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The critical exponent functions

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Abstract. The *critical exponent* of a finite or infinite word w over a given alphabet is the supremum of the reals α for which w contains an α-power. We study the maps associating to every real in the unit interval the inverse of the critical exponent of its base-n expansion. We strengthen a combinatorial result by J.D. Currie and N. Rampersad to show that these maps are left- or right-Darboux at every point, and use dynamical methods to show that they have infinitely many nontrivial fixed points and infinite topological entropy. Moreover, we show that our model-case map is topologically mixing.

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1. Introduction

The base-n expansion of real numbers has often been used to define sets and functions having interesting properties. In many cases, it has been through this path that remarkable counterexamples to appealing but mistaken intuition in the theory of real functions has been identified. For instance the Cantor set, which is basic for the construction of counterexamples in analysis and topology, can be very simply defined by means of base-3 expansion. Early examples of transcendental numbers have been defined through simple recursive procedures based on the decimal expansion ([15]) and the rich theory of normal numbers, which has been very fruitful in measure theory and measurable dynamics, exploits the base-n expansion at its very beginning (a very nice reference work is [17]).

In the last decades, attention has been devoted to the concept of *critical exponent* of a word w over a given alphabet, that is the supremum of the reals α for which w contains an α-power. The study of critical exponents is natural for understanding the repetition threshold of an alphabet (see [5, p. 126]). The long-standing problem of finding the repetition threshold as a function of the alphabet size was recently solved by M. Rao ([18]) (and independently by J. D. Currie and N. Rampersad ([8])), who covered the last cases of a general conjecture by F. Dejean ([9]), and the answer is that the repetition threshold for size n is $\frac{n}{n-1}$ for $n = 2$ and $n \geq 5$, while it is $7/4$ for

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$n = 3$ and $7/5$ for $n = 4$. The critical exponent and related concepts (such as the Diophantine exponent of a sequence) proved significant in symbolic dynamics ([2]) and transcendental number theory ([1]).

In this paper we explore the relations between the combinatorial concept of critical exponent and the analytical and dynamical properties of objects defined through it. We do so by considering, as a prototype, the *n-critical exponent function* $\kappa_n : [0,1] \to [0,1]$, i.e. the map associating to every real in the unit interval the inverse of the critical exponent of its (infinite) base-n expansion (we assume $1/\infty = 0$). We mainly focus on the model case κ_2. In particular, we strengthen a combinatorial result by J. D. Currie and N. Rampersad ([7]) to prove, using dynamical methods, that the critical exponent functions have infinite topological entropy and infinitely many fixed points.

The very fine dependence shown by the critical exponent on the combinatorial properties of the word makes the analytical and dynamical properties of functions defined through it quite complicated in principle. For instance, while simple normality of reals is concerned with asymptotic densities, and thus is unaffected by changes limited to prefixes of any finite length, the critical exponent may depend sharply on a single digit of an infinite word. This is the main reason for which methods developed for the study of dynamics of highly irregular maps (typically having dense discontinuities) prove useful in our framework. The most relevant feature of our approach is indeed the interplay between methods and results from combinatorics on words and quite recent results on the study of the dynamics of Baire 1 maps ([21,22]) and Darboux maps ([16]). In particular, we establish the existence of nontrivial fixed points for κ_n, which is in itself a genuinely combinatorial property, by exploiting dynamical properties of Darboux–Baire 1 maps.

The structure of the paper is as follows. In Section 2 we present the notation and the basic concepts that we use throughout the paper. In Section 3 we define the critical exponent and prove a combinatorial result which will be used throughout. In Section 4 we study the model case κ_2, in which the critical exponent function is defined using base-2 expansion, investigating its basic analytical and dynamical properties. In Section 5 we generalize most of our results to the critical exponent function in base n, indicated by κ_n. In Section 6 we introduce and study the map defined as the pointwise supremum of κ_n, which is independent of the choice of a base. Finally, in Section 7 some open questions are presented.

2. Notation and preliminaries

For an alphabet $\mathscr{A} = \{a_0, a_1, \dots, a_n\}$, we denote by $\mathscr{A}^*$ and $\mathscr{A}^\omega$ the set of the finite and infinite words over $\mathscr{A}$ respectively. We use the symbol ϵ for the empty word and denote by $\mathscr{A}^+$ the set of finite words different from ϵ. We set

$$\mathscr{A}^\infty = \mathscr{A}^* \cup \mathscr{A}^\omega.$$

If $w \in \mathscr{A}^\infty$, we denote by w_i its i^{th} digit. An element of an indexed set of words will be denoted by $w_{(i)}$. We indicate by $\ell : \mathscr{A}^* \to \mathbb{N}$ the map associating to the nonempty finite word $a = a_1 \dots a_n$ the natural number n, while we set $\ell(\epsilon) = 0$.

Let $y, w \in \mathscr{A}^\infty$. We say that y is a *subword* (in the literature the term *factor* is also used with the same meaning) of w if there exist $w_{(1)} \in \mathscr{A}^*$ and $w_{(2)} \in \mathscr{A}^\infty$ such that $w = w_{(1)} y w_{(2)}$. We say that $y \in \mathscr{A}^+$ is a *prefix* (*suffix*) of w if $w = y w_{(2)}$ ($w = w_{(1)} y$). For every $w \in \mathscr{A}^\omega$, we denote by $\mathscr{L}(w)$ the language of w, namely the set of all finite subwords of w.

For every $w \in \mathscr{A}^*$ and $n \in \mathbb{N}$, we denote by w^n the concatenation of n-copies of w, namely

$$w^n := \underbrace{ww\dots w}_{n\text{-times}}.$$

We denote by $\delta\colon \mathscr{A}^\infty \to \mathscr{A}^\infty$ the deletion operator that removes the first digit of a word. Hence, if $w = w_0w_1w_2w_3\cdots \in \mathscr{A}^\infty$, we have

$$\delta(w) = \delta(w_0w_1w_2w_3\dots) = w_1w_2w_3\dots,$$

while $\delta(\epsilon) = \epsilon$. Accordingly, for every $n \in \mathbb{N}$, the operator δ^n removes the first n letters of a word, hence

$$\delta^n(w) = \delta^n(w_0w_1\dots w_nw_{n+1}\dots) = w_nw_{n+1}w_{n+2}\dots.$$

For every two real numbers $a < b$, we denote by $]a,b[$ the open interval with endpoints a and b. For each $x \in [0,1]$, we denote by $w_x \in \{0,1\}^\omega$ the binary expansion of x, that is

$$x = (0.w_x)_2 = \sum_{i=0}^{\infty} \frac{(w_x)_i}{2^i}.$$

If $x \in (0,1]$ is a dyadic rational, we consider its binary expansion w_x whose digits are ultimately 1, for instance $w_{\frac{1}{2}} = 01111\dots$.

Definition 1. *A function $f\colon [0,1] \to \mathbb{R}$ is said to be* of Baire class 0 *if it is continuous, and for every positive integer N it is said to be of Baire class N if there exists a sequence (f_n) of functions of Baire class $N-1$ which converges pointwise to f. For every $N \geq 1$, we denote by $\mathscr{B}_N$ the set of all functions of Baire class N defined over* $[0,1]$.

Definition 2. *A function $f\colon [0,1] \to \mathbb{R}$ is a* Darboux function *if it satisfies the intermediate value property. More precisely, f is a Darboux function if for all $x, y \in [0,1]$ and α between $f(x)$ and $f(y)$, there exists a number z between x and y such that $f(z) = \alpha$.*

If $f\colon [0,1] \to [0,1]$, for every $n \in \mathbb{N}$ we denote by $f^n\colon [0,1] \to [0,1]$ the n^{th} iteration of f, hence

$$f^n(x) = \underbrace{f\big(f\big(f(\dots(f}_{n \text{ times}}(x)))\big)\big).$$

3. The critical exponent and a combinatorial result

We recall here the definition of critical exponent of a word (cf. [5]). Let us fix an alphabet $\mathscr{A}$. For positive integers p and q, a word $w \in \mathscr{A}^+$ is a *p/q-power* if it is of the form $w = x^n y$, where $x \in \mathscr{A}^+$, $y \in \mathscr{A}^*$ is a prefix of x, $\ell(w) = p$ and $\ell(x) = q$ (cf. [9]). Let $\alpha \in \mathbb{Q}^+$. A word $w \in \mathscr{A}^\infty \setminus \epsilon$ *avoids α-powers* if none of its subwords is an r-power for any rational $r \geq \alpha$. Otherwise, we say that w *contains an α-power.*

Definition 3. *The* critical exponent *of a word $E\colon \mathscr{A}^\infty \to \mathbb{R}$ is defined as*

$$E(w) = \begin{cases} \sup\left\{r \in \mathbb{Q} : w \text{ contains an } r\text{-power}\right\}, & \text{if } w \neq \epsilon, \\ 0, & \text{if } w = \epsilon. \end{cases}$$

The *repetition threshold* of an alphabet $\mathscr{A}$ is the least possible critical exponent for infinite words over $\mathscr{A}$. The answer to the question about the repetition threshold of $\{0,1\}$ was given by the well-known Thue–Morse sequence, defined below, which has proven significant in as different fields as combinatorics of words, number theory, differential geometry and Morse theory ([3]).

Definition 4. *The* Thue–Morse sequence *is the element $w_\tau = w_0w_1w_2\dots w_n\cdots \in \{0,1\}^\omega$ given by*

$$\begin{cases} w_0 = 0, \\ w_{2n} = w_n, \\ w_{2n+1} = 1 - w_n. \end{cases}$$

The Thue–Morse sequence starts as follows

$$w_\tau = 0\ 1\ 10\ 1001\ 10010110\ 1001011001101001\ \dots.$$

It has been proved (see the classical paper [23]) that $E(w_\tau) = 2$.

Remark 5. The Thue–Morse sequence can be also defined recursively using the bitwise negation as follows. The first digit is $w_0 = 0$. Then, if the first 2^n digits are given, then the next 2^n digits are their bitwise negation. More precisely, if we have

$$w_0,\ w_1, \dots w_{2^n-1},$$

then

$$w_k = 1 - w_{k-2^n}, \quad \forall\ k = 2^n, 2^n+1, \dots 2^{2n}-1.$$

Remark 6. Let $w \in \{0,1\}^+$ and $y \in \{0,1\}^\omega$ such that $E(y) = \alpha > E(w)$. If $E(wy) > \alpha$, then there exists a finite subword v of wy such that $E(v) > \alpha$. Moreover, v is not a subword of w or a subword of y. As a consequence, $v = sp$, with s a suffix of w and p a prefix of y.

The next theorem guarantees the existence of binary words constrained to a given prefix w having any critical exponent compatible with w. This theorem is a strengthening of [7, Theorem 6], which corresponds to the particular case $w = \epsilon$.

Theorem 7. *Let* $w \in \{0,1\}^*$. *For every* $\alpha > \max\{2, E(w)\}$, *there exists* $y \in \{0,1\}^\omega$ *such that* $E(wy) = \alpha$.

Proof. As usual in this context, the tricky part is to make sure that there is no suffix of w which, concatenated with a prefix of y, is a β-power, with $\beta > \alpha$.

By the construction given in the proof of [7, Theorem 6], there exists an infinite binary word z such that

$$E(z) = \alpha, \quad \text{and} \quad E\left(\delta^n(z)\right) = \alpha, \quad \forall\ n \in \mathbb{N}.$$

Moreover, for every m sufficiently large, we can choose z from a sequence $(z_{(m)})_m$, where

$$z_{(m)} = w_{\tau,(m)}\, w_{\tau,(m)}\, z_{(3,m)} \tag{1}$$

and $w_{\tau,(m)}$ is the prefix of length 2^m of the Thue–Morse sequence, with $z_{(3,m)}$ an infinite binary word[1].

The idea of the proof is to find $n, m \in \mathbb{N}$ such that

$$E\left(w\delta^n\left(z_{(m)}\right)\right) = \alpha.$$

Indeed, the word $wz_{(m)}$ could have a critical exponent greater than α. By Remark 6, this issue could occur only if $w = \widetilde{w}b$, where $\widetilde{w}$ and b are a prefix and a suffix of w respectively, and $z_{(m)} = pbp\widetilde{b}\dots$, where p is a prefix of $z_{(m)}$ and $\widetilde{b}$ is a prefix of b, so that

$$wz_{(m)} = \widetilde{w}\,\underbrace{\overbrace{b|\ p\ b\ \dots p\ b\ p}^{\lfloor\alpha\rfloor\text{ times}}\ \widetilde{b}}_{=q}\dots,$$

with $E(q)$ greater than α.

To avoid this problem, we proceed as follows. Let $\bar{b}$ be the longest suffix of w which is a subword of the Thue–Morse sequence. Notice that $\bar{b}$ is well defined, since the last digit of w is of course inside Thue–Morse and w is a finite word. We write $w = \bar{w}\bar{b}$. By (1), there exists $m_w \in \mathbb{N}$ such that, for every $m \ge m_w$, the first 2^m digits of $z_{(m)}$ contain at least one occurrence of $\bar{b}$. Then, for every $m \ge m_w$, $z_{(m)}$ can be written as in (1) with

$$w_{\tau,(m)} = z_{(1)}\bar{b}z_{(2,m)},$$

[1] In the notation of [7], set $s_0 = m$, $r_0 = 3$, $t_0 = 2^{m-1} + c$, with c such that β_0 is obtainable, and remove the first $2^m - (t_0)$ digits so as to start with the beginning of the Thue–Morse sequence.

where $z_{(1)}$ is the longest prefix of the Thue–Morse sequence such that $z_{(1)}\bar{b}$ contains only one occurrence of $\bar{b}$. For the following steps, notice that this construction gives only a lower bound for the length of $z_{(2,m)}$, hence we are still free to extend it as we desire.

Removing the prefix $z_{(1)}\bar{b}$ from z, we obtain $y_{(m)} = z_{(2,m)}w_{\tau,(m)}z_{(3,m)}$. By construction, there exists $n \in \mathbb{N}$ such that $y_{(m)} = \delta^n(z)$ for every $m \geq m_w$. As a consequence, $E(y_{(m)}) = \alpha$. We will show that, choosing properly the length of $z_{(2,m)}$, we obtain $E(wy_{(m)}) = \alpha$. In other words, we will prove the existence of an $m_0 \geq m_w$ such that

$$E\left(wy_{(m_0)}\right) = \alpha. \tag{2}$$

Arguing by contradiction, assume that $E(wy_{(m)}) > \alpha$ for every $m \geq m_w$. By Remark 6, for every $m \geq m_w$ there exist a suffix $s_{(m)}$ of w and a prefix $p_{(m)}$ of $y_{(m)}$ such that

$$E\left(s_{(m)}p_{(m)}\right) > \alpha.$$

Set $u_{(m)} = s_{(m)}p_{(m)}$. The suffix $s_{(m)}$ of w cannot be a subword of the Thue-Morse sequence. Otherwise, by definition of $y_{(m)}$, $u_{(m)}$ should be a subword of $z_{(m)}$, hence $E(z_{(m)}) \geq E(u_{(m)}) > \alpha$, which is a contradiction. As a consequence, $s_{(m)}$ is not a subword of $z_{(2,m)}$. Since $E(u) > \alpha > 2$, $s_{(m)}$ must appear at least twice in $u_{(m)}$, so $p_{(m)}$ is of the form

$$p_{(m)} = x_{(m)}s_{(m)}\ldots x_{(m)}\widetilde{s}_{(m)},$$

with $\widetilde{s}_{(m)}$ a prefix of $s_{(m)}$ and $z_{(2,m)}$ a subword of $x_{(m)}$. As a consequence, we have $\ell(x_{(m)}) > \ell(z_{(2,m)})$ and

$$E\left(u_{(m)}\right) = E\left(p_{(m)}\right) + \frac{\ell\left(s_{(m)}\right)}{\ell\left(x_{(m)}s_{(m)}\right)} \leq E\left(p_{(m)}\right) + \frac{\ell(w)}{\ell\left(x_{(m)}s_{(m)}\right)} < E(p) + \frac{\ell(w)}{\ell\left(z_{(2,m)}\right)}.$$

Thus,

$$\alpha \leq \limsup_{m\to\infty} E\left(u_{(m)}\right) \leq \limsup_{m\to\infty} E\left(p_{(m)}\right). \tag{3}$$

To obtain a contradiction and complete the proof, it remains to show that

$$\limsup_{m\to\infty} E\left(p_{(m)}\right) < \alpha. \tag{4}$$

Since $p_{(m)}$ is a subword of $y_{(m)}$, $E(p_{(m)}) \leq \alpha$. Hence, if $\alpha \notin \mathbb{N}$, then

$$E\left(p_{(m)}\right) = \lfloor\alpha\rfloor + \frac{\ell\left(\widetilde{s}_{(m)}\right)}{\ell\left(x_{(m)}s_{(m)}\right)} < \lfloor\alpha\rfloor + \frac{\ell(w)}{\ell\left(z_{(2,m)}\right)},$$

and therefore

$$\limsup_{m\to\infty} E\left(p_{(m)}\right) = \lfloor\alpha\rfloor < \alpha.$$

If $\alpha \in \mathbb{N}$, $\alpha \geq 3$, then

$$E(p) = \alpha - 1 + \frac{\ell\left(\widetilde{s}_{(m)}\right)}{\ell\left(x_{(m)}s_{(m)}\right)},$$

so

$$\limsup_{m\to\infty} E\left(p_{(m)}\right) = \alpha - 1 < \alpha.$$

By (3) and (4) we obtain a contradiction, so there exists an $m_0 \geq m_w$ such that (2) holds, and we are done. $\square$

Remark 8. By the construction of the proof of Theorem 7, the length of $z_{(2,m)}$ can be chosen arbitrarily large. More formally, let $w \in \{0,1\}^*$ and $\alpha > \max\{2, E(w)\}$. Using the same notation of the proof of Theorem 7, for every $m \geq m_w$, the word $y_{(m)}$ is well defined and $E(y_{(m)}) = \alpha$. The proof of 7 shows not only that there exists an $m_0 \geq m_w$ such that $E(wy_{(m_0)}) = \alpha$ holds, but also that there exists an increasing sequence $(m_k)_k \subset \mathbb{N}$ such that $m_k \geq m_w$ and $E(wy_{(m_k)}) = \alpha$. In the following, such a sequence will be denoted by $(y_{(\alpha,k)})_k \subset \{0,1\}^\omega$. We notice that every $y_{(\alpha,k)}$ is of the form

$$y_{(\alpha,k)} = z_{(2,m_k)}\, w_{\tau,(m_k)}\, z_{(3,m_k)}.$$

Being $(m_k)_k$ an increasing sequence, the length of $z_{(2,m_k)}$ can be arbitrarily large.

4. The model case κ_2

In this section, we define and analyze the 2-critical exponential function, which will be denoted by κ_2. Its definition is based on the critical exponent of a word, and many properties we shall present stem from Theorem 7. We remind that, given $x \in [0,1]$, $w_x \in \{0,1\}^\omega$ is its binary expansion and $\mathscr{L}(w_x)$ is the set of all the finite subwords of w_x.

Definition 9. *The critical exponent function $\kappa_2\colon [0,1] \to [0,1]$ is defined as*

$$\kappa_2(x) := \inf_{u \in \mathscr{L}(w_x)} \frac{1}{E(u)}.$$

Remark 10. Using the convention $1/\infty = 0$, we can write $\kappa_2(x) = 1/E(w_x)$.

4.1. *Analytical properties for κ_2*

Proposition 11. *The function κ_2 is upper semi-continuous and of Baire class* 1.

Proof. Let us fix $x \in [0,1]$ and $\varepsilon > 0$. By definition of κ_2, there exists a prefix of w_x, say $w_\varepsilon \in \{0,1\}^+$, such that

$$\frac{1}{E(w_\varepsilon)} \le \kappa_2(x) + \varepsilon.$$

If $x = 0.a01^\infty = 0.a10^\infty$ ($a \in \{0,1\}^*$) is a nonzero dyadic rational, then, for any integer $M > 0$, the binary expansion of any real close enough to x will have a prefix of type $a10^M$ or $a01^M$. Therefore, $\lim_{n\to\infty} \kappa_2(x_n) = 0 = \kappa_2(x)$, so κ_2 is continuous at x. A similar argument of course applies if $x = 0$. Otherwise, there exist $N_\varepsilon > 0$ and a sequence $(y_n)_{n \ge N_\varepsilon} \subset \{0,1\}^\omega$ such that

$$w_{x_n} = w_\varepsilon y_n, \quad \forall\, n \ge N_\varepsilon.$$

As a consequence,

$$\kappa_2(x_n) = \inf_{u \in \mathscr{L}(w_{x_n})} \frac{1}{E(u)} \le \frac{1}{E(w_\varepsilon)} \le \kappa_2(x) + \varepsilon, \quad \forall\, n \ge N_\varepsilon.$$

By the arbitrariness of ε, we obtain

$$\limsup_{n\to\infty} \kappa_2(x_n) \le \kappa_2(x),$$

and by the arbitrariness of the sequence $(x_n)_n$ we obtain that κ_2 is upper semi-continuous. By a classical result (cf. [4]), every semi-continuous function is of Baire class 1, so $\kappa_2 \in \mathscr{B}_1$. □

Let us define the sets

$$C_2 := \{x \in I : \kappa_2(x) = 0\}$$

and

$$D_2 := \{x \in I : \kappa_2(x) \ne 0\}.$$

Since the binary expansion of every rational number in $[0,1]$ has a period, which is a finite word repeated infinitely many times, then

$$\mathbb{Q} \cap [0,1] \subset C_2.$$

In the following Propositions 12-16, we will show that C_2 coincides in fact with the continuity set of κ_2 and prove some of its topological and measure-theoretic properties.

Proposition 12. *The function κ_2 is continuous at $x \in [0,1]$ if and only if $x \in C_2$.*

Proof. Since $\kappa_2(x) \geq 0$ for every $x \in [0,1]$, Proposition 11 implies that if $x \in C_2$, then κ_2 is continuous at x. On the other hand, for every $x \notin C_2$ there exists a sequence $(x_n)_n \subset \mathbb{Q} \cap [0,1] \subset C_2$ such that $x_n \to x$. Therefore,

$$\lim_{n\to\infty} \kappa_2(x_n) = 0 \neq \kappa_2(x),$$

and κ_2 is not continuous at x. □

Remark 13. By Proposition 12, the function κ_2 is not quasi-continuous. Indeed, the graph of the restriction of every quasi-continuous function to its continuity set is dense in the graph of the function (see [19, p. 632]), whereas $\kappa_2|_{C_2}$ is identically zero.

Before proving the next result, we introduce the following notation. We say that a number $x \in [0,1]$ is normal in base 2 if every word $w \in \{0,1\}^+$ has density $2^{-\ell(w)}$ in w_x (cf. [17]). We set

$$\mathcal{N}_2 := \{x \in I : x \text{ is a normal number in base } 2\}.$$

Proposition 14. *The set C_2 is a co-meagre set and it has full Lebesgue measure.*

Proof. Since the set of continuity of every real function is a G_δ set, Proposition 12 implies that C_2 is a G_δ. Moreover, C_2 is dense, since $\mathbb{Q} \cap [0,1] \subset C_2$. Then, D_2 is an F_σ subset of $[0,1]$. As a consequence, it is either meagre or else has nonempty interior. Since for every interval $]a,b[\subset [0,1]$ we have $]a,b[\cap C_2 \neq \emptyset$, we obtain that D_2 has empty interior, hence it is meagre. Thus, C_2 is co-meagre.

Since $\mathcal{N}_2$ has full Lebesgue measure (for an elementary proof see [12]) and $\mathcal{N}_2 \subset C_2$, C_2 has full Lebesgue measure. □

Proposition 15. *The sets C_2 and D_2 are dense in* $[0,1]$.

Proof. Since $\mathbb{Q} \cap [0,1] \subset C_2$, the set C_2 is dense in $[0,1]$. To prove that D_2 is dense in $[0,1]$, let us fix $x \in [0,1]$. For every $\varepsilon > 0$, there exists a prefix of w_x, say w_ε, such that

$$\left|(0.w_\varepsilon z)_2 - x\right| \leq \varepsilon, \quad \forall\, z \in \{0,1\}^\omega.$$

Hence, we obtain

$$x_\varepsilon = (0.w_\varepsilon w_\tau)_2 \in D_2 \cap [x-\varepsilon, x+\varepsilon].$$

By the arbitrariness of x and ε, we obtain the thesis. □

Proposition 16. *The sets C_2 and D_2 are neither closed nor open in* $[0,1]$.

Proof. Since C_2 and D_2 are complementary sets, it suffices to show that C_2 and D_2 are not closed. To show that C_2 is not closed, let us consider $x_0 \in D_2$ and a sequence $(x_k)_k$ of dyadic rationals which converges to x_0. Then $x_k \in C_2$ for every k, and $x_k \to x_0 \notin C_2$. On the other hand, to show that D_2 is not closed, let us consider the sequence $(x_k)_k \subset [0,1]$ defined as $x_k = (0.0^k w_\tau)_2$. Then, $x_k \in D_2$ for every k and $\lim_{k\to\infty} x_k = 0 \notin D_2$. □

Remark 17. The number $x_\tau = (0.w_\tau)_2 \in [0,1]$ is such that $\kappa_2(x_\tau) = 1/2$, and x_τ is a maximum of κ_2, since there is not a word $w \in \{0,1\}^\omega$ such that $E(w) < 2$.

Proposition 18. *The image of κ_2 is $[0,1/2]$. Moreover, the function κ_2 has the following symmetry:*

$$\kappa_2(x) = \kappa_2(1-x), \quad \forall\, x \in [0,1].$$

Proof. By [7, Theorem 6], for each $\alpha > 2$ there is an infinite binary word, say $w_\alpha \in \{0,1\}^\omega$, with critical exponent α. Therefore, $x_\alpha = (0.w_\alpha)_2 \in [0,1]$ is such that $\kappa_2(x_\alpha) = 1/\alpha$. By the arbitrariness of $\alpha > 2$, we obtain $]0,1/2[\subset \kappa_2(I)$. Since $\kappa_2(x_\tau) = 1/2$ and C_2 is non-empty, we have $\kappa_2([0,1]) = [0,1/2]$.

To show the desired symmetry, it suffices to notice that for every $x \in [0,1]$ the word w_{1-x} is the bitwise negation of w_x. □

For every function $f\colon [0,1]\to\mathbb{R}$, we denote by $R^-(f,x)$ and $R^+(f,x)$ the left and right range of f at x, respectively. More precisely,

$$R^-(f,x) := \left\{\alpha\in\mathbb{R} : f^{-1}(\alpha)\cap(x-\delta,x)\neq\emptyset,\ \forall\,\delta>0\right\},$$
$$R^+(f,x) := \left\{\alpha\in\mathbb{R} : f^{-1}(\alpha)\cap(x,x+\delta)\neq\emptyset,\ \forall\,\delta>0\right\}.$$

Sufficiently well-behaved functions will have typically "small" left and right ranges at every point. For instance, if the left (right) limit of f at x exists, then its left (right) range at x consists at most of one point. On the contrary, the map κ_2 has "large" left and right ranges at every point $x\in D_2$, as proved in the following proposition. We will exploit this to get other properties of κ_2, such as the existence of almost-fixed points and fixed points for κ_2.

Proposition 19. *If $x\in C_2$, then*

$$R^-\left(\kappa_2,x\right)\cup R^+\left(\kappa_2,x\right)=\{0\}. \tag{5}$$

Otherwise, if $x\in D_2$, then at least one of the following holds:

$$\left[0,\kappa_2(x)\right[\,\subset R^-\left(\kappa_2,x\right) \quad or \quad \left[0,\kappa_2(x)\right[\,\subset R^+\left(\kappa_2,x\right). \tag{6}$$

To prove the previous proposition, we need the following lemma.

Lemma 20. *Let $x\in[0,1]$ and let us denote by $w_{(n)}$ the finite prefix of w_x of length n. Then for every n at least one of the following holds:*

(a) *for every $\alpha>\max\{2,E(w_{(n)})\}$ there exists a word $y_{\alpha,n}\in\{0,1\}^\omega$ such that $E(w_{(n)}y_{\alpha,n})=\alpha$ and $(w_{(n)}y_{\alpha,n})_2<x$;*

(b) *for every $\alpha>\max\{2,E(w_{(n)})\}$ there exists a word $y_{\alpha,n}\in\{0,1\}^\omega$ such that $E(w_{(n)}y_{\alpha,n})=\alpha$ and $(w_{(n)}y_{\alpha,n})_2>x$.*

Proof. Let us fix $n\geq 1$. Using the same notation of the proof of Theorem 7, for every $m\geq m_{w_{(n)}}$, let $w_{\tau,(m)}$ be the prefix of the Thue–Morse sequence of length 2^m and $z_{(2,m)}$ such that

$$w_{\tau,(m)}=z_{(1)}\bar{b}z_{(2,m)},$$

being $z_{(1)}$ and $\bar{b}$ uniquely determined by $w_{(n)}$. Moreover, let us denote by $z_{(2)}$ the infinite word over $\{0,1\}$ such that

$$w_\tau=z_{(1)}\bar{b}z_{(2)}.$$

Notice that $z_{(2,m)}\to z_{(2)}$ as $m\to\infty$. By Remark 8, for every $\alpha>\{\max 2,E(w_{(n)})\}$ there exists an increasing sequence $(m_k)_k\subset\mathbb{N}$ such that, setting

$$y_{(\alpha,k)}=z_{(2,m_k)}w_{\tau,(m_k)}z_{(3,m_k)},$$

we have $E(w_{(n)}y_{(\alpha,k)})=\alpha$, for every $k\in\mathbb{N}$. Notice that z_{2,m_k} does not depend on α. We divide the remaining part of the proof in two cases:

(i) if $w_x\neq w_{(n)}z_{(2)}$, then these two words differ on at least one digit. Let $j>n$ be the index of the first different digit, hence $(w_x)_j\neq(w_{(n)}z_{(2)})_j$. Let $z_{(2,j)}$ be the prefix of $z_{(2)}$ with $\ell(w_{(n)}z_{(2,j)})=j$. Then either $(0.w_{(n)}z_{(2,j)}r)_2<x$ for all $r\in\{0,1\}^\omega$ or $(0.w_{(n)}z_{(2,j)}r)_2>x$ for all $r\in\{0,1\}^\omega$. As a consequence, since $z_{(2,m_k)}\to z_{(2)}$ as $k\to\infty$ and since for every α we can choose $y_{(\alpha,k)}$ such that $\ell(w_{(n)}z_{(2,m_k)})>j$, we obtain the thesis.

(ii) if $w_x=w_{(n)}z_{(2)}$, then w_x is definitely equal to the Thue–Morse sequence w_τ. By the bitwise negation construction of the Thue–Morse sequence presented in Remark 5, for every $m\geq m_{w_{(n)}}$ we have

$$w_x=w_{(n)}z_{(2,m)}1001\ldots.$$

At the same time, for every $\alpha > \max\{2, E(w_{(n)})\}$ and $m \geq m_{w_{(n)}}$ the word $y_{(\alpha,m)}$ is given by $w_{(n)} z_{(2,m)} w_{\tau,(m)} z_{(3,m)}$, hence $w_{(n)} y_{(\alpha,m)} = w_{(n)} z_{(2,m)} 0110\ldots$. As a consequence, for every α we obtain $(w_{(n)} y_{(\alpha,m)})_2 < (w_x)_2 = x$, so a) holds.

□

Remark 21. Let $x_\tau = (0.w_\tau)_2$. Then for all $n \in \mathbb{N}$ the construction given at the point (ii) of the previous proof applies. As a consequence, denoting by $w_{\tau,n}$ the prefix of w_τ such that $\ell(w_{\tau,n}) = n$, for every $n \in \mathbb{N}$ and every $\alpha > 2$ there exists an infinite word $y_{(\alpha,n)}$ such that $E(w_{\tau,n} y_{(\alpha,n)}) = \alpha$ and $(w_{\tau,n} y_{(\alpha,n)})_2 < x_\tau$.

Proof of Proposition 19. If $x \in C_2$, κ_2 is continuous at x by Proposition 12. As a consequence, for every $\alpha \neq 0$ there exists a neighbourhood I_α of x such that $\alpha \notin \kappa_2(I_\alpha)$. This implies that $\alpha \notin R^-(\kappa_2, x) \cup R^+(\kappa_2, x)$. On the other hand, for every neighbourhood of x there exists a number $y \neq x$ such that $\kappa_2(y) = 0$ and (5) holds.

Let $x \in D_2$, hence $\kappa_2(x) > 0$. Let us fix $n \in \mathbb{N}$ and denote by $w_{(n)}$ the finite prefix of w_x of length n. Moreover, let us set $A = [0, \kappa_2(x)[$. Since $E(w_x) \geq E(w_{(n)})$, by Lemma 20, there exists a set of words $(y_{\alpha,n})_{\alpha \in A} \subset \{0,1\}^\omega$ such that $\kappa_2((w_{(n)} y_{\alpha,n})_2) = \alpha$ and either

$$\left(0.w_{(n)} y_{\alpha,n}\right)_2 < x, \quad \forall\, \alpha \in A, \tag{7}$$

or

$$\left(0.w_{(n)} y_{\alpha,n}\right)_2 > x, \quad \forall\, \alpha \in A. \tag{8}$$

Therefore, we can define the sets

$$L = \{n \in \mathbb{N} :\ (7)\ \text{holds}\} \quad \text{and} \quad R = \{n \in \mathbb{N} :\ (8)\ \text{holds}\}.$$

Since $L \cup R = \mathbb{N}$, at least one between L and R has infinite elements. Without loss of generality, let us assume that it is L. Then, for every $\varepsilon > 0$ there exists a number $n_\varepsilon \in L$ such that the interval characterized by w_{n_ε}, namely

$$I_{w_{n_\varepsilon}} = \left\{\left(w_{n_\varepsilon} y\right)_2 : y \in \{0,1\}^\omega\right\},$$

satisfies

$$I_{w_{n_\varepsilon}} \subset \,]x-\varepsilon, x+\varepsilon[\,.$$

Since $n_\varepsilon \in L$, for every $\alpha \in [0, \kappa_2(x)[$ there exists y_{α,n_ε} such that

$$E\left(0.w_{(n)} y_{\alpha,n}\right) = \frac{1}{\alpha}, \quad \text{and} \quad \left(0.w_{(n)} y_{\alpha,n}\right)_2 < x,$$

hence

$$\kappa_2\left(\left(0.w_{(n)} y_{\alpha,n}\right)_2\right) = \alpha, \quad \text{and} \quad \left(0.w_{(n)} y_{\alpha,n}\right)_2 \in I_{w_{n_\varepsilon}} \cap \left[0, x\right[\subset \left]x-\varepsilon, x\right[.$$

By the arbitrariness of $\alpha \in [0, \kappa_2(x)[$ and $\varepsilon > 0$, we have $R^-(\kappa_2, x) = [0, \kappa_2(x)[$. Otherwise, if R is an infinite set, the same argument shows that $R^+(\kappa_2, x) = [0, \kappa_2(x)[$. □

By Remark 21 and Proposition 19, we obtain the following result.

Corollary 22. *Let $x_\tau = (0.w_\tau)_2$. Then $R^-(\kappa_2, x_\tau) = [0, 1/2[$.*

4.2. *Dynamical properties for κ_2*

We give the following definition (cf. [16]).

Definition 23. *Let $f\colon [0,1] \to \mathbb{R}$. A point $x \in [0,1]$ is said an* almost fixed point *of f, and we write $x \in \mathrm{aFix}(f)$, if and only if*

$$x \in \mathrm{Int}\left(R^-(f,x)\right) \cup \mathrm{Int}\left(R^+(f,x)\right). \tag{9}$$

If $x=0$ or $x=1$, then we only consider $R^+(f,x)$ or $R^-(f,x)$, respectively.

Corollary 24. *A point $x \in [0,1]$ is an almost fixed point for κ_2 if and only if $\kappa_2(x) > x$.*

Proof. If x is an almost fixed point, then there exists a sequence $(x_n)_n \subset [0,1]$ which converges to x and such that $\kappa_2(x_n) = x$, for every $n \in \mathbb{N}$. By the upper semi-continuity of κ_2 we have

$$x = \limsup_{n\to\infty} \kappa_2(x_n) \le \kappa_2(x).$$

It remains to show that $x \neq \kappa_2(x)$. By (9), there exists an $\varepsilon > 0$ such that $[x-\varepsilon, x+\varepsilon] \subset R^-(\kappa_2, x)$, or $[x-\varepsilon, x+\varepsilon] \subset R^+(\kappa_2, x)$. Assuming by contradiction that $x = \kappa_2(x)$, this implies the existence of a sequence $(x_n)_n \subset [0,1]$ which converges to x and such that $\kappa_2(x_n) = x+\varepsilon = \kappa_2(x)+\varepsilon$. Using again the upper semi-continuity of κ_2 we obtain

$$\kappa_2(x)+\varepsilon = x+\varepsilon = \limsup_{n\to\infty} \kappa_2(x_n) \le \kappa_2(x),$$

which is absurd.

If $\kappa_2(x) > x$, then Proposition 19 implies that in every neighbourhood of x there exists a number y such that $\kappa_2(y) = x$. Thus x is an almost fixed point. □

Corollary 25. *The set* $\mathrm{aFix}(\kappa_2)$ *is uncountable.*

Proof. Since $x_\tau = (0.w_\tau)_2 = (0.01101\ldots)_2 < 1/2$, we have that $\kappa_2(x_\tau) = 1/2 > x_\tau$. Fix a $\delta > 0$ such that $x_\tau + \delta < 1/2$. By Corollary 22, for every $\alpha \in [x_\tau+\delta, 1/2[$ there exists a number $y_\alpha \in]x_\tau-\delta, x_\tau[$ such that $\kappa_2(y_\alpha) = \alpha$. As a consequence, $\kappa_2(y_\alpha) > y_\alpha$. By Corollary 24, y_α is an almost fixed point. Since α can be chosen from an uncountable set, and $\alpha \neq \beta$ implies that $y_\alpha \neq y_\beta$, we are done. □

[16, Theorem 3.2] ensures that, assuming the Darboux property, in every neighbourhood of an almost fixed point of a Baire 1 function there exists a fixed point. It is not known to us whether κ_2 has the Darboux property. However, by Proposition 19, it is left- or right-Darboux at every point. This in fact suffices to prove an analogous existence result for fixed points.

Corollary 26. *For every $x_0 \in aFix(\kappa_2)$ and every $\varepsilon > 0$, there exists $z \in]x_0-\varepsilon, x_0+\varepsilon[$ such that*

$$\kappa_2(z) = z.$$

Proof. Let us fix $x_0 \in aFix(\kappa_2)$ and $\varepsilon > 0$. Using Proposition 19, we can assume either $R^-(\kappa_2, x_0) = [0, \kappa_2(x_0)[$ or $R^+(\kappa_2, x_0) = [0, \kappa_2(x_0)[$. If $R^-(\kappa_2, x_0) = [0, \kappa_2(x_0)[$, the following procedure proves the existence of a fixed point $c \in]x_0-\varepsilon, x_0[$. Otherwise, an analogous procedure shows the existence of a fixed point $c \in]x_0, x_0+\varepsilon[$.

Let us assume $R^-(\kappa_2, x) = [0, \kappa_2(x_0)[$. By Corollary 24, $\kappa_2(x_0) > x_0$, so there exists two points x_1, x_2 such that

$$0 < x_0-\varepsilon < x_1 < x_2 < x_0 < \kappa_2(x_0)$$

Hence, $[x_1, x_0] \subset R^-(\kappa_2, x_0)$. We denote by K the set of points on the diagonal of $[0,1]\times[0,1]$ between (x_2, x_2) and (x_0, x_0), namely

$$K = \left\{\left(x_2 + t(x_0-x_2), x_2 + t(x_0-x_2)\right) \in [0,1]\times I : t \in [0,1]\right\}.$$

We remark that K is a compact subset of $[0,1]\times[0,1]$. By Proposition 19, there exist $y_1, y_0 \in]x_2, x_0[$ such that $\kappa_2(y_1) = x_1$ and $\kappa_2(y_0) = x_0$. Since κ_2 is a function of Baire class one, there exists a sequence of continuous functions, say $(f_n)_n$, that converges pointwise to κ_2. Since $x_1 < x_2 < y_1$ and $y_0 < x_0$, for n sufficiently large we have $f_n(y_1) < y_1$ and $f_n(y_0) > y_0$. Since (f_n) is a sequence of continuous functions, for every n sufficiently large there exists z_n such that $(z_n, f_n(z_n)) \in K$. Since K is a compact set, going if necessary to a subsequence, there exists $z \in [x_2, x_0]$ such that $z_n \to z$. By the pointwise convergence of $(f_n)_n$ to κ_2, we obtain

$$\kappa_2(z) = \lim_{n\to\infty} \kappa_2(z_n) = \lim_{n\to\infty} z_n = z,$$

hence z is a fixed point for κ_2, and $c \in]x_0 - \varepsilon, x_0[$. □

Since aFix(κ_2) is uncountable and every neighbourhood of an almost fixed point contains a fixed point, we have the following result.

Corollary 27. *The set of the fixed points of κ_2 is infinite.*

We address now the problem of estimating the complexity of the dynamics of κ_2 from the point of view of its topological entropy. We recall that the topological entropy h of an interval map f can be written as ([6, 10]):

$$h(f) = \lim_{\varepsilon\to 0}\left(\limsup_{n\to\infty} \frac{1}{n} \log|(n,\varepsilon)|\right), \tag{10}$$

where $|(n,\varepsilon)|$ indicates the maximum cardinality of an ε-separated set in the metric d_n defined below:

$$d_n(x_1, x_2) := \max\left\{\left|f^i(x_1) - f^i(x_2)\right| : 0 \le i \le n\right\}. \tag{11}$$

The topological entropy can therefore be interpreted as an estimate of the exponential increase rate of the number of topologically distinguishable orbits when the resolution power diverges.

We first establish a technical result concerning the existence of horseshoes, which are defined as follows.

Definition 28. *Let $f : [0,1] \to [0,1]$. For each $m \in \mathbb{N}$, $m \ge 2$, an* m-horseshoe *for f is an ordered pair (J,D), where $J \subset [0,1]$ is an interval and D is a family of pairwise disjoint closed intervals $I_1, \dots, I_m \subset J$ such that $J \subset f(I_k)$, for all $k = 1, \dots, m$.*

Remark 29. Some works give a slightly more general definition of m-horseshoe, allowing the closed intervals $I_1, \dots, I_m$ to be adjacent.

Proposition 30. *The function κ_2 has an m-horseshoe for each $m \ge 2$.*

Proof. Noting that $x_\tau < 7/16$, we set $J = [0, 7/16]$. By Proposition 19, there exists a δ such that $I = [x_\tau - \delta, x_\tau + \delta] \subset J$ and

$$J \subsetneq \kappa_2(I). \tag{12}$$

If I satisfies (12), there exist two numbers $x_1, x_2 \in I^o$ (the interior of I) such that $\kappa_2(x_1) > 7/16$, $\kappa_2(x_2) > 7/16$ and $x_1 \neq x_2$. By Proposition 19, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that $I_1 = [x_1 - \delta_1, x_1 + \delta_1]$ and $I_2 = [x_2 - \delta_2, x_2 + \delta_2]$ are disjoint subsets of I and they both satisfy (12). Since we can repeat this construction as many times we need, the thesis follows by induction. □

It is well known in topological dynamics of interval maps that the existence of horseshoes of every order implies infinite topological entropy (cf. [20, Proposition 4.6]). We remark that, although usually continuity is assumed (as for instance in the nice reference work [20]), this implication does not depend on any regularity assumption on the function, as it is immediately clear considering the expression (10) for the entropy. Therefore, the following result holds.

Proposition 31. *The function κ_2 has infinite topological entropy.*

We recall that a point is called *transitive* if its orbit is dense, and that a *Li–Yorke pair* is a pair of points (x,y) such that

$$\limsup_{n\to\infty}\left|\kappa_2^n(x)-\kappa_2^n(y)\right|>0 \quad \text{and} \quad \liminf_{n\to\infty}\left|\kappa_2^n(x)-\kappa_2^n(y)\right|=0.$$

When continuity is assumed for interval maps, positive topological entropy implies Li–Yorke chaos and topological mixing implies transitivity ([20]). In this case, however, the implications do rely on the regularity assumption. In the following, we prove some results which imply topological mixing, but only establish a sort of "finite version" of transitivity and Li–Yorke chaos, as we will see.

Lemma 32. *For all $n\in\mathbb{N}$, $n\geq 1$, and for all $y\in[0,1/2^n]$, there exists $x\in[0,1/2^{2^n}]$ such that $\kappa_2(x)=y$.*

Proof. If $y=0$, then $x=0$. If $y\in]0,1/2^n]$, then $1/y\geq 2^n$. By Theorem 7, there exists $\alpha_y\in\{0,1\}^\omega$ such that $E(0^{2^n}\alpha_y)=1/y$. As a consequence, the number $x=(0.0^{2^n}\alpha_y)_2$ is such that $\kappa_2(x)=y$ and $x\in[0,1/2^{2^n}]$. □

Proposition 33. *For each non-empty open interval $I\subset[0,1]$, there exists an $n\in\mathbb{N}$ such that $\kappa_2^n(I)=[0,1/2]$.*

Proof. Let I be a non-empty open interval. By Proposition 15, there exists a number $x\in I$ such that $\kappa_2(x)>0$. By Theorem 7, there exists a neighbourhood $U\subset I$ of x such that $[0,\kappa_2(x)]\subset\kappa_2(U)$. Let us use the following notation for the tetration operation:

$$^{n}2:=\begin{cases}1, & \text{if } n=0,\\ 2^{\,^{n-1}2}, & \text{if } n>0,\end{cases}$$

hence

$$^{n}2=\underbrace{2^{2^{\cdot^{\cdot^{2}}}}}_{n\text{-times}}.$$

Let $n\in\mathbb{N}$ be such that $1/(^{n}2)<\kappa_2(x)$. Hence,

$$\left[0,1/\left(^{n}2\right)\right]\subset\left[0,\kappa_2(x)\right]\subset\kappa_2(U)\subset\kappa_2(I),$$

and by Lemma 32 we have $[0,1/(^{n-1}2)]\subset\kappa_2([0,1/(^{n}2)])\subset\kappa_2^2(I)$. By an iterative procedure involving Lemma 32 we have

$$\left[0,1/\left(^{n-k+1}2\right)\right]\subset\kappa_2^k(I),\quad \forall\, k=1,\dots,n,$$

hence $[0,1/(^{1}2)]=[0,1/2]\subset\kappa_2^n(I)$. Since also $\kappa_2^n(x)\subset[0,1/2]$, we are done. □

The following result directly follows from Proposition 33.

Corollary 34. *The function*

$$\bar{\kappa}_2:=\kappa_2|_{[0,1/2]}:[0,1/2]\to[0,1/2]$$

is topologically mixing, namely for every two non-empty open sets $U,V\subset[0,1/2]$, there exists $N\in\mathbb{N}$ such that $\bar{\kappa}_2^n(U)\cap V\neq\emptyset$, for all $n\geq N$.

Before stating the next result, we need the following definition.

Definition 35. *For all $w\in\{0,1\}^+$, we define the cylinder characterized by w as the open interval*

$$I_w:=\left](0.w)_2,(0.w)_2+2^{-\ell(w)}\right[.$$

We remark that if $x\in I_w$, then there exists $z_x\in\{0,1\}^\omega$ such that $x=(0.wz_x)_2$.

Lemma 36. *For every* $w_{(1)}, w_{(2)} \in \{0,1\}^+$ *such that* $(0.w_{(1)})_2, (0.w_{(2)})_2 < 1/2$, *there exist* $x_1, x_2 \in I_{w_{(1)}}$ *and a number* $n \in \mathbb{N}$ *such that:*

(i) $\kappa_2^{n+2}(x_1), \kappa_2^{n+2}(x_2) \in I_{w_{(2)}}$;
(ii) $\kappa_2^n(x_2) - \kappa_2^n(x_1) > 1/8$.

Proof. Since $(0.w_{(2)})_2 < 1/2$, by Theorem 7, there exist $\alpha_1, \alpha_2 \in \{0,1\}^\omega$ such that

$$1/E(00\alpha_1), 1/E(01\alpha_2) \in I_{w_{(2)}}.$$

Let $z_{(1)} = (0.00\alpha_1)_2$ and $z_{(2)} = (0.01\alpha_2)_2$. Since $z_{(1)} < 1/4$, by Lemma 32 there exists $y_1 \in [0, 1/16]$ such that $\kappa_2(y_1) = z_{(1)}$. Moreover, using again Theorem 7, there exists $\beta_1 \in \{0,1\}^\omega$ such that $1/E(010\beta_2) = z_{(2)}$. As a consequence, setting $y_2 = (0.010\beta_2)_2$ we have $\kappa_2^2(y_1), \kappa_2^2(y_2) \in I_{w_{(2)}}$ and $y_2 - y_1 > 1/4 - 1/16 > 1/8$. Therefore, it remains to prove that there exist $x_1, x_2 \in I_{w_{(1)}}$ and a number $n \in \mathbb{N}$ such that $\kappa_2^n(x_1) = y_1$ and $\kappa_2^n(x_2) = y_2$. Let $\mu = \max\{2, E(w_{(1)})\}$. By Theorem 7, for any $\alpha > \mu$ there exists a word $w \in \{0,1\}^\omega$ such that $E(w_{(1)}w) = \alpha$. As a consequence, we have $[0, 1/\mu[\subset \kappa_2(I_{w_{(1)}})$. Since $w_{(1)}$ is a finite word, $1/\mu > 0$. By Proposition 33, there exists an $n \in \mathbb{N}$ such that $\kappa_2^{n-1}([0, 1/\mu[) = [0, 1/2]$, hence $\kappa_2^n(I_{w_{(1)}}) = [0, 1/2]$. As a consequence, there exist $x_1, x_2 \in I_{w_{(1)}}$ such that $\kappa_2(x_1) = y_1$ and $\kappa_2(x_2) = y_2$, and we are done. □

The result proven in Lemma 36 can be interpreted as the existence of "finite-type transitivity" and "finite-type Li–Yorke pairs", in the sense that, by a backward iterative procedure, it ensures that we can find:

(1) points whose orbit visits an arbitrarily large (but only finite) set of open intervals of arbitrarily small size;
(2) pairs of points which, for as many iterates as we want (but only finitely many), are, alternatively, arbitrarily close and at some finite distance.

Whether it can be established the existence of proper Li–Yorke chaos, and/or of a point with a dense orbit, is an open question for the authors. In fact, even the existence of an infinite orbit, although extremely plausible on general ground, is not directly achievable through the technique developed in the previous lemmas. Notice that the topologically generic (in the C^0 topology) bounded Baire 1 map is not Li–Yorke chaotic (as proven in [21]).

We conclude this section with a remark concerning dimensional aspects. Let us recall that the box-counting dimension of a subset $A \subset I$ can be defined as:

$$\dim_B(A) = \lim_{n\to\infty} \frac{\log N(n)}{n \log 2}, \tag{13}$$

where $N(n)$ is the number of intervals of length 2^{-n}, having dyadic rationals as endpoints, needed to cover the set A. This is equivalent to the standard definition of box-counting dimension, in which no constraint is imposed on the endpoints of the covering intervals, (see [11, p. 41]).

For rational numbers $\alpha < 7/3$, the number of words of length n which are α-power free grows polynomially with n (cf. [13])[2]. This, using (13), immediately implies that

$$\dim_H\left(\left\{x \in [0,1] : \kappa_2(x) > \frac{3}{7}\right\}\right) \le \dim_B\left(\left\{x \in [0,1] : \kappa_2(x) > \frac{3}{7}\right\}\right) = 0,$$

where $\dim_H$ indicates the Hausdorff dimension. It is unknown to the authors which is $\dim_H(D_2)$.

[2] The authors thank Boris Adamczewski for having indicated this result.

5. The functions κ_n

In this section we extend our investigation to maps generated using bases other than 2. We will derive most of the properties from the ones established for κ_2, although the deduction is not trivial as, similarly to what happens for properties linked to the expansion of reals in different bases (such as, for instance, normality), no easy procedure allows a direct generalization.

Let $n \in \mathbb{N}$, $n \geq 2$, and consider $\mathscr{A}_n = \{a_0, a_1, \dots, a_{n-1}\}$. For every $x \in [0,1]$, we denote by $w_{x,n} \in \mathscr{A}_n^\omega$ the expansion of x in base n, namely

$$x = \left(0.w_{x,n}\right)_n.$$

Since the definition of critical exponent does not depend on the chosen alphabet, we give the following definition.

Definition 37. *For every integer $n \geq 2$, the function $\kappa_n \colon [0,1] \to [0,1]$ is defined by:*

$$\kappa_n(x) = \inf_{u \in \mathscr{L}(w_{x,n})} \frac{1}{E(u)}.$$

We denote by C_n the set of the zeros of κ_n, hence $C_n = \{x \in I : \kappa_n(x) = 0\}$, and we set $D_n = [0,1] \setminus C_n$.

Remark 38. As for the κ_2 function, the following properties hold: κ_n is upper semi-continuous; $\mathscr{N}_n \subset C_n$, where $\mathscr{N}_n$ is the set of points in $[0,1]$ that are normal real numbers in base n; C_n is a co-meagre set of full Lebesgue measure; κ_n is continuous at x if and only if $x \in C_n$.

For all $n \in \mathbb{N}$, $n \geq 2$, we denote by $r_n \in [0,1]$ the supremum over the set of $y \in [0,1]$ such that there exists $x \in [0,1]$ with $\kappa_n(x) \geq y$. More formally,

$$r_n = \left\{ y \in [0,1] : \left\{ x \in I : \kappa_n(x) \geq y \right\} \neq \emptyset \right\}, \quad \forall\, n \in \mathbb{N}, n \geq 2. \tag{14}$$

Using the Dejean's theorem [9], proved by M. Rao [18], we have the following proposition.

Proposition 39. *Let r_n be defined by* (14)*. Then*

$$r_n = \begin{cases} \frac{4}{7}, & \textit{if } n = 3, \\ \frac{5}{7}, & \textit{if } n = 4, \\ \frac{n-1}{n}, & \textit{otherwise.} \end{cases}$$

Remark 40. By the results given in [7] and [24], for $n = 2,3$ we have $\kappa_n([0,1]) = [0, r_n]$, namely

$$\kappa_2([0,1]) = [0, 1/2], \quad \kappa_3([0,1]) = [0, 4/7].$$

However, at the best of our knowledge, there are not similar results for $n \geq 4$.

Since for every $\alpha \in]1, \infty[$ it is possible to find a finite alphabet and an infinite word w on it such that α is its critical exponent (c.f. [14]), we have the following result.

Proposition 41. *For every $\alpha \in [0,1)$, there exist $n \geq 2$ and $x \in [0,1]$ such that $\kappa_n(x) = \alpha$.*

Proof. If $\alpha = 0$, then $\kappa_n(0) = \alpha$, for all $n \geq 2$. Hence, using the result of [14], there exists an $n \geq 2$ and a word $w \in \{a_0, a_1, \dots, a_{n-1}\}^\omega$ such that $E(w) = 1/\alpha \in]1, \infty[$. Hence, setting $x = (0.w)_n$, we have $\kappa_n(x) = 1/E(w) = \alpha$. □

Some analytical and dynamical properties of the general function κ_n can be obtained by considering its restriction on the set of points whose n-base expansion is an infinite binary word, more precisely on the set

$$A_2^n := \left\{ x \in [0,1] : w_{x,n} \in \{0,1\}^\omega \right\}.$$

In particular, Proposition 19 can be used to prove the following.

Corollary 42. *Let $n \geq 2$. If $x \in C_n$, then*

$$R^-\left(\kappa_n, x\right) \cup R^+\left(\kappa_n, x\right) = \{0\}. \tag{15}$$

If $x \in A_2^n \cap D_n$, then at least one of the following holds:

$$\left[0, \kappa_n(x)\right[\subset R^-\left(\kappa_n, x\right) \quad or \quad \left[0, \kappa_n(x)\right[\subset R^+\left(\kappa_n, x\right). \tag{16}$$

Proof. If $n = 2$, then the thesis is equivalent to Proposition 19. So let us assume $n > 2$.

As in Proposition 19, if $x \in C_n$, then the upper semi-continuity and the density of $\mathbb{Q} \cap [0,1]$ in $[0,1]$ implies (15).

If $x \in A_2^n \cap D_n$, then $w_{x,n} \in \{0,1\}^\omega$ and $\kappa_n(x) \in]0,1/2]$. Let $y = (0.w_{x,n})_2$, hence $\kappa_2(y) = \kappa_n(x)$. By Proposition 19, either $[0,\kappa_2(y)[\subset R^-(\kappa_2, y)$ or $[0,\kappa_2(y)[\subset R^+(\kappa_2, y)$. Without loss of generality, let us assume that $[0,\kappa_2(y)[\subset R^-(\kappa_2, y)$ (the other case is similar). In this case, we shall prove that $[0,\kappa_n(x)[\subset R^-(\kappa_n, x)$. This is equivalent to prove that for every $\alpha \in [0,\kappa_n(x)[$, there exists a sequence $(x_{(\alpha,k)})_k \subset [0,1]$ such that $x_{(\alpha,k)} < x$ and $\kappa_n(x_{(\alpha,k)}) = \alpha$ for all k, and $\lim_{k\to\infty} x_{(\alpha,k)} = x$. Let us fix $\alpha \in [0,\kappa_n(x)[= [0,\kappa_2(y)[$. Since $[0,\kappa_2(y)[\subset R^-(\kappa_2, y)$, there exists a sequence $(y_{(\alpha,k)})_k \subset [0,1]$ such that $y_{(\alpha,k)} < y$ and $\kappa_2(y_{(\alpha,k)}) = \alpha$ for all k, and $\lim_{k\to\infty} y_{(\alpha,k)} = y$. Now, set $x_{(\alpha,k)} = (0.w_{y_{(\alpha,k)}})_n$. Therefore, $\kappa_n(x_{(\alpha,k)}) = \alpha$ and $x_{(\alpha,k)} < x$ for all k.

From $x \in A_2^n \cap D_n$ it follows $\kappa_n(x) = \kappa_2(y) > 0$, so y is not rational, and in particular it is not a dyadic rational. This implies that $w_{y_{(\alpha,k)}}$ converges to $w_{x,n}$ in standard product topology on the Cantor space $\{0,1\}^\omega$. This implies that $\lim_{k\to\infty} x_{(\alpha,k)} = x$. Then, (16) holds. □

Corollary 42 plays for κ_n the same role of Proposition 19 for κ_2. Indeed, it implies the following results for the function κ_n.

- The existence of an uncountable set of almost-fixed points: as for κ_2, if $\kappa_n(x) > x$ then x is an almost-fixed point (see Corollary 24). Let $x_{\tau,n} = (0.w_\tau)_n$. Then $\kappa_n(x_{\tau,n}) = 1/2 > x_{\tau,n}$, so $x_{\tau,n}$ is an almost fixed point. With a similar procedure exploited in the proof of Corollary 25, Corollary 42 implies the existence of an uncountable set of almost-fixed points for κ_n.
- The existence of an infinite set of fixed-points: Corollary 26 holds since the function κ_2 is left- or right-Darboux at every point, is of Baire class one and it has almost-fixed points. Since each κ_n verifies these properties, results analogous to Corollary 26 and Corollary 27 hold.
- The existence of horseshoes of every order: Proposition 30 is based on Proposition 19 and on the fact that $x_\tau < 7/16$. Since $x_{\tau,n} = (0.w_\tau)_n < x_\tau < 7/16$, a similar result holds for κ_n. So also the next result holds.
- For every $n \geq 2$, κ_n has infinite topological entropy.

As for what concerns the generalizations of results 32-36, they are more difficult, since they are concerned with multiple iterations of κ_n. Indeed, for $p > 1$, $\kappa_n^p((0.w)_n)$ depends on n in a complicated way.

6. The function κ

We finally want to study an object again defined through the critical exponent, but independent of the choice of a particular base. The simplest way to achieve that is the following.

Definition 43. *We define $\kappa\colon [0,1] \to [0,1]$ as*

$$\kappa(x) = \sup\left\{\kappa_n(x) : n \in \mathbb{N}, n \geq 2\right\}.$$

Remark 44. The function κ clearly vanishes on absolutely normal real numbers. As a consequence, κ is Lebesgue-measurable and $\int_0^1 \kappa(x)\mathrm{d}x = 0$.

Proposition 45. *Set $C_\kappa = \{x \in I : \kappa(x) = 0\}$. Then $I \backslash C_\kappa$ is an uncountable set.*

Proof. It is based on the fact that every real number greater than 1 is a critical exponent(see [14]). More formally, for every $\alpha > 1$, there exists $n > 1$ and $w_\alpha \in \{a_0, a_1, \dots, a_{n-1}\}^\infty$ such that $\alpha = E_n(0.w_\alpha)$. As a consequence, being $\kappa(x) = \sup_{n>1} \kappa_n(x)$, we have $\bigcup_{\alpha>1}\{0.w_\alpha\} \subset I\backslash C_\kappa$, so $I \setminus C_\kappa$ is uncountable. ☐

Proposition 46. *The set C_κ is co-meagre.*

Proof. By definition of κ, $x \in C_\kappa$ if and only if $\kappa_n(x) = 0$ for every $n > 1$. Hence

$$C_\kappa = \bigcap_{n\in\mathbb{N},\, n\geq 2} C_n,$$

and

$$I\backslash C_\kappa = \bigcup_{n\in\mathbb{N},\, n\geq 2} ([0,1]\backslash C_n).$$

Since the countable union of meagre sets is meagre, $I\backslash C_\kappa$ is meagre, thus C_κ is a co-meagre set. ☐

Proposition 47. *For all $x \in [0,1]$, κ is not continuous in x.*

Proof. If $\kappa(x) \neq 0$, then for every $\varepsilon > 0$, there exists $x_\varepsilon \in]x-\varepsilon, x+\varepsilon[\cap \mathbb{Q}$. As a consequence,

$$\kappa(x) - \kappa(x_\varepsilon) = \kappa(x) > 0, \quad \forall\, \varepsilon > 0,$$

hence κ is not continuous in x. Now, we consider $x \in [0,1]$ such that $\kappa(x) = 0$ and we are going to prove that for every $\varepsilon > 0$ there exists an $x_\varepsilon \in]x-\varepsilon, x+\varepsilon[$ such that $\kappa(x_\varepsilon) \geq 1/2$. For every $\varepsilon > 0$, there exists an $\bar{n} \geq 2$ and an $a_i \in \mathscr{A}_{\bar{n}}$ such that, for every $w \in \mathscr{A}_{\bar{n}}^\infty$, we have $(0.a_i\, w)_{\bar{n}} \in]x-\varepsilon, x+\varepsilon[$. Since the word w_τ defined using the Thue–Morse sequence

$$w_\tau = a_0 a_1\; a_1 a_0\; a_1 a_0 a_0 a_1\; a_0 a_1 a_1 a_0 a_1 a_0 a_0 a_1 \dots,$$

belongs to $\mathscr{A}_n^\infty$ for all $n \geq 2$, we obtain

$$x_\varepsilon = \left(0.a_i\; w_\tau\right)_{\bar{n}} \in]x-\varepsilon, x+\varepsilon[, \quad \text{and} \quad \kappa(x_\varepsilon) \geq \kappa_{\bar{n}}(x_\varepsilon) = \frac{1}{2}.$$

As a consequence, κ is not continuous in x, and we are done. ☐

Lemma 48. *For every $n \in \mathbb{N}$, the Baire class of order n is closed with respect to the maximum between a finite number of functions. In other words, if $f_1, \dots, f_k \in \mathscr{B}_n$, with $k \in \mathbb{N}, k \geq 1$, then $h = \max\{f_1, \dots, f_k\} \in \mathscr{B}_n$.*

Proof. Set $n = 0$. Since the maximum of two continuous functions is a continuous function, $\mathscr{B}_0$ is closed with respect the maximum between two elements. Now suppose that $\mathscr{B}_{n-1}$ is closed with respect the maximum between two elements. Let $f, g \in \mathscr{B}_n$ and $h = \max\{f, g\}$. By definition, there exist two sequences $(f_k), (g_k) \subset \mathscr{B}_{n-1}$ such that

$$f(x) = \lim_{k\to\infty} f_k(x), \quad g(x) = \lim_{k\to\infty} g_k(x), \quad \forall\, x.$$

We define $h_k = \max\{f_k, g_k\}$, which is in $\mathscr{B}_{n-1}$ by induction hypothesis. Since we have

$$h(x) = \max\left\{f(x), g(x)\right\} = \lim_{k\to\infty} \max\left\{f_k(x), g_k(x)\right\} = \lim_{k\to\infty} h_k(x),$$

then $h \in \mathscr{B}_n$. This proves that $\mathscr{B}_n$ is closed with respect to the maximum between two functions, $\forall\, n \in \mathbb{N}$.

To extend this result to the maximum between a finite number of functions, it suffices to notice that

$$\max\left\{f_1, f_2, \dots, f_k\right\} = \max\Big\{\max\dots\left\{\max\{\max\left\{f_1, f_2\right\}, f_3\}\dots\right\}, f_k\Big\}.$$

☐

Proposition 49. *The function κ belongs to $\mathscr{B}_2 \setminus \mathscr{B}_1$.*

Proof. By Proposition 47, $k \notin \mathscr{B}_1$. For every $n \in \mathbb{N}$, $n \geq 2$, let us define $h_n \colon [0,1] \to [0,1]$ as

$$h_n(x) = \max\{\kappa_2(x), \ldots, \kappa_n(x)\}.$$

By Lemma 48, $h_n \in \mathscr{B}_1$. Since

$$\kappa(x) = \sup_{n \geq 2} \kappa_n(x) = \lim_{n \to \infty} h_n(x),$$

then $\kappa \in \mathscr{B}_2$. □

7. Some open questions

This work opens some questions that link combinatorial, analytical and dynamical properties. Some of them are listed below.

- What is the Hausdorff dimension of D_2 (and in general of D_n)?
- Is κ_2 (and in general κ_n) a Darboux function?
- Are the fixed points of κ_2 (and in general of κ_n) uncountable?
- Does there exist a transitive point for κ_2 (and, in general, for κ_n)?
- Does there exist a Li–Yorke pair for κ_2 (and, in general, for κ_n)?
- Does κ attain the value 1?

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