for $P_k$ in terms of $P$,

$$P_k(s) = \sum_{n_1+2n_2+\cdots+kj=1}^{1} \prod_{j=1}^{k} \left( P(js/j) \right)^{n_j}. \quad (2)$$

Here the above sum ranges over all partitions of $k$. Also see [7, Proposition 2.1].

In practice, we may rapidly compute $P_k$ (and $P_k'$) using recursion relations.

\textbf{Lemma 3.} For $k \geq 1$ let $P_k(s) = \sum_{\Omega(n)=k} n^{-s}$ and $P_1(s) = P(s) = \sum_{p} p^{-s}$. We have

$$P_k(s) = \frac{1}{k} \sum_{j=1}^{k} P_{k-j}(s)P(js) \quad \text{and} \quad P_k'(s) = \sum_{j=1}^{k} P_{k-j}(s)P'(js). \quad (3)$$

\textbf{Proof.} The recursion (3) for $P_k$ is given in [6, Proposition 3.1], and is equivalent to (2).

The recursion (3) for $P_k'$ is obtained by differentiating (2),

$$P_k'(s) = \sum_{n_1+2n_2+\cdots+kj=1}^{1} \prod_{j=1}^{k} \frac{1}{(n_1-1)!} (P(js/j)^{n_j})^j.$$

\hfill $\square$
As observed empirically in [1], the Dirichlet series $P_2(s) - P(s) = \frac{1}{2} [P(s)^2 + P(2s)] - P(s)$ has a unique root at $s = \sigma = 1.1403 \cdots$, through which it passes from positive to negative. We prove this more generally for $k \leq 20$.

**Lemma 4.** For $2 \leq k \leq 20$, the Dirichlet series $P_k(s) - P(s)$ has a unique root at $s = \sigma_k > 1$, through which it passes from positive to negative.

**Proof.** For each $k \geq 1$, $P_k(s)$ is monotonically decreasing in $s > 1$. As such there is a unique $s_k > 1$ for which $P(s)$ passes through $(k!)^{1/(k-1)}$ from above. Using the main term in (2), $P_k(s) > P(s)^k/k!$ which is larger than $P(s)$ iff $P(s)^{k-1} > k!$ if $s < s_k$ by definition. That is,

$$P_k(s) - P(s) > 0 \quad \text{for} \quad s \in (1, s_k).$$

Also there is a unique $s_k' > 1$ for which $P_{k-1}(s)$ passes through 1 from above. Using the first term in the recursion (3), $-P_k'(s) = -P'(s)P_{k-1}(s)$ which is larger than $-P'(s) > 0$ iff $P_{k-1}(s) > 1$ if $s < s_k'$ by definition. That is,

$$P_k(s) - P(s) > 0 \quad \text{for} \quad s \in (1, s_k').$$

For $c < 2^k$, $P_k(s)e^s$ is monotonically decreasing in $s > 1$, and so is $P_k(s)/(2^{-s} + 3^{-s}) = \{1/[P_k(s)2^s] + 1/[P_k(s)3^s]\}^{-1}$. Thus there is a unique $t_k > 1$ for which $P_k(s)/(2^{-s} + 3^{-s})$ passes through 1 from above. Now by definition $P(s)/(2^{-s} + 3^{-s}) > 1 = P_k(t_k)/(2^{-t_k} + 3^{-t_k})$, which is larger than $P_k(s)/(2^{-s} + 3^{-s})$ iff $s > t_k$ by monotonicty. That is,

$$0 < P_k(s) < P(s) \quad \text{for} \quad s \in (t_k, \infty).$$

In summary $P_k - P$ has at most one root in $(1, s_k')$, and no roots in $(1, s_k) \cup (t_k, \infty)$. We directly compute that $s_k < t_k < s_k'$ for $2 \leq k \leq 20$, and so $P_k(s) - P(s)$ has a unique root $\sigma_k \in (s_k, t_k)$ as claimed.

We deduce the following corollary, which gives (the first part of) Theorem 1 when $k = 2$.

**Corollary 5.** For $2 \leq k \leq 20$, $f(N_k, h) - f(N_1, h)$ has a unique root at $h_k > 0$, through which it passes from negative to positive.

**Proof.** For $h \geq 0$, recall $f(N_k, h) = \int_1^\infty P_k(s)e^{(1-s)h} \, ds$ by (1). Now by Lemma 4, $P_k(s) - P(s)$ passes from positive to negative at $s = \sigma_k > 1$. Thus

$$f(N_k, h) - f(N_1, h) e^{(\sigma_k-1)h} = \int_1^\infty [P_k(s) - P(s)] e^{(\sigma_k-s)h} \, ds$$

is difference of two integrals with positive integrands, which are monotonically increasing and decreasing in $h \geq 0$, respectively. Hence the difference is monotonically increasing in $h \geq 0$ and $f(N_k, 0) - f(N_1, 0) < 0$ by Zhang [9], so the result follows.

For the second part of Theorem 1, note for any fixed $h \geq 0$ we have $\log n + h \sim \log n$ for $n \in \mathbb{N}_k$ as $k \to \infty$, since $n \geq 2^k$. Thus $f(N_k, h) \sim f(N_k, 0)$ as $k \to \infty$. Hence by [6, Theorem 2.2],

$$\lim_{k \to \infty} f(N_k, h) = \lim_{k \to \infty} f(N_k) = 1.$$

Note $1 - f(N_1, h)$ passes from negative to positive at a unique root $h_\infty = 0.277 \cdots$. So for each $h > h_\infty$, we have $f(N_k, h) - f(N_1, h) > 0$ for $k$ sufficiently large (and similarly for the converse). This completes the proof of Theorem 1.
2.1. Computations

For \( k \leq 20 \), we compute the unique roots \( \sigma_k \) and \( h_k \) of \( P_k(s) - P(s) \) and \( f(\mathbb{N}_k, h) - f(\mathbb{N}_1, h) \), respectively, as well as verify that the auxiliary parameters (as defined in Lemma 4) satisfy \( s_k < \sigma_k < t_k < s'_k \). We similarly compute the root of \( 1 - f(\mathbb{N}_1, h) \) as \( h_\infty = 0.277 \cdots \).

In our computations, we express \( P_k \) in terms of \( P \) using the recursion in (3). In turn by Möbius inversion \( P(s) = \sum_{m \geq 1} (\mu(m)/m) \log \zeta(ms) \), so \( P \) is obtained via well-known rapid computation of \( \zeta \). Finally, we compute \( f(\mathbb{N}_k, h) \) from its integral form (1). The data are displayed in the table below, obtained using Mathematica (for technical convenience, we first compute \( y_k = \log h_k \)).

We believe a unique of root \( h_k \) exists as in Corollary 5 for all \( k > 20 \) as well. This would enable a strengthening of Theorem 2 to \( f(\mathbb{N}_k, h) > f(\mathbb{N}_1, h) \) for all values of \( k > 1, h \geq h_2 \) (so far we only establish this for \( \{1 < k \leq 20, h \geq h_2\} \) or \( \{k > 1, h = h_2\} \)). Uniqueness of \( h_k \) would follow if \( \sigma_k \) is unique, as in Lemma 4, for all \( k \). In turn it would suffice to show \( t_k < s'_k \) for all \( k \) (note \( s_k < t_k \) holds automatically by (4), (6)), though it is not clear how to establish such an inequality in general.

Moreover, it appears both \( h_k \) and \( \sigma_k \) are monotonically decreasing in \( k \). This may be related to some empirical trends for \( f(\mathbb{N}_k) \), found in a recent disproof of a conjecture of Banks–Martin, see [1, 6].

\[\]

\[1\]P[k_Integer, s_] := If[k == 1, PrimeZetaP[s],
Expand[(Sum[PrimeZetaP[1 + j s] * PrimeZetaP[k - j, s], j, 1, k - 1] + PrimeZetaP[1, k s]) / k]]
FindRoot[PrimeZetaP[1, s] == (k!)^(1/(k - 1)), s, 1 + 1/k^3]
FindRoot[PrimeZetaP[k, t] / (2^(-t) + 3^(-t)) == 1, t, 1 + 1/k^3]
FindRoot[PrimeZetaP[k - 1, s] == 1, s, 1 + 1/k^3]
FindRoot[PrimeZetaP[k, sigma] == PrimeZetaP[1, sigma] / (sigma, sigma, 1 + 1/k^3)]
FindRoot[NIntegrate[(PrimeZetaP[1, s] - PrimeZetaP[k, s]) / y^s, s, 1, Infinity, WorkingPrecision->30, AccuracyGoal->13, PrecisionGoal->13], y, 1 + 1/k^3]
3. Proof of Theorem 2

**Proof.** We have already verified the claim directly for \( k \leq 20 \), since in this case \( h_k \leq h_2 = 1.04 \ldots \).

For \( k > 20 \), the proof strategy is similar to that of [6, Theorem 5.5]. That is, the integral \( f(N_k, h) = \int_1^\infty P_k(s)e^{(1-s)h} \, ds \) has its mass concentrated near 1 as \( k \to \infty \), so it suffices to truncate the integration to [1, 1.01] say, as a lower bound. Thus by (1),

\[
 f(N_k, h_2) = \int_1^\infty P_k(s)e^{(1-s)h_2} \, ds > e^{-0.01h_2} \int_1^{1.01} P_k(s) \, ds. \tag{7}
\]

Next, we may lower bound \( P_k(s) \) by \( P(s)^k/k! \), which constitutes the first of the terms in the identity (2), one per partition of \( k \). Note the terms of partitions built from small parts contribute the most mass. So by also including the terms for the partitions \( k = 1 \cdot (k - j) + j \) and \( k = 1 \cdot (k - j - 2) + j \) for \( j \leq 6 \), we shall obtain a sufficiently tight lower bound to deduce the result. Indeed, we have

\[
 \int_1^{1.01} P_k(s) \, ds \geq \frac{1}{k!} \int_1^{1.01} P(s)^k \, ds + \sum_{j=2}^6 \frac{\int_1^{1.01} P(s)^{k-j} P(j) \, ds}{j!} + \frac{\int_1^{1.01} P(s)^{k-4} P(2s)^2 \, ds}{2k^{2}(k-4)!} + \sum_{j=3}^6 \frac{\int_1^{1.01} P(s)^{k-j-2} P(2s) P(j) \, ds}{2j \cdot (k-j-2)!}. \tag{8}
\]

From [6, (5.10)], we have

\[
 0 < P(s) - \log \left( \frac{\alpha}{s-1} \right) < 1.4(s-1), \quad \text{for } s \in [1,2],
\]

where \( \alpha = \exp(-\sum_{m \geq 2} P(m)/m) = .7292 \ldots \). Thus for every \( k \geq 1 \), since \( \log(\alpha/101) > 4 \),

\[
 \int_1^{1.01} P(s)^k \, ds > \int_1^{1.01} \log \left( \frac{\alpha}{s} \right)^k \, ds = k \int_0^\infty u^k e^{-u} \, du > a \Gamma(k+1,4) > .729 k!,
\]

where \( \Gamma(k+1,4) \) the incomplete Gamma function, and noting \( \Gamma(k+1,4)/k! \) is monotonically increasing in \( k \). Also note

\[
 \int_0^1 s \log \left( \frac{\alpha}{s} \right)^k \, ds = \alpha^2 \int_0^\infty u^k e^{-2u} \, du = \frac{\alpha^2}{2k+1} k!.
\]

Using the first order Taylor approximation \( P(j) > P(j) + P'(j)(s-1) \) for \( j \geq 2 \),

\[
 \int_1^{1.01} P(s)^{k-j} P(j) \, ds > P(j) \int_0^{0.01} \log \left( \frac{\alpha}{s} \right)^{k-j} \, ds + P'(j) \int_0^{1.01} s \log \left( \frac{\alpha}{s} \right)^{k-j} \, ds
\]

\[
 > .729 (k-j)! \left[ P(j) + \frac{\alpha P'(j)}{2k-j-1} \right]
\]

by (10). Similarly,

\[
 \int_1^{1.01} P(s)^{k-j-2} P(2s) P(j) \, ds
\]

\[
 > P(2) P(j) \int_0^{0.01} \log \left( \frac{\alpha}{s} \right)^{k-j-2} \, ds + \left[ P'(2) P(j) + P(2) P'(j) \right] \int_0^{1.01} s \log \left( \frac{\alpha}{s} \right)^{k-j-2} \, ds
\]

\[
 > .729 (k-j-2)! \left[ P(2) P(j) + \frac{\alpha}{2k-j-1} \left[ P'(2) P(j) + P(2) P'(j) \right] \right].
\]

Hence plugging back into (8),

\[
 f(N_k, h_2)e^{0.01h_2} > \int_1^{1.01} P(s)^k \, ds > \ell_k,
\]
for the explicit lower bound

\[ \ell_k := .729 \left[ 1 + \frac{6}{j} \sum_{j=2}^{k} P(j) + \alpha P'(j) 2^{k-j} \right] + \frac{1}{8} \left( P(2)^2 + \frac{\alpha P(2) P'(2)}{2^{k-4}} \right) \\
+ \frac{6}{j} \sum_{j=3}^{k} \frac{1}{2j} \left( P(2) P(j) + \frac{\alpha}{2^{k-j-1}} \left[ P'(2) P(j) + P(2) P'(j) \right] \right). \]

Note \( \ell_k \) is clearly increasing in \( k \) (recall \( P'(s) < 0 \)). Hence for \( k > 20 \) we have

\[ f(N_k, h_2) > e^{-0.01h_2} \ell_k > e^{-0.01h_2} \ell_{20} > .98 > .91 > f(N_1, h_2). \]  \hspace{1cm} (11)

Here we compute \( \ell_{20} = 0.99069 \cdots \) and \( f(N_1, h_2) = 0.908599 \cdots \). This completes the proof of Theorem 2. \( \square \)

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References