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Cristhian Montoya

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Remarks on local controllability for the Boussinesq system with Navier boundary condition

Remarque sur la contrôlabilité locale du système de Boussinesq avec la condition de frontière de Navier

Cristhian Montoya^a

^a Universidad Técnica Federico Santa María, Casilla 110-V, Valparaíso, Chile

E-mail: cristhian.montoya@usm.cl

Abstract. This note deals with the local exact controllability to a particular class of trajectories for the Boussinesq system with nonlinear Navier–slip boundary conditions and internal controls having vanishing components. Briefly speaking, in two dimensions, the local exact controllability property is obtained using only one control in the heat equation, whereas two scalar controls are required in three dimensions.

Résumé. Cette note concerne la contrôlabilité locale d'une classe particulière de trajectoires, ceci pour le système de Boussinesq avec la condition de Navier non linéaire et certains contrôles internes. En bref, la propriété de contrôlabilité exacte locale s'obtient en dimension deux en n'utilisant que le contrôle associé à l'équation de la chaleur. En revanche, deux contrôles scalaires sont nécessaires pour obtenir notre résultat dans le cas de dimension trois

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1. Introduction

The interaction of incompressible fluids with a diffusion process can be modeled by a coupled system between the Navier–Stokes and heat equations, usually called Boussinesq system. On bounded domains, both heat and the velocity field can show a different behaviour on its boundary. In this paper, nonlinear Navier–type boundary conditions for the fluid flow and homogeneous Neumann conditions for the diffusion equation are considered in order to study the local exact controllability for the Boussinesq system with few scalar controls.

Henceforth, let Ω be a nonempty bounded connected open subset of \mathbb{R}^N ($N = 2$ or $N = 3$) of class C^∞ . Let $T > 0$ and let $\omega \subset \Omega$ be a (small) nonempty open subset which is the control domain. Here, we will use the notation $Q := \Omega \times (0, T)$, $\Sigma := \partial\Omega \times (0, T)$ and n the outward unit normal vector to Ω . Moreover, C denotes a generic positive constant which may depend on Ω and ω .

In this Note, we will consider the Boussinesq system with Navier–slip and Neumann conditions

$$\begin{cases} y_t - \nabla \cdot (Dy) + (y, \nabla)y + \nabla p = u\chi_\omega + \theta e_N, & \nabla \cdot y = 0 & \text{in } Q, \\ \theta_t - \Delta\theta + y \cdot \nabla\theta = v1_\omega & & \text{in } Q, \\ y \cdot n = 0, (\sigma(y, p) \cdot n)_{tg} + f(y)_{tg} = 0, & \nabla\theta \cdot n = 0 & \text{on } \Sigma, \\ y(\cdot, 0) = y_0(\cdot), \theta(\cdot, 0) = \theta_0(\cdot) & & \text{in } \Omega, \end{cases} \quad (1)$$

as well as the linearized Boussinesq system (around a target flow of the form $((0, \bar{p}, \bar{\theta}))$)

$$\begin{cases} y_t - \nabla \cdot (Dy) + \nabla p = h_1 + u\chi_\omega + \theta e_N, & \nabla \cdot y = 0 & \text{in } Q, \\ \theta_t - \Delta\theta + y \cdot \nabla\bar{\theta} = h_2 + v1_\omega & & \text{in } Q, \\ y \cdot n = 0, (\sigma(y, p) \cdot n)_{tg} + (A(x, t)y)_{tg} = 0, & \nabla\theta \cdot n = 0 & \text{on } \Sigma, \\ y(\cdot, 0) = y_0(\cdot), \theta(\cdot, 0) = \theta_0(\cdot) & & \text{in } \Omega, \end{cases} \quad (2)$$

where $y = y(x, t)$ is the velocity field of the fluid, $\theta = \theta(x, t)$ their temperature, v and $u = (u_1, \dots, u_N)$ stand for the controls, which are acting in a arbitrary fixed domain $\omega \times (0, T)$, where χ_ω is a smooth positive function such that $\chi_\omega = 1$ in ω' , $\omega' \Subset \omega$, and 1_ω is the indicator function. Here, the gravity vector field is given by $e_N = (0, 1)$ for $N = 2$, or $e_N = (0, 0, 1)$ for $N = 3$. Moreover, $f: \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a nonlinear regular function given, $\sigma(y, p) := -p\text{Id} + Dy$ is the stress tensor, A is a $N \times N$ matrix-valued function in a suitable space, and tg stands for the tangential component of the corresponding vector field, i.e., $y_{tg} = y - (y \cdot n)n$.

In the context of controllability, the first results for the Boussinesq system were made by Fursikov and Imanuvilov in [8] and [9]. The work by S. Guerrero [10] shows the local exact controllability to the trajectories of the Boussinesq system with Dirichlet boundary conditions, meanwhile, the same author proven in [11] the local exact controllability to the trajectories for the Navier–Stokes with Navier–slip boundary conditions. In both papers $N + 1$ distributed scalar controls supported in small sets are considered.

Additionally, recent works have been developed for controllability problems with a reduced number of controls. For instance, N. Carreño and S. Guerrero in [2] have proven the local null controllability for the Navier–Stokes system with Dirichlet conditions and $N - 1$ scalar controls. The recent work made by S. Guerrero and C. Montoya shows that the local null controllability property is achieved for the N -dimensional Navier–Stokes system with Navier–slip conditions and $N - 1$ scalar controls [12]. The methodology in the previous articles are Carleman estimates. In the three dimensional case of the Navier–Stokes system with Dirichlet conditions, J-M. Coron and P. Lissy developed in [4] a new strategy to prove the local null controllability using only one scalar control.

Concerning the N -dimensional Boussinesq system with Dirichlet conditions, in [6] the authors proved that the local exact controllability to the trajectories can be achieved with $N - 1$ scalar controls, under certain geometric assumption on the control domain. N. Carreño showed the local controllability of the N -Boussinesq system using $N - 1$ scalar controls, without conditions on the control domain [1].

Our theorems extend the results of [2] and [12]. Taking into account the relation between the observability and controllability property, it will be appropriate to consider the following adjoint

system related to (2):

$$\begin{cases} -\varphi_t - \nabla \cdot (D\varphi) + \nabla \pi = g - \psi \nabla \bar{\theta}, & \nabla \cdot \varphi = 0 & \text{in } Q, \\ -\psi_t - \Delta \psi = g_0 + \varphi \cdot e_N & & \text{in } Q, \\ \varphi \cdot n = 0, (\sigma(\varphi, \pi) \cdot n)_{t_g} + (A^t(x, t)\varphi)_{t_g} = 0, & \nabla \psi \cdot n = 0 & \text{on } \Sigma, \\ \varphi(\cdot, T) = \varphi^T(\cdot), \psi(\cdot, T) = \psi^T(\cdot) & & \text{in } \Omega, \end{cases} \quad (3)$$

where g, φ^T, g_0 and ψ^T satisfying adequate regularity assumptions. We will introduce several spaces and hypotheses over $\bar{\theta}$ which will be needed in order to have suitable Carleman estimates for the solution of (3):

$$\begin{aligned} W &= \{u \in H^1(\Omega)^N : \nabla \cdot u = 0 \text{ in } \Omega, u \cdot n = 0 \text{ on } \partial\Omega\}, \\ H &= \{u \in L^2(\Omega)^N : \nabla \cdot u = 0, \text{ in } \Omega, u \cdot n = 0 \text{ on } \partial\Omega\}, \\ P_\varepsilon^1 &= H^{5/4+\varepsilon}(0, T; L^2(\partial\Omega)^{N \times N}), \quad P^2 = L^2(0, T; H^{5/2}(\partial\Omega)^{N \times N}), \quad \forall \varepsilon > 0, \\ Y_m &:= L^2(0, T; H^{2m}(\Omega)^N) \cap H^m(0, T; L^2(\Omega)^N), \quad m = 1, 2. \end{aligned}$$

and

$$\bar{\theta} \in L^\infty(0, T; W^{3,\infty}(\Omega)), \quad \nabla \bar{\theta}_t \in L^\infty(Q)^N. \quad (4)$$

Here, the target flow $(0, \bar{p}, \bar{\theta})$ satisfies the problem

$$\begin{cases} \nabla \bar{p} = \bar{\theta} e_N, & \bar{\theta}_t - \Delta \bar{\theta} = 0 & \text{in } Q, \\ \nabla \bar{\theta} \cdot n = 0 & & \text{on } \Sigma, \\ \bar{\theta}(\cdot, 0) = \bar{\theta}_0(\cdot) & & \text{in } \Omega. \end{cases} \quad (5)$$

One of the main result in this Note concerns the local controllability to a particular class of trajectories of (1). This result is presented in the following theorem.

Theorem 1. *Assume $f \in C^4(\mathbb{R}^N; \mathbb{R}^N)$ with $f(0) = 0$ and $i \in \{1, \dots, N-1\}$ fixed. Let $(0, \bar{p}, \bar{\theta})$ be a solution to (5) satisfying (4). Then, for every $T > 0$ and $\omega \subset \Omega$, there exists $\delta > 0$ such that, for every $(y_0, \theta_0) \in [H^3(\Omega)^N \cap W] \times H^1(\Omega)$ satisfying*

$$(Dy_0 \cdot n)_{t_g} + (f(y_0))_{t_g} = 0 \text{ on } \partial\Omega \quad \text{and} \quad \|(y_0, \theta_0) - (0, \bar{\theta}_0)\|_{[H^3(\Omega)^N \cap W] \times H^1(\Omega)} \leq \delta, \quad (6)$$

we can find controls $v \in L^2(\omega \times (0, T))$ and $u \in L^2(0, T; H^2(\omega)^N) \cap H^1(0, T; L^2(\omega)^N)$ with $u_i \equiv 0$ and $u_N \equiv 0$ such that the corresponding solution (y, p, θ) to (1) satisfies

$$y(\cdot, T) = 0 \quad \text{and} \quad \theta(\cdot, T) = \bar{\theta}(\cdot, T) \quad \text{in } \Omega. \quad (7)$$

The second main in this Note provides a new Carleman inequality for the linear Boussinesq system given in (3), see Section 2, see Theorem 2. Finally, in Section 3, the main ideas of the proof of Theorem 1 are presented.

2. A new Carleman inequality

Our main result in this section is a new Carleman estimate for the solution of (3). Before presenting such an inequality, several weight functions are needed:

$$\begin{aligned} \alpha(x, t) &= \frac{e^{2\lambda\|\eta\|_\infty} - e^{\lambda\eta(x)}}{(t(T-t))^{11}}, & \xi(x, t) &= \frac{e^{\lambda\eta(x)}}{(t(T-t))^{11}}, & \alpha^*(t) &= \max_{x \in \bar{\Omega}} \alpha(x, t), \\ \zeta^*(t) &= \min_{x \in \bar{\Omega}} \xi(x, t), & \hat{\alpha}(t) &= \min_{x \in \bar{\Omega}} \alpha(x, t), & \hat{\zeta}(t) &= \max_{x \in \bar{\Omega}} \xi(x, t). \end{aligned} \quad (8)$$

Here, $\eta \in C^2(\bar{\Omega})$ and satisfies that

$$|\nabla \eta| > 0 \text{ in } \bar{\Omega} \setminus \omega_0, \quad \eta > 0 \text{ in } \Omega \quad \text{and} \quad \eta \equiv 0 \text{ on } \partial\Omega,$$

where $\omega_0 \Subset \omega_1 \Subset \omega' \Subset \omega$ is a nonempty open set. The existence of such a function η is proved in [7].

The new Carleman inequality is given in the following theorem.

Theorem 2. Assume $A \in P_\varepsilon^1 \cap P^2$ for some $\varepsilon > 0$ and $(0, \bar{p}, \bar{\theta})$ satisfying (4)–(5). There exists a constant λ_0 , such that for any $\lambda \geq \lambda_0$ there exist two constants $C(\lambda) > 0$ increasing on $\|A\|_{P_\varepsilon^1 \cap P^2}$ and $s_0(\lambda) > 0$ such that for any $j \in \{1, 2\}$, any $a > 0$, any $g \in L^2(Q)^N$, any $g_0 \in L^2(Q)$, any $\varphi^T \in H$ and any $\psi^T \in L^2(\Omega)$, the solution of (3) satisfies

$$\begin{aligned} & s^3 \iint_Q e^{-2(1+a)s\alpha^*} (\xi^*)^3 |\varphi|^2 dx dt + s^5 \iint_Q e^{-2(1+a)s\alpha^*} (\xi^*)^5 |\psi|^2 dx dt \\ & \leq C \left(\iint_Q e^{-2asa^*} (|g|^2 + |g_0|^2) dx dt + (N-2) s^7 \int_0^T \int_{\omega'} e^{-4s\hat{a}+2(1-a)s\alpha^*} (\hat{\xi})^{12} |\varphi_j|^2 dx dt \right. \\ & \quad \left. + s^{13} \int_0^T \int_{\omega} e^{-8s\hat{a}+(6-2a)s\alpha^*} (\hat{\xi})^{24} |\psi|^2 dx dt \right), \quad (9) \end{aligned}$$

for every $s \geq s_0$.

Sketch of the proof. Without loss of generality, we consider $N = 3$. However, the strategy can be easily adapted to the general case. Our arguments are based in [2, 3, 6, 12]. From (3) and using the decomposition $\rho\varphi = w + z$, $\rho\pi = \pi_z + \pi_w$ and $\rho\psi = \tilde{\psi}$, where $\rho(t) = e^{-asa^*}$ and $a > 0$, it is very easy to verify that (w, π_w) , (z, π_z) and $\tilde{\psi}$ are solutions to the systems

$$\begin{cases} -w_t - \nabla \cdot (Dw) + \nabla \pi_w = \rho g; & -z_t - \nabla \cdot (Dz) + \nabla \pi_z = -\rho' \varphi - \tilde{\psi} \nabla \bar{\theta} & \text{in } Q, \\ \nabla \cdot w = 0; & \nabla \cdot z = 0 & \text{in } Q, \\ w \cdot n = 0, (\sigma(w, \pi_w) \cdot n)_{tg} + (A^t(x, t)w)_{tg} = 0; & z \cdot n = 0, (\sigma(z, \pi_z) \cdot n)_{tg} + (A^t(x, t)z)_{tg} = 0 & \text{on } \Sigma, \\ w(\cdot, T) = 0; & z(\cdot, T) = 0 & \text{in } \Omega, \end{cases}$$

and

$$\begin{cases} -\tilde{\psi}_t - \Delta \tilde{\psi} = \rho g_0 + \rho \varphi_3 - \rho' \psi & \text{in } Q, \\ \nabla \tilde{\psi} \cdot n = 0 & \text{on } \Sigma, \\ \tilde{\psi}(\cdot, T) = 0 & \text{in } \Omega, \end{cases} \quad (10)$$

We will use the Carleman inequality for parabolic equations with Neumann conditions [7] for the system (10) in order to estimate the global terms associated to $\tilde{\psi}$. Thus, if ω_1 is an open set such that $\omega_1 \Subset \omega$, there exists $\bar{\lambda} > 0$ such that for any $\lambda > \bar{\lambda}$ there exists a positive constant C depending on $\lambda, \Omega, \omega_1, \|\theta\|_{L^\infty(0, T; W^{3, \infty}(\Omega))}$ such that

$$\begin{aligned} & \iint_Q e^{-2s\alpha} (s\xi |\tilde{\psi}_t|^2 + s\xi \sum_{\ell, m=1}^3 |\partial_{\ell m} \tilde{\psi}|^2 + s^3 \xi^3 |\nabla \tilde{\psi}|^2 + s^5 \xi^5 |\tilde{\psi}|^2) dx dt \\ & \leq C \left(\iint_Q e^{-2s\alpha} s^2 \xi^2 (|\rho g_0|^2 + |\varphi_3|^2 + |\rho'|^2 |\rho|^{-2} |\tilde{\psi}|^2) dx dt + s^5 \int_0^T \int_{\omega_1} e^{-2s\alpha} \xi^5 |\tilde{\psi}|^2 dx dt \right), \quad (11) \end{aligned}$$

for every $s \geq C$.

On the other hand, for the first and third scalar component of z , we use the same arguments of the proof of Proposition 3.1 of [12]. After that, we can deduce for z and $\tilde{\psi}$ the inequality

$$\begin{aligned} I(s, z) + J(s, \tilde{\psi}) & \leq C \left(\|\rho g\|_{L^2(Q)^3}^2 + \|\rho g_0\|_{L^2(Q)}^2 + s^5 \int_0^T \int_{\omega_1} e^{-2s\alpha} \xi^5 |\tilde{\psi}|^2 dx dt \right. \\ & \quad \left. + \sum_{k=1, k \neq 2}^3 \left[\int_0^T \int_{\omega_1} e^{-2s\alpha} (s^5 \xi^5 |z_k|^2 + s^3 \xi^3 |\nabla z_k|^2) dx dt + \int_0^T \int_{\omega_2} e^{-2s\alpha} \xi^2 |\nabla \partial_k \pi_z|^2 dx dt \right] \right), \quad (12) \end{aligned}$$

where $J(s, \tilde{\psi})$ denotes the left-hand side of (11) and $I(s, z)$ is defined by

$$\begin{aligned} I(s, z) & := \sum_{k=1, k \neq 2}^3 s^5 \iint_Q e^{-2s\alpha} \xi^5 |z_k|^2 dx dt + s^3 \iint_Q e^{-2s\alpha} \xi^3 |\nabla z_k|^2 dx dt + s^3 \iint_Q e^{-2s\alpha} \xi^3 |z_2|^2 dx dt \\ & \quad + \|s^{1/2} e^{-s\alpha^*} (\xi^*)^{9/22} z\|_{Y_1}^2 + \|s^{-1/2} e^{-s\alpha^*} (\xi^*)^{-15/22} z\|_{L^2(0, T; H^4(\Omega^3) \cap H^2(0, T; L^2(\Omega^3))}^2 \\ & \quad + \|s^{1/2} e^{-s\alpha^*} (\xi^*)^{9/22} \pi_z\|_{L^2(0, T; H^1(\Omega))}^2. \end{aligned}$$

Here, ω_2 is an open set such that $\omega_1 \Subset \omega_2 \Subset \omega$. The rest of the proof is oriented towards the absorption of the local pressure term in (12). However, we have omitted these details since analogous arguments can be found in [12, Section 3]. Let us remark that the regularity over $\bar{\theta}$ given in (4) is used in several estimates associated to the pressure term. The other local terms can be estimated in an easier way. Therefore, those local estimates lead to the desired Carleman inequality (9). \square

3. Local controllability for the Boussinesq system

The proof of Theorem 1 follows the ideas of [2] and [12]. Henceforth, we consider $N = 3$ and a control function for the movement equation in (1) and (2) of the form $u = (u_1, 0, 0)$. Again, the arguments can be easily adapted to the general case. Thus, in a first step a null controllability result for (2) with an appropriate right-hand side h_1, h_2 is mentioned. Here, the idea is to look for a solution in an appropriate weighted functional space. Let us define the operators

$$L_1 w := w_t - \nabla \cdot Dw \quad \text{and} \quad L_2 w := w_t - \Delta w$$

and also the space E as follows:

$$\begin{aligned} \{(y, p, u_1, \theta, v) : & e^{as\beta^*} y, e^{2s\hat{\beta} - (1-a)s\beta^*} (\hat{\gamma})^{-6} (u_1, 0, 0)\chi_\omega, \tilde{\rho}(\partial_t u_1, 0, 0) \in L^2(Q)^3, e^{as\beta^*} \theta \in L^2(Q), \\ & e^{4s\hat{\beta} - (3-a)s\beta^*} (\hat{\gamma})^{-12} v1_\omega \in L^2(Q), \tilde{\rho} u_1 \in L^2(0, T; H^2(\Omega)), \text{supp } u_1 \subset \omega \times (0, T), \\ & e^{as\beta^*} (\gamma^*)^{-12/11} y \in Y_1, e^{as\beta^*} (\gamma^*)^{-12/11} \theta \in L^2(0, T; H^2(\Omega)) \cap H^1(0, T; L^2(\Omega)), \\ & e^{(a+1)s\beta^*} (\gamma^*)^{-3/2} (L_1 y + \nabla p - (u_1, 0, 0)\chi_\omega - \theta e_3) \in L^2(Q)^3, \\ & e^{(a+1)s\beta^*} (\gamma^*)^{-5/2} (L_2 \theta + y \cdot \nabla \bar{\theta} - v1_\omega) \in L^2(Q)\} =: E, \end{aligned}$$

where $\tilde{\rho} := e^{4s\hat{\beta} + 2(1-a)s\beta^*} (\hat{\gamma})^{-12} e^{-(1+a)s\beta^*} (\gamma^*)^{9/22}$ and whose weight functions are given by

$$\begin{aligned} \beta(x, t) &= \frac{e^{2\lambda\|\eta\|_\infty} - e^{\lambda\eta(x)}}{\ell^{11}(t)}, & \gamma(x, t) &= \frac{e^{\lambda\eta(x)}}{\ell^{11}(t)}, & \beta^*(t) &= \max_{x \in \bar{\Omega}} \beta(x, t), \\ \gamma^*(t) &= \min_{x \in \bar{\Omega}} \gamma(x, t), & \hat{\beta}(t) &= \min_{x \in \bar{\Omega}} \beta(x, t), & \hat{\gamma}(t) &= \max_{x \in \bar{\Omega}} \gamma(x, t). \end{aligned} \quad (13)$$

In this case, $\ell \in C^2([0, T])$ is a positive function in $[0, T)$ such that $\ell(t) > t(T-t)$ for all $t \in [0, T/4]$ and $\ell(t) = t(T-t)$ for all $t \in [T/2, T]$.

Proposition 3. *Let s and λ be like in Theorem 2 and $(0, \bar{p}, \bar{\theta})$ satisfy (5). Assume that*

$$y_0 \in W, \theta_0 \in H^1(\Omega), e^{(a+1)s\beta^*} (\gamma^*)^{-3/2} h_1 \in L^2(Q)^3 \quad \text{and} \quad e^{(a+1)s\beta^*} (\gamma^*)^{-5/2} h_2 \in L^2(Q). \quad (14)$$

Then, there exists controls u_1 and v such that, if (y, p, θ) is the associated solution to (2), we have $(y, p, u_1, \theta, v) \in E$. In particular $y(\cdot, T) = 0$ and $\theta(\cdot, T) = 0$ in Ω .

The rest of the proof of Theorem 1 relies on two fixed point theorems, namely, one for the nonlinearity posed on the boundary condition, and another one, for the convective term in (1). We will mention only these results since the methodology given in [12] can be adapted to (1). To do that, let us consider the following system:

$$\begin{cases} y_t - \nabla \cdot (Dy) + \nabla p = h_1 + (u_1, 0, 0)\chi_\omega + \theta e_3, & \nabla \cdot y = 0 & \text{in } Q, \\ \theta_t - \Delta \theta + y \cdot \nabla \bar{\theta} = h_2 + v1_\omega & & \text{in } Q, \\ y \cdot n = 0, (\sigma(y, p) \cdot n)_{t_g} + (f(y))_{t_g} = 0, & \nabla \theta \cdot n = 0 & \text{on } \Sigma, \\ y(\cdot, 0) = y_0(\cdot), \theta(\cdot, 0) = \theta_0(\cdot) & & \text{in } \Omega. \end{cases} \quad (15)$$

Theorem 4. *Let us assume that $f \in C^4(\mathbb{R}^3; \mathbb{R}^3)$ with $f(0) = 0$. Then, for every $T > 0$ and $\omega \subset \Omega$, there exists $\delta > 0$ such that, for every $a > 0$ and for every $(y_0, \theta_0) \in H^3(\Omega)^3 \cap W \times H^1(\Omega)$, $h_1 \in Y_1, h_2 \in L^2(Q)$ satisfying $e^{(a+1)s\beta^*} (\gamma^*)^{-3/2} h_1 \in L^2(Q)^3$ and $e^{(a+1)s\beta^*} (\gamma^*)^{-5/2} h_2 \in L^2(Q)$,*

$$\|h_1\|_{Y_1} + \|h_2\|_{L^2(Q)} + \|y_0\|_{H^3(\Omega)^3 \cap W} + \|\theta_0\|_{H^1(\Omega)} \leq \delta \quad (16)$$

and (6), there exists controls $v \in L^2(0, T; L^2(\omega))$ and $u_1 \in L^2(0, T; H^2(\omega)) \cap H^1(0, T; L^2(\omega))$ and an associated solution (y, p, θ) of (15) satisfying $(y, \theta) \in Y_2 \times L^2(0, T; H^2(\Omega)) \cap H^1(0, T; L^2(\Omega))$ and such that $(y, p, u_1, \theta, v) \in E$.

Theorem 5. *Suppose that $\mathcal{B}_1, \mathcal{B}_2$ are Banach spaces and*

$$\mathcal{A} : \mathcal{B}_1 \rightarrow \mathcal{B}_2$$

is a continuously differentiable map. We assume that for $b_1^0 \in \mathcal{B}_1, b_2^0 \in \mathcal{B}_2$ the equality

$$\mathcal{A}(b_1^0) = b_2^0 \quad (17)$$

holds and $\mathcal{A}'(b_1^0) : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ is an epimorphism. Then there exists $\delta > 0$ such that for any $b_2 \in \mathcal{B}_2$ which satisfies the condition

$$\|b_2^0 - b_2\|_{\mathcal{B}_2} < \delta$$

there exists a solution $b_1 \in \mathcal{B}_1$ of the equation

$$\mathcal{A}(b_1) = b_2.$$

Let us set

$$y = \tilde{y}, \quad p = \bar{p} + \tilde{p} \quad \text{and} \quad \theta = \bar{\theta} + \tilde{\theta}.$$

For $a = 2 > 1$, we apply Theorem 5 with the spaces

$$\mathcal{B}_1 := \{(y, p, u_1, \theta, v) \in E : y \in Y_2\},$$

$$\mathcal{B}_2 := \{(h_1, y_0, h_2, \theta_0) \in Z_1 \times [H^3(\Omega)^3 \cap W] \times Z_2 \times H^1(\Omega) : h_1, h_2, y_0, \theta_0 \text{ satisfies (16)}\},$$

and where

$$Z_1 := L^2(e^{3s\beta^*} (\gamma^*)^{-3/2}(0, T); L^2(\Omega)^3), \quad \text{and} \quad Z_2 := L^2(e^{3s\beta^*} (\gamma^*)^{-5/2}(0, T); L^2(\Omega)).$$

By defining the operator $\mathcal{A} : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ by

$$\mathcal{A} \rightarrow (L_1 \tilde{y} + (\tilde{y} \cdot \nabla) \tilde{y} + \nabla \tilde{p} - \tilde{\theta} e_3 - (u_1, 0, 0) \chi_\omega, \tilde{y}_0, L_2 \tilde{\theta} + \tilde{y} \cdot \nabla \tilde{\theta} + \tilde{y} \cdot \nabla \bar{\theta} - \nu 1_\omega, \tilde{\theta}_0),$$

for every $(\tilde{y}, \tilde{p}, u_1, \tilde{\theta}, v) \in \mathcal{B}_1$, one can easily check the conditions for \mathcal{A} in order to complete the proof of Theorem 1.

Some open problems

It would be interesting to know if the local controllability to the trajectories with $N - 1$ scalar controls holds for $\bar{y} \neq 0$ and ω like in Theorem 1. However, is not clear at all and therefore is an open problem even for the Navier–Stokes system.

On the other side, could be reasonable to expect results of the same kind whether one considers nonlinear conditions such as $\nabla \theta \cdot n + g(\theta) = 0$, where g is a suitable function to study.

Recently, Coron et al. have proved a global exact controllability result for the Navier–Stokes and Navier–type conditions (for small time), see [5]. A challenging problem would be to use the Boussinesq system proposed in this Note in order to apply and prove analogous results to [5].

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